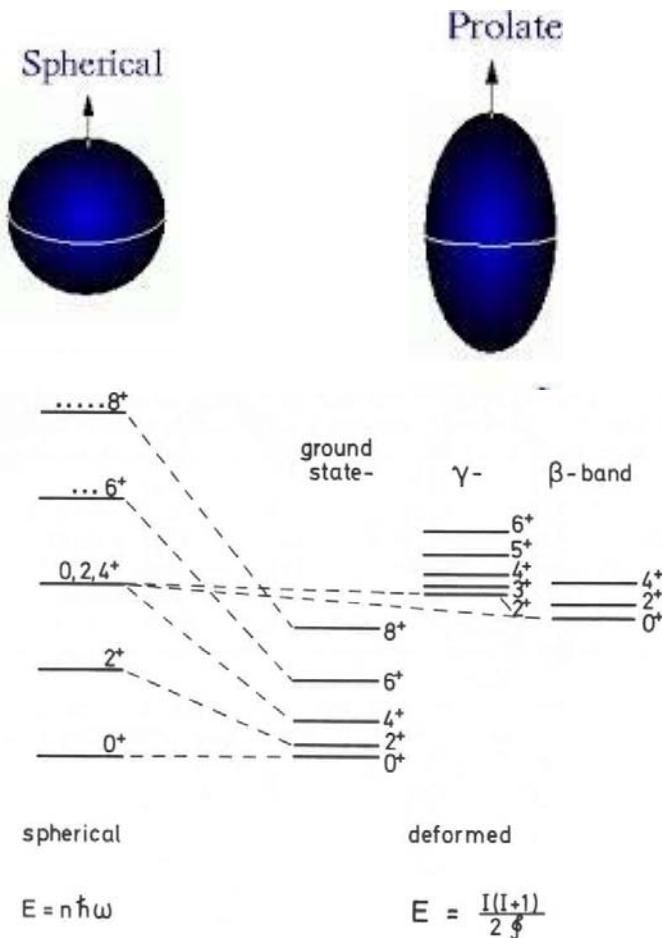


# Geometric Symmetries and Shapes in Light Nuclei

- Introduction
- Algebraic Cluster Model
- Special solutions
- Quantum phase transitions
- Applications:  $^{12}\text{C}$  and  $^{16}\text{O}$
- Energies and em couplings
- Summary and conclusions



