Approximants of QCD Green's Functions
Evaluation of the HVP contribution to $g_\mu - 2$

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25th May 2016

4th Workshop on Muon $g-2$

EDM anf Flavour Violation in the LHC Era

Motivation

Study of QCD two-point functions of color singlet local operators (*with possible insertions of soft operators*).
Integrals of these Green's functions over their euclidean momenta *(with appropriate weights)* govern the hadronic contributions to many electromagnetic and weak interaction processes.

Two simple examples *(with no soft insertions)*

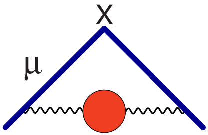
- Hadronic Vacuum Polarization two-point function (HVP)

$$\Pi_{\mu\nu}(q) = i\int d^4x\, e^{iq\cdot x}\langle 0|T\left(J_\mu(x)J_\nu(0)\right)|0\rangle = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(Q^2)\,,$$

- The Left-Right two-point function (LR) *(in the chiral limit)*

$$\Pi_{LR}^{\mu\nu}(q) = 2i\int d^4x\, e^{iq\cdot x}\langle 0|\mathrm{T}\left(L^\mu(x)R^\nu(0)^\dagger\right)|0\rangle = (q^\mu q^\nu - g^{\mu\nu}q^2)\Pi_{LR}(Q^2)\,.$$

They provide excellent theoretical laboratories to test non perturbative approaches.



Muon Anomaly from HVP (*C. Bouchiat-L. Michel '61*)

Standard Formulation in terms of the **Hadronic Spectral Function**

$$\frac{1}{2}(g_\mu - 2)_{\rm Hadrons} \equiv a_\mu^{\rm HVP} = \frac{\alpha}{\pi}\int_{4m_\pi^2}^{\infty}\frac{dt}{t}\int_0^1 dx\, \frac{x^2(1-x)}{x^2+\frac{t}{m_\mu^2}(1-x)}\,\frac{1}{\pi}{\rm Im}\Pi(t)$$

where ($m_e \to 0$)

$$\sigma(t)_{[e^+e^- \to (\gamma) \to {\rm Hadrons}]} = \frac{4\pi^2\alpha}{t}\frac{1}{\pi}{\rm Im}\Pi(t)$$

The Underlying Physics is well understood
We need, however, a very accurate evaluation.

$a_\mu^{\rm HVP}$ as an Integral over the Euclidean HVP Two-Point Function

Dispersion Relation:

$$-\Pi(Q^2)=\int_0^\infty \frac{dt}{t}\underbrace{\frac{Q^2}{t+Q^2}}\frac{1}{\pi}{\rm Im}\Pi(t)\,,\quad Q^2=-q^2\geq 0\,.$$

Euclidean Representation (*Lautrup- de Rafael '69*)

$$a_\mu^{\rm HVP}=\frac{\alpha}{\pi}\int_0^1 dx\,(1-x)\int_0^\infty \frac{dt}{t}\underbrace{\frac{\frac{x^2}{1-x}m_\mu^2}{t+\frac{x^2}{1-x}m_\mu^2}}\frac{1}{\pi}{\rm Im}\Pi(t)\,,$$

$$a_\mu^{\rm HVP}=\frac{\alpha}{\pi}\int_0^1 dx(1-x)\left[-\Pi\left(\frac{x^2}{1-x}m_\mu^2\right)\right]\,,\quad Q^2\equiv\frac{x^2}{1-x}m_\mu^2\,.$$

Euclidean Representation in terms of the Adler-like Function

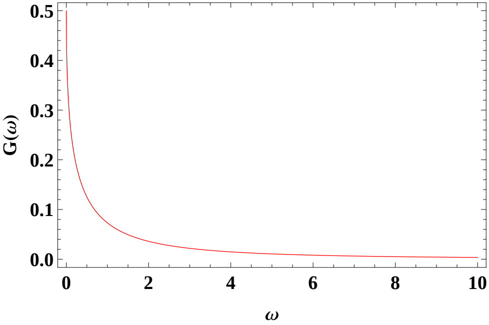
$$a_\mu^{\rm HVP}=\frac{\alpha}{\pi}\frac{1}{2}\int_0^1 dx\;x(2-x)\mathcal{A}\left(Q^2\equiv\frac{x^2}{1-x}m_\mu^2\right)\,,$$

$$\mathcal{A}(Q^2)=-m_\mu^2\frac{\partial\Pi(Q^2)}{\partial Q^2}=m_\mu