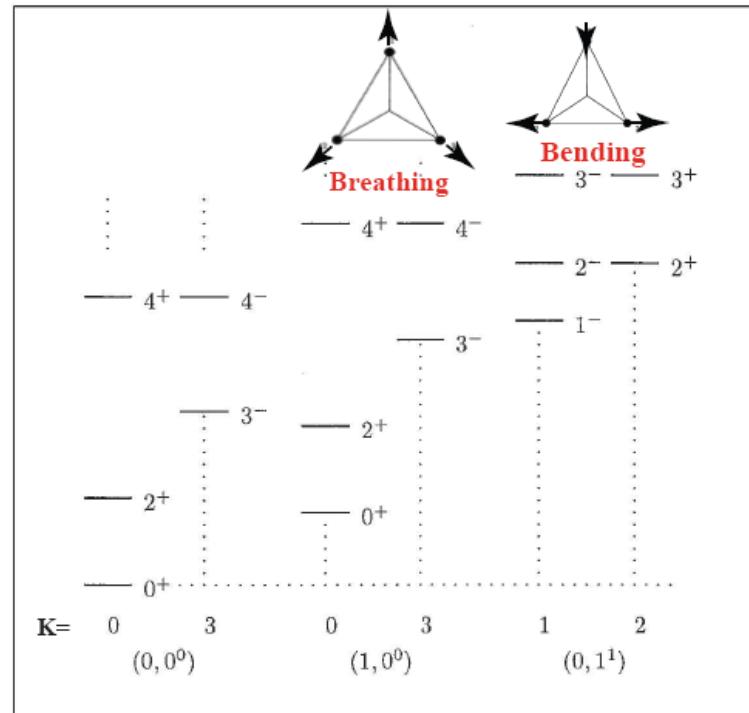
# Nuclear Shapes

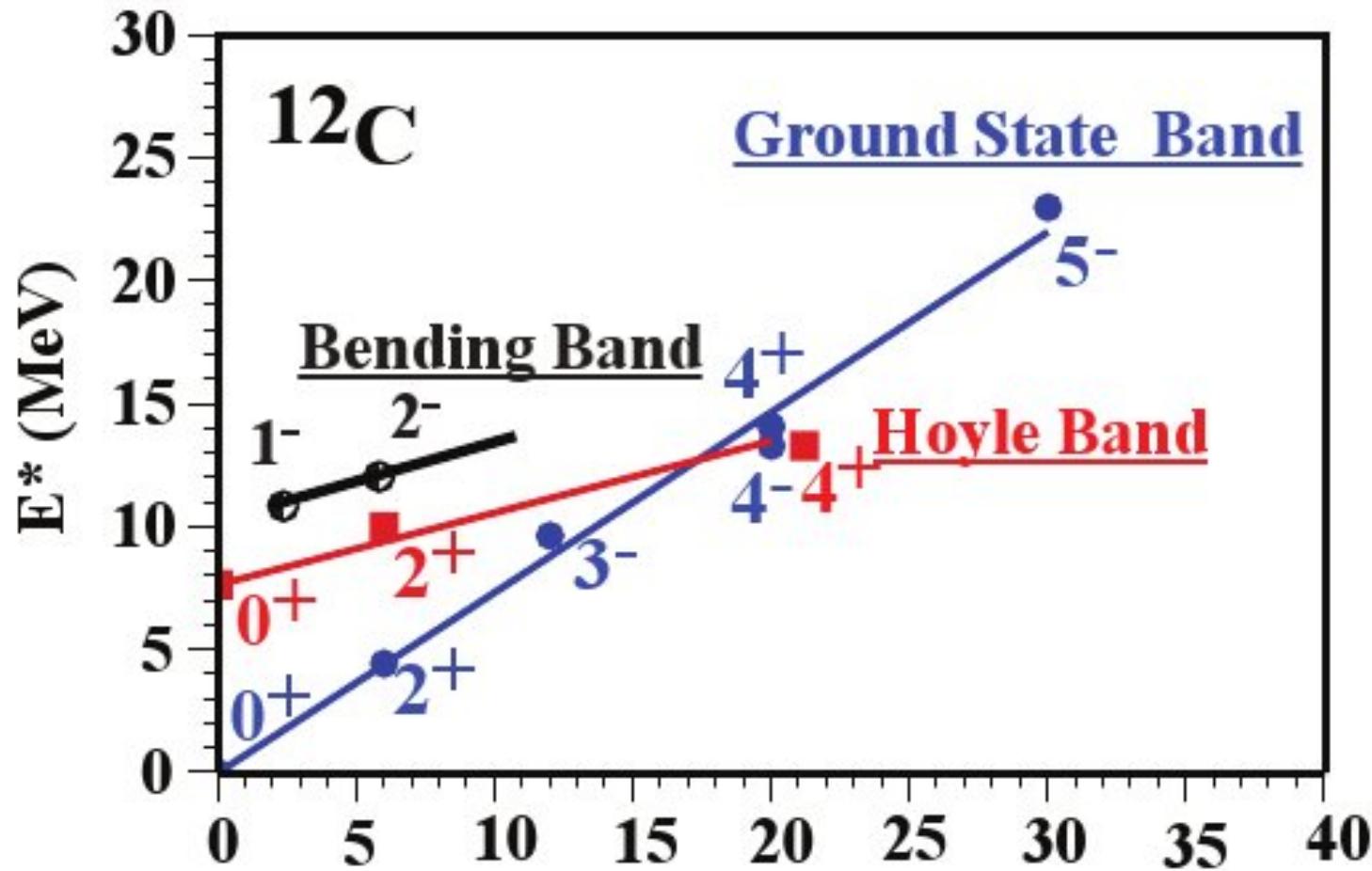


## Rotation-Vibration Spectrum of the Three Alpha Triangular Spinning Top

**U(7) Model**

R. Bijker and F. Iachello; Ann. Phys. **298**(2002)334





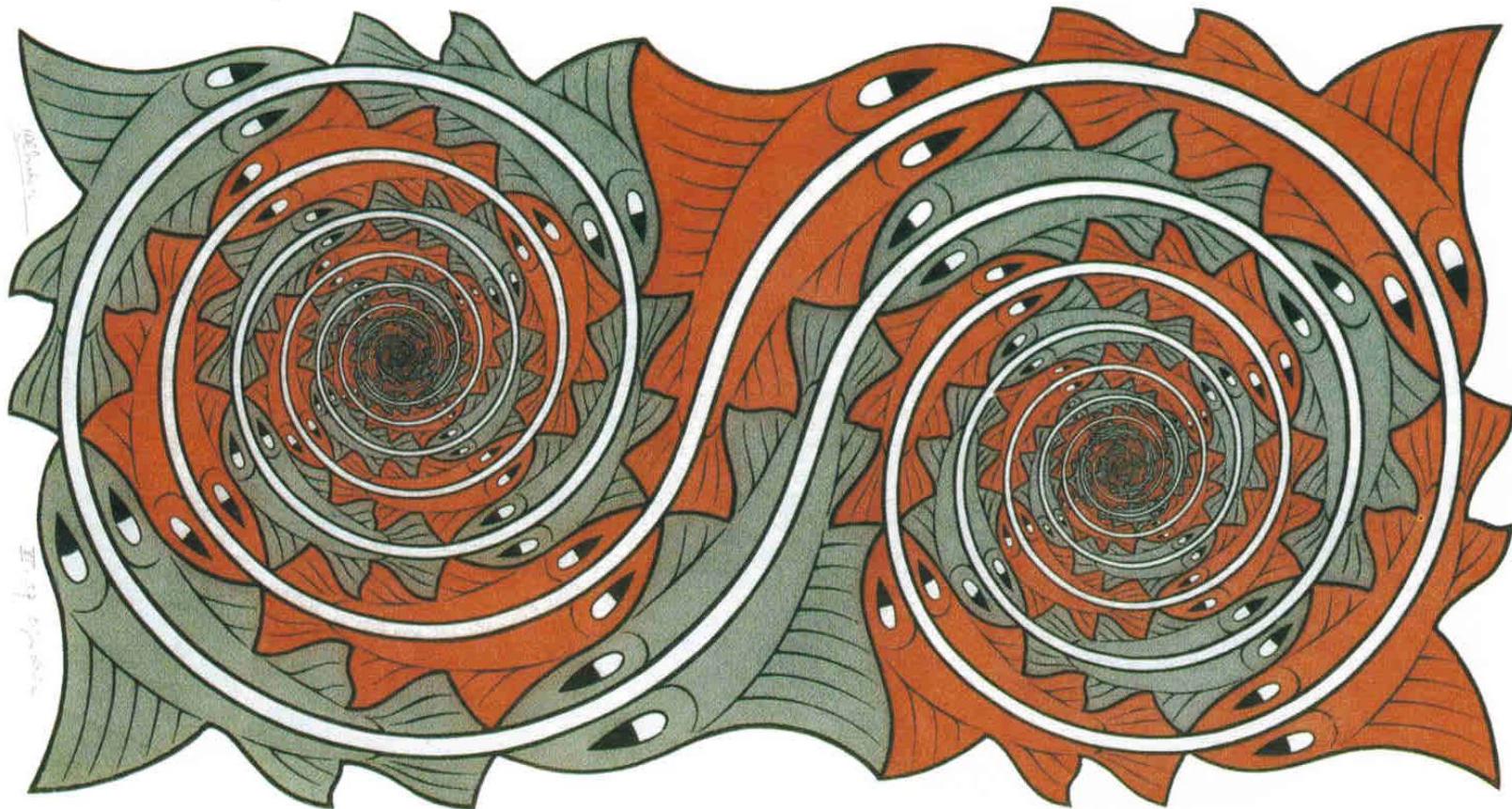
Itoh et al, PRC 84, 054308 (2011)  
 Freer et al, PRC 86, 034320 (2012)  
 Zimmerman et al, PRL 110, 152502 (2013)

$J(J+1)$

Marín-Lámbarri, Bijker et al,  
 PRL 113, 012502 (2014)

# Recent Theoretical Work

- Alpha-cluster model (Wheeler 1937, Brink 1966, Robson, 1978)
- AMD (Kanada-Enyo, PTP, 2007)
- FMD model (Chernykh et al, PRL, 2007)
- BEC-like cluster model (Funaki et al, PRC, 2009)
- Ab initio no-core shell model (Roth et al, PRL, 2011)
- Lattice EFT (Epelbaum et al, PRL, 2011, 2012)
- No-core symplectic model (Dreyfuss et al, PLB, 2013)
- **Algebraic Cluster Model (2002, 2014, 2017)**
- and many others
- Recent reviews: Freer & Fynbo, PPNP 78, 1 (2014), Jenkins & Courtin, JPG 42, 034010 (2015)



# Algebraic Cluster Model (ACM)

- Interacting boson model for few-body systems
- For  $v$  dof introduce a SGA of  $U(v+1)$
- 2-body system:  $U(4)$  model
- 3-body system:  $U(7)$  model
- $k$ -body system:  $U(3k-2)$  model
- Applications: hadrons, molecules,  
**alpha-cluster nuclei**

# ACM for k-body Systems

3k-3 relative degrees of freedom: Jacobi vectors

$$\vec{p}_j = \frac{1}{\sqrt{j(j+1)}} \left( \sum_{i=1}^j \vec{r}_i - j \vec{r}_{j+1} \right), \quad j = 1, 2, \dots, k-1$$

Introduce k-1 dipole bosons and a scalar boson

$$b_{j,m}^\dagger, s^\dagger, \quad j = 1, 2, \dots, k-1$$

One- and two-body  $S_k$  invariant Hamiltonian

$$\begin{aligned} H = & \epsilon_0 s^\dagger \tilde{s} - \epsilon_1 \sum_i b_i^\dagger \cdot \tilde{b}_i + u_0 s^\dagger s^\dagger \tilde{s} \tilde{s} \\ & - u_1 \sum_i s^\dagger b_i^\dagger \cdot \tilde{b}_i \tilde{s} + v_0 \sum_i [b_i^\dagger \cdot b_i^\dagger \tilde{s} \tilde{s} + \text{h.c.}] \\ & + \sum_L \sum_{iji'j'} v_{iji'j'}^{(L)} [b_i^\dagger \times b_j^\dagger]^{(L)} \cdot [\tilde{b}_{i'} \times \tilde{b}_{j'}]^{(L)} \end{aligned}$$

Mixing between  
different  
oscillator shells

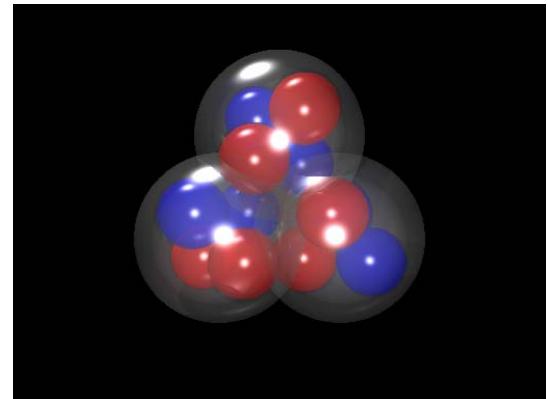
# 3-body Clusters: Oblate Top

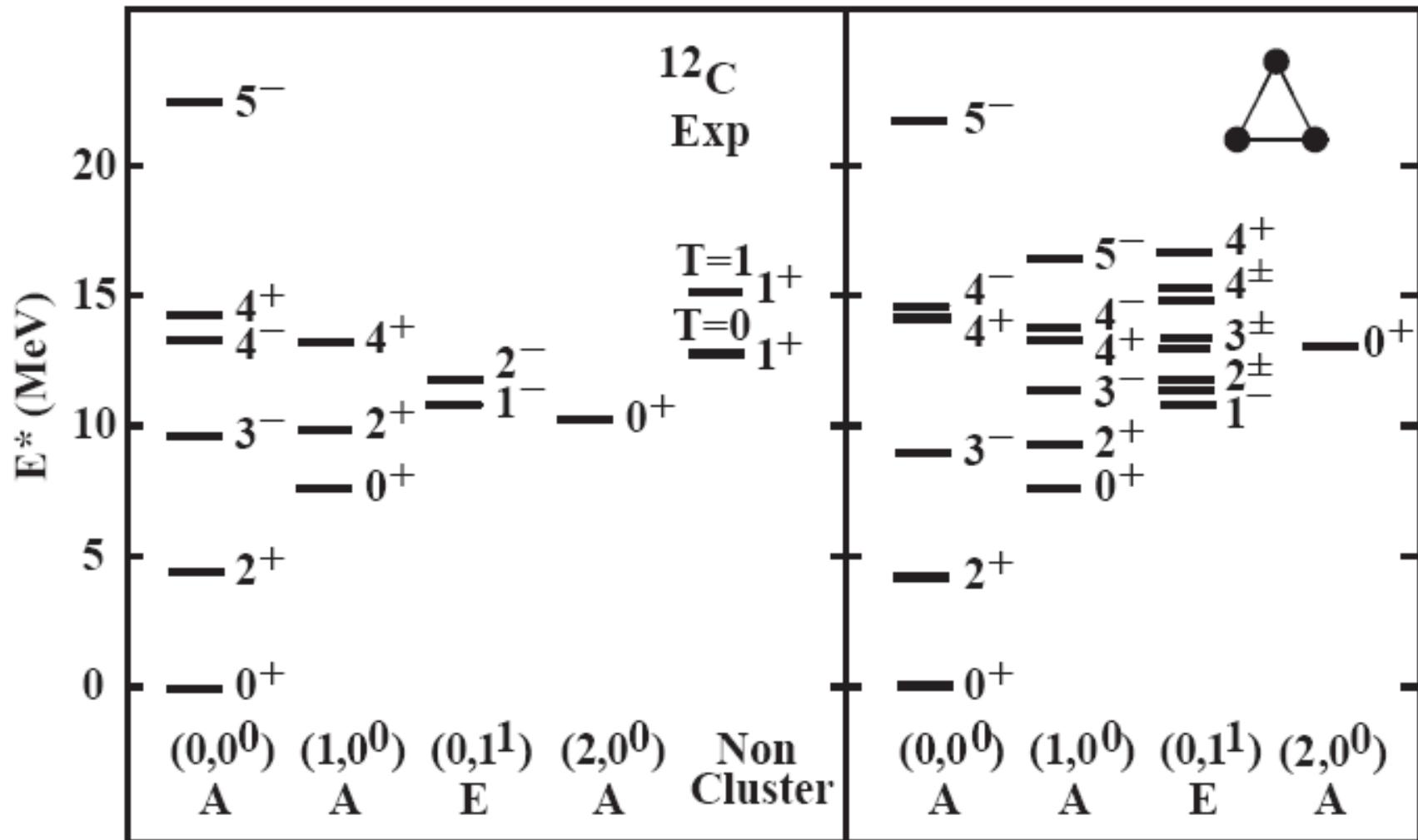
$$\begin{aligned} H = & \xi_1 (R^2 s^\dagger s^\dagger - b_\rho^\dagger \cdot b_\rho^\dagger - b_\lambda^\dagger \cdot b_\lambda^\dagger) (\text{h.c.}) \\ & + \xi_2 [(b_\rho^\dagger \cdot b_\rho^\dagger - b_\lambda^\dagger \cdot b_\lambda^\dagger) (\text{h.c.}) + 4(b_\rho^\dagger \cdot b_\lambda^\dagger) (\text{h.c.})] \\ & + \kappa_1 \vec{L} \cdot \vec{L} + \kappa_2 (b_\rho^\dagger \cdot \tilde{b}_\lambda - b_\lambda^\dagger \cdot \tilde{b}_\rho) (\text{h.c.}) \end{aligned}$$

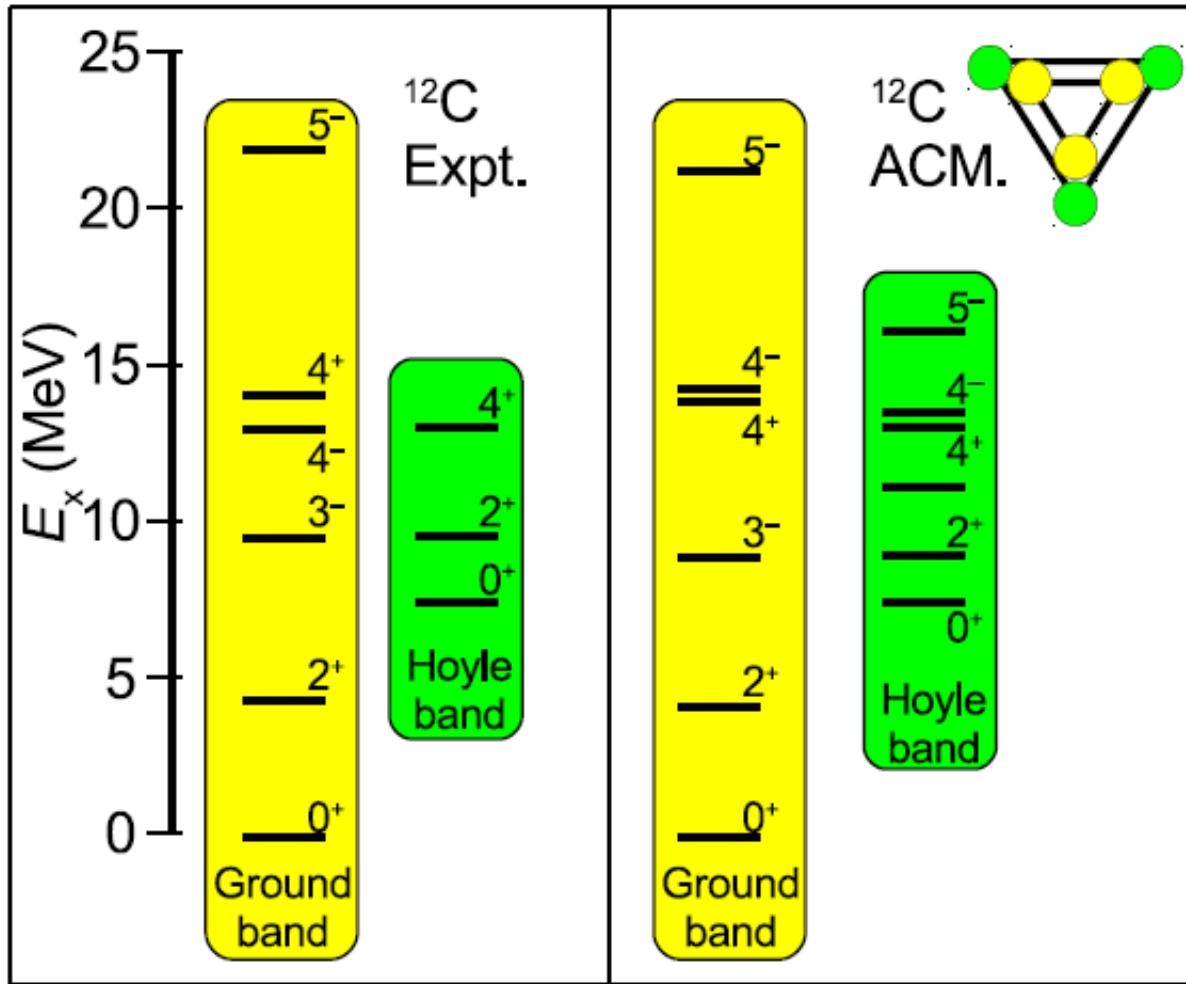
$R^2 = 0$	: anharmonic oscillator
$R^2 = 1, \xi_1 > 0, \xi_2 = 0$	: deformed oscillator
$R^2 \neq 0, \xi_1, \xi_2 > 0$	: oblate top

Equilibrium shape:  
equilateral triangle

Bijker, Iachello & Leviatan, AP 236, 69 (1994)  
Bijker & Iachello, AP 298, 334 (2002)







$$\langle r^2 \rangle_{\text{gs}}^{1/2} = 2.47 \text{ fm}$$

$$\langle r^2 \rangle_{\text{H}}^{1/2} = 3.45 \text{ fm}$$

Some evidence for negative parity strength between 11-14 MeV!

Freer et al, PRC 76, 034320 (2007)

t: experimentally observed states currently assigned to the group

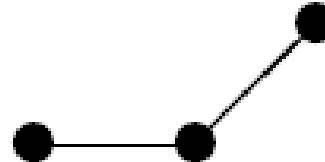
Tzany Kokalova, JPCS 569, 012010 (2014)

# Alpha-Cluster Configurations



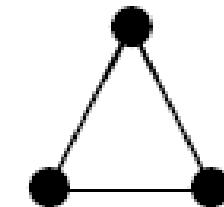
$$D_{\infty h}$$

Linear



$$C_{2v}$$

Bent



$$D_{3h}$$

Triangular

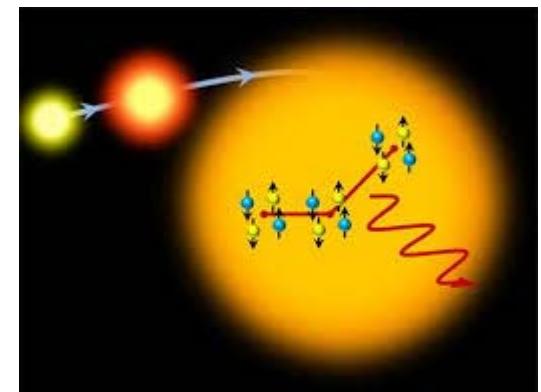
Examine rotational structure!

Morinaga, PR 101, 254 (1956)

Epelbaum et al, PRL 109, 252501 (2012)

Wheeler (1937), Robson (1978)

Bijker & Iachello, AP 298, 334 (2002)

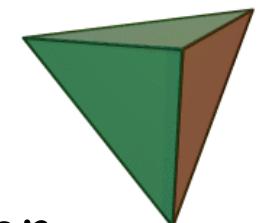




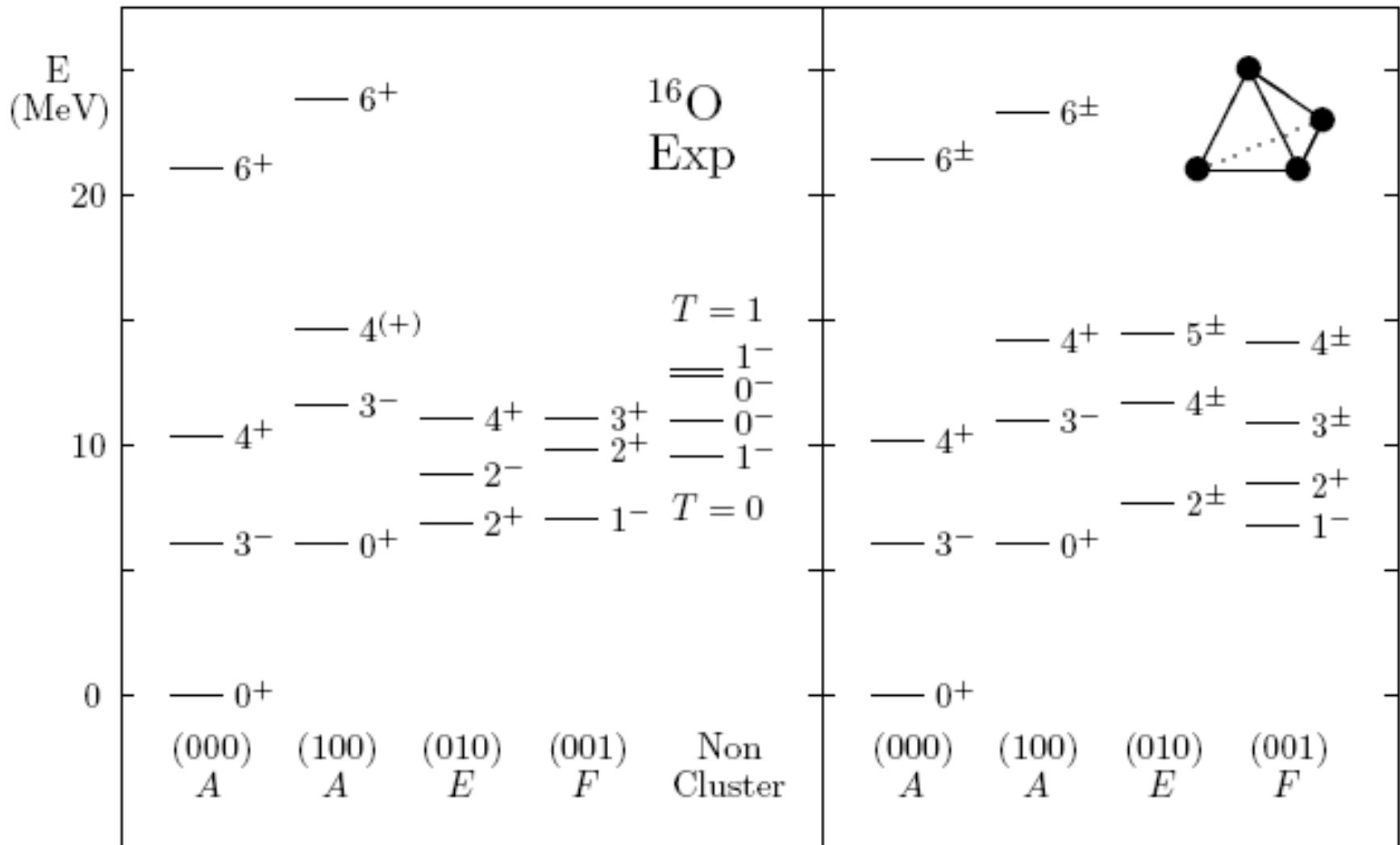
# 4-body Clusters: Spherical Top

$$\begin{aligned}
 H_{\text{vib}} = & \xi_1 (R^2 s^\dagger s^\dagger - b_\rho^\dagger \cdot b_\rho^\dagger - b_\lambda^\dagger \cdot b_\lambda^\dagger - b_\eta^\dagger \cdot b_\eta^\dagger) (\text{h.c.}) \\
 & + \xi_2 [(-2\sqrt{2} b_\rho^\dagger \cdot b_\eta^\dagger + 2b_\rho^\dagger \cdot b_\lambda^\dagger) (\text{h.c.}) \\
 & \quad + (-2\sqrt{2} b_\lambda^\dagger \cdot b_\eta^\dagger + (b_\rho^\dagger \cdot b_\rho^\dagger - b_\lambda^\dagger \cdot b_\lambda^\dagger)) (\text{h.c.})] \\
 & + \xi_3 [(2b_\rho^\dagger \cdot b_\eta^\dagger + 2\sqrt{2} b_\rho^\dagger \cdot b_\lambda^\dagger) (\text{h.c.}) \\
 & \quad + (2b_\lambda^\dagger \cdot b_\eta^\dagger + \sqrt{2} (b_\rho^\dagger \cdot b_\rho^\dagger - b_\lambda^\dagger \cdot b_\lambda^\dagger)) (\text{h.c.}) \\
 & \quad + (b_\rho^\dagger \cdot b_\rho^\dagger + b_\lambda^\dagger \cdot b_\lambda^\dagger - 2b_\eta^\dagger \cdot b_\eta^\dagger) (\text{h.c.})]
 \end{aligned}$$

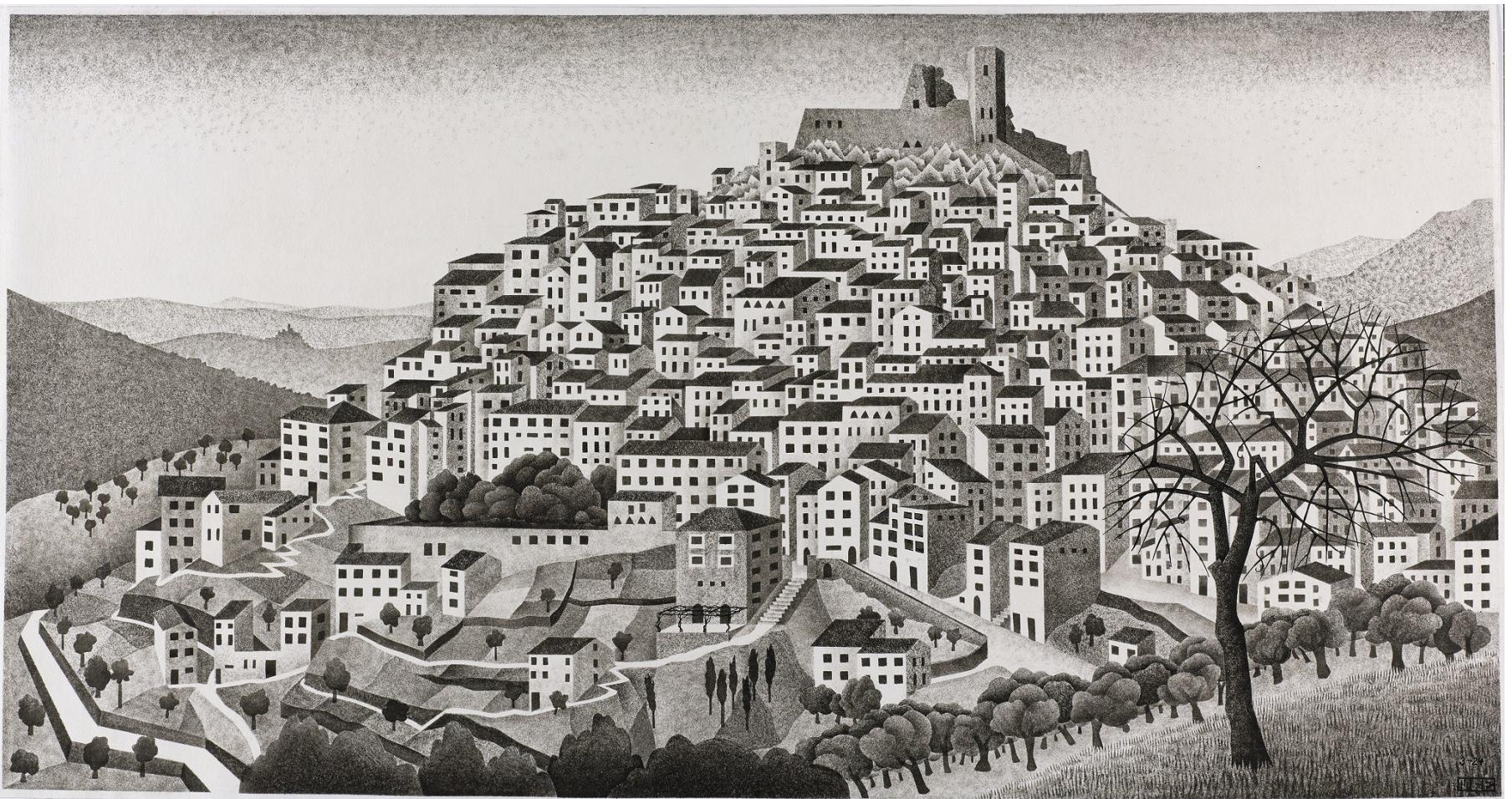
$R^2 = 0$	: anharmonic oscillator
$R^2 = 1, \xi_1 > 0, \xi_2 = \xi_3 = 0$	: deformed oscillator
$R^2 \neq 0, \xi_1, \xi_2, \xi_3 > 0$	: spherical top



Equilibrium shape: Regular tetrahedron



Bijker & Iachello, PRL 112, 152501 (2014)  
NPA 957, 154 (2017)

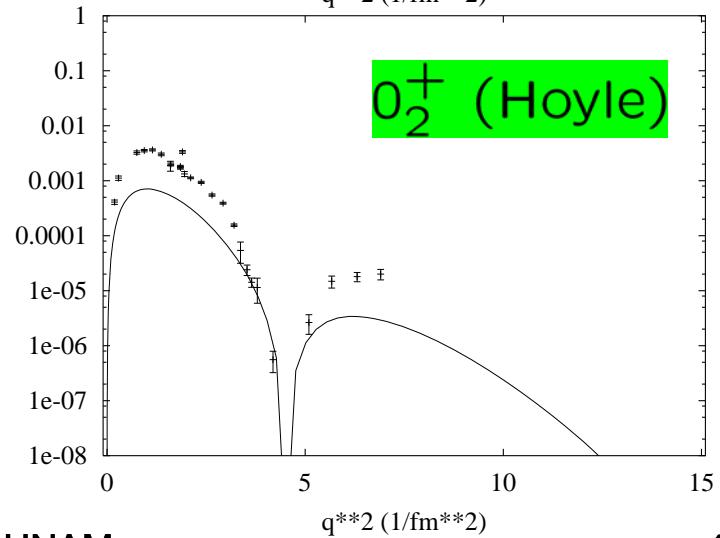
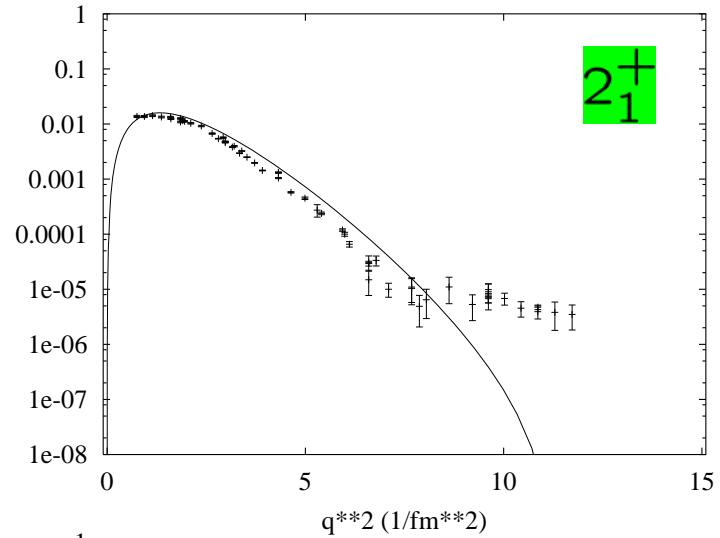
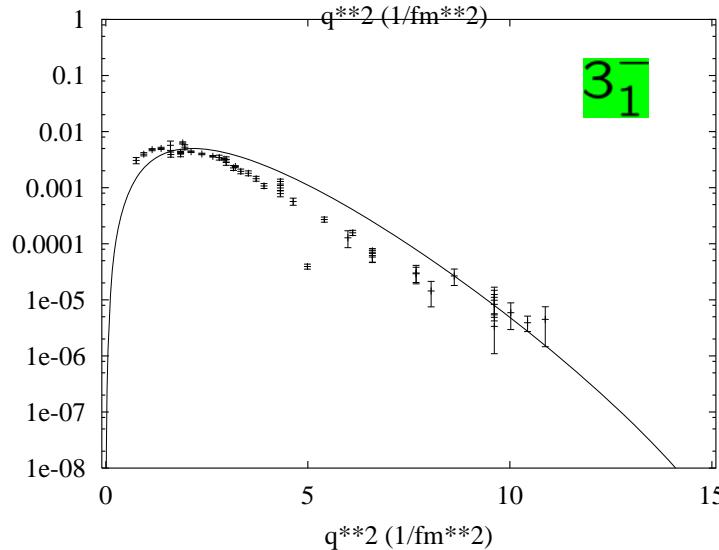
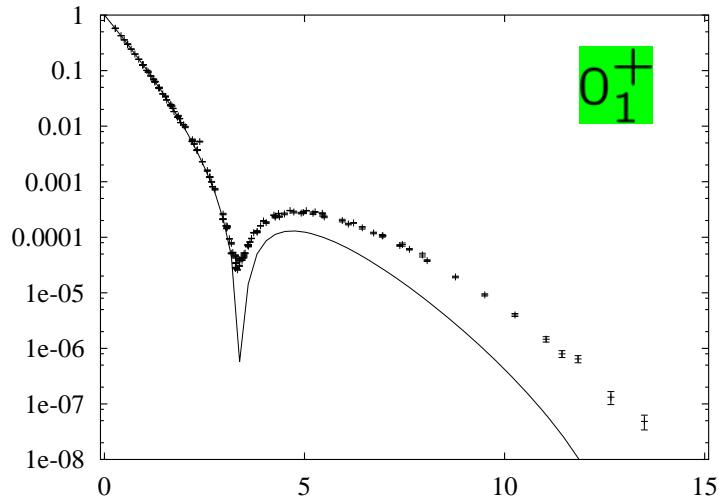


# Electric Transitions

$$\begin{aligned}
 \rho(\vec{r}) &= \frac{Ze}{k} \left( \frac{\alpha}{\pi} \right)^{3/2} \sum_{i=1}^k e^{-\alpha(\vec{r}-\vec{r}_i)^2} \\
 \mathcal{F}_L(q) &\rightarrow c_L j_L(q\beta) e^{-q^2/4\alpha} \\
 B(EL; 0^+ \rightarrow L^P) &\rightarrow \frac{(Ze)^2 c_L^2 \beta^{2L}}{4\pi} \\
 c_L^2 &= \begin{cases} \frac{2L+1}{2} [1 + P_L(-1)] & 2\alpha\text{-cluster} \\ \frac{2L+1}{3} \left[ 1 + 2P_L(-\frac{1}{2}) \right] & 3\alpha\text{-cluster} \\ \frac{2L+1}{4} \left[ 1 + 3P_L(-\frac{1}{3}) \right] & 4\alpha\text{-cluster} \end{cases}
 \end{aligned}$$

$$\langle r^2 \rangle^{1/2} = \left[ -6 \left. \frac{d\mathcal{F}_0(q)}{dq^2} \right|_{q=0} \right]^{1/2} = \sqrt{\frac{3}{2\alpha} + \beta^2}$$

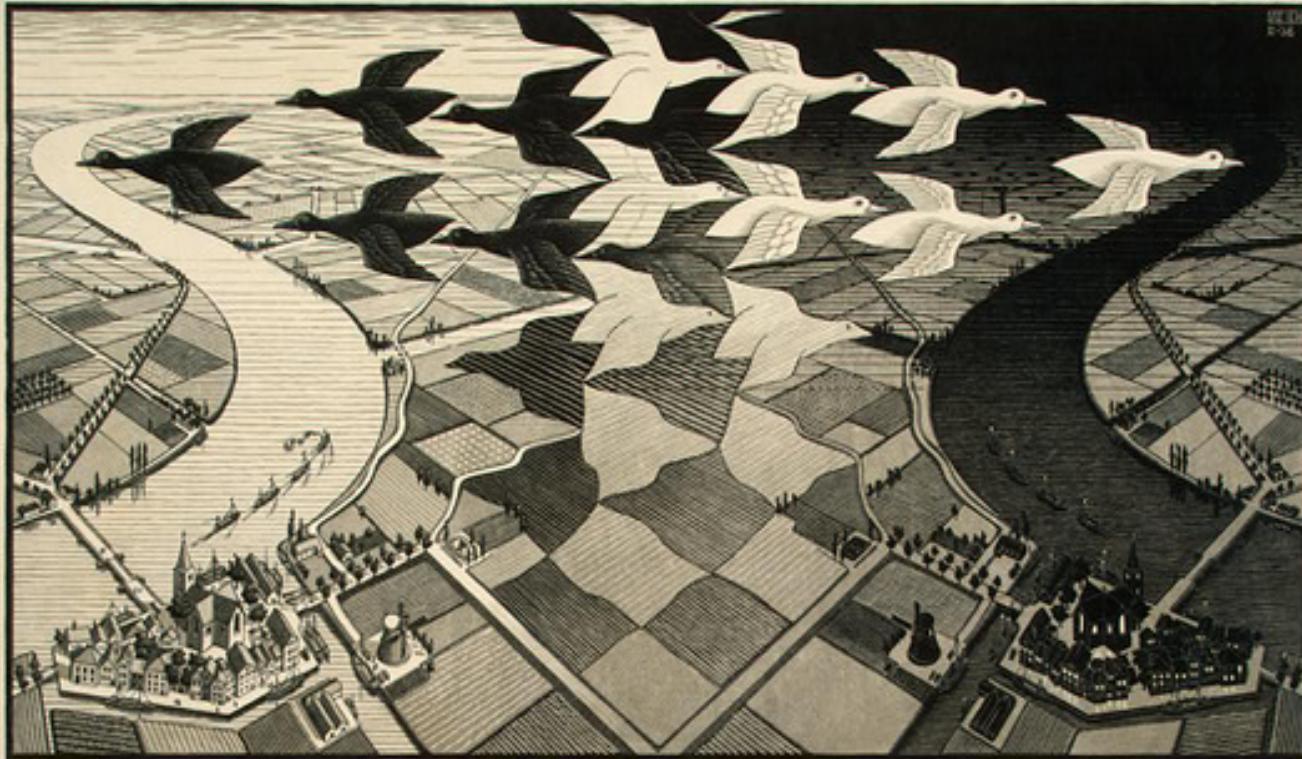
# $^{12}\text{C}$ Form Factors



# Electric Transitions

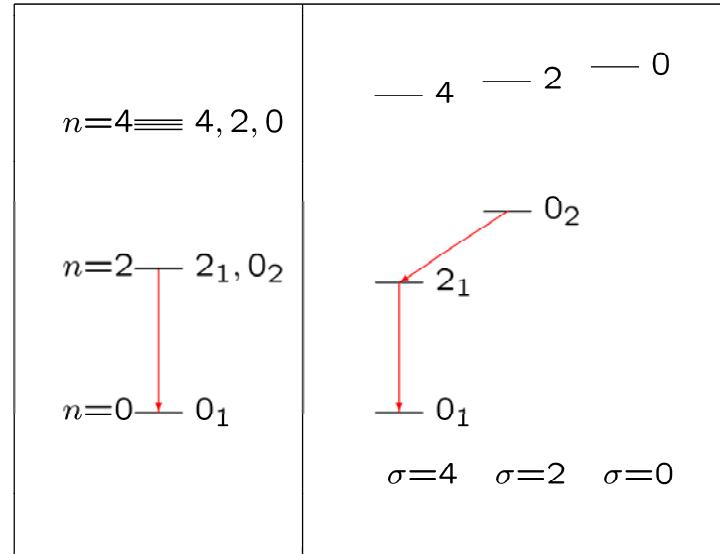
		Th.	Exp.	
73	$^{12}\text{C}$	$B(E2; 2_1^+ \rightarrow 0_1^+)$	8.4	$7.6 \pm 0.4$
		$B(E3; 3_1^- \rightarrow 0_1^+)$	44	$103 \pm 17$
44		$B(E4; 4_1^+ \rightarrow 0_1^+)$	73	$e^2 \text{fm}^8$
		$B(E2; 0_2^+ \rightarrow 2_1^+)$	1.3	$13.1 \pm 1.8$ fm <sup>2</sup>
		$\langle r^2 \rangle^{1/2}$	2.468	$2.468 \pm 0.12$ fm
16	$^{16}\text{O}$	$B(E3; 3_1^- \rightarrow 0_1^+)$	215	$205 \pm 10$
		$B(E4; 4_1^+ \rightarrow 0_1^+)$	425	$378 \pm 133$
	$^{16}\text{O}$	$B(E6; 6_1^+ \rightarrow 0_1^+)$	9626	$e^2 \text{fm}^{12}$
		$\langle r^2 \rangle^{1/2}$	2.710	$2.710 \pm 0.015$ fm

Bijker & Iachello, AP 298, 334 (2002); PRL 112, 152501 (2014)  
 NPA 957, 154 (2017)



# B(E2) values

Rotational and vibrational energies of the same order



	Harm.	Osc.	Def.	Osc.	$^{12}\text{C}$
$\frac{E_4}{E_2}$		2		$\frac{10}{3}$	3.17
$\frac{B(E2; 0_2^+ \rightarrow 2_1^+)}{B(E2; 2_1^+ \rightarrow 0_1^+)}$		$\frac{40}{3}$		0	$1.72 \pm 0.25$

# Quantum Phase Transition

$$H = (1 - \chi)H_{\text{sph}} + \chi H_{\text{obl}}$$

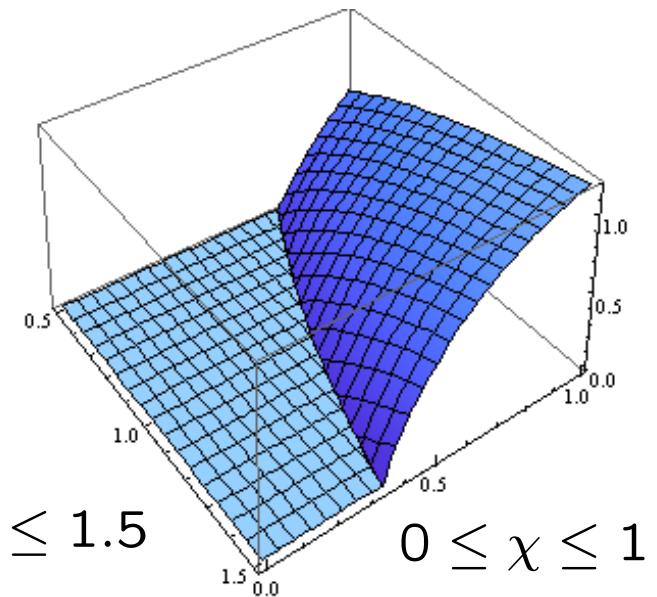
$$\begin{aligned} V_{\text{Cl}} = & \frac{1-\chi}{2}q^2 + \frac{\chi}{4}\left[R^2 - \frac{1}{2}q^2(1+R^2)\right]^2 \\ & + \frac{\xi q^4}{4}(\cos^2 2\eta + \sin^2 2\eta \cos^2 2\zeta) \end{aligned}$$

$$\begin{aligned} q_0^2 &= \begin{cases} 0 & \chi \leq \chi_c \\ \frac{2R^2}{1+R^2} + \frac{4(\chi-1)}{\chi(1+R^2)^2} & \chi \geq \chi_c \end{cases} \\ \chi_c &= \frac{1}{1+R^2(1+R^2)/2} \end{aligned}$$

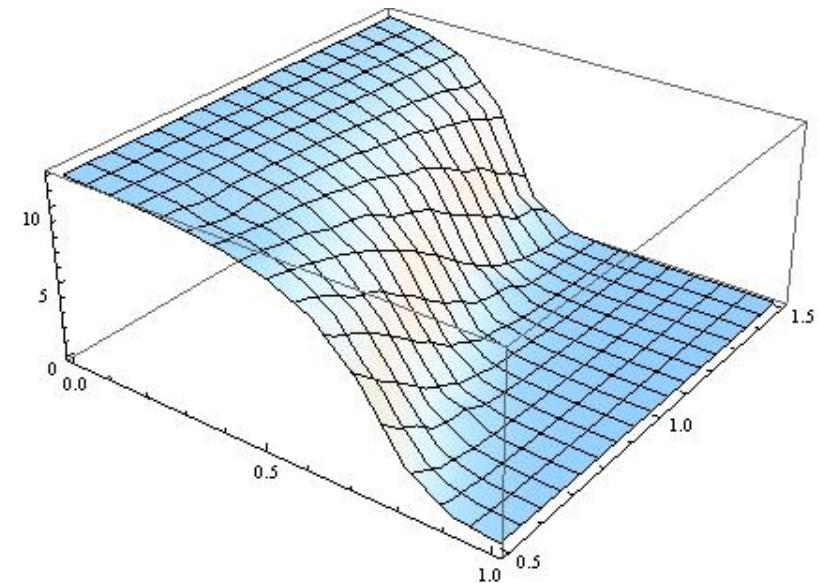
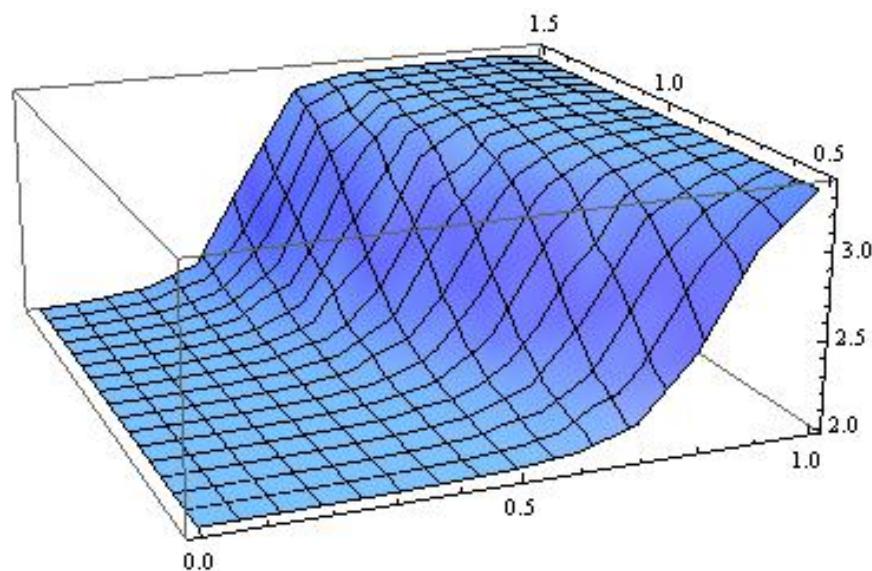
$$0.5 \leq R^2 \leq 1.5$$

2nd order phase transition

$$\begin{aligned} \rho &= q \sin \eta \\ \lambda &= q \cos \eta \\ \vec{\rho} \cdot \vec{\lambda} &= \rho \lambda \cos 2\zeta \end{aligned}$$



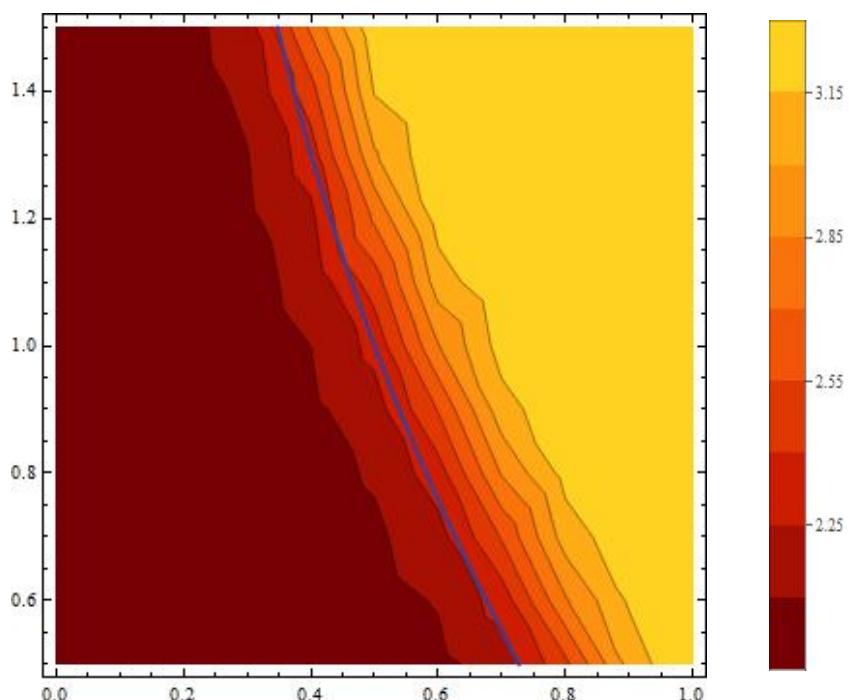
# Energies and $B(E2)$ values



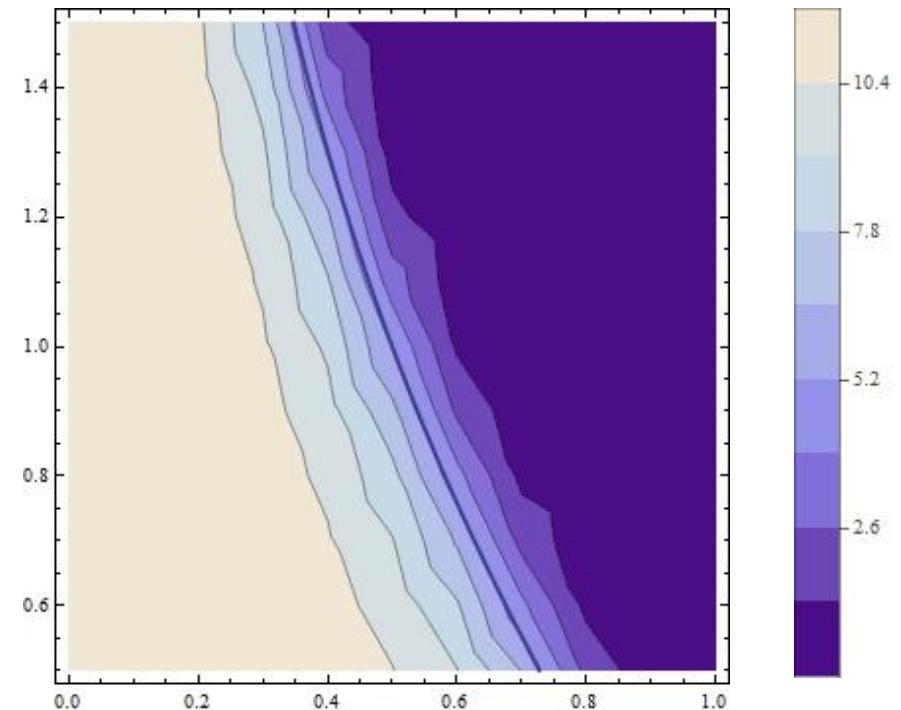
Omar Díaz, Tesis de Licenciatura, 2016

# Energies and $B(E2)$ values

Deformed



Deformed



Spherical



# Summary and Conclusions

- Algebraic Cluster Model
- $U(3k-2)$  SGA for k-body systems
- Discrete and continuous symmetries
- Special solutions: spherical and deformed oscillators, oblate top, spherical top
- Rotational bands: fingerprints of geometric configurations of alpha particles
- Applications in molecular, nuclear, hadron physics

# Alpha-Cluster Nuclei

- Oblate top with triangular symmetry for  $^{12}C$
- Ground state band: triangular
- Hoyle band: bent-arm, triangular?
- Search for negative parity states  $3^-$ ,  $4^-$   
(Darmstadt)
- Shape-phase transitions: non-rigid configuration
- ACFM: Odd-mass cluster nuclei
- Spherical top with tetrahedral symmetry for  $^{16}O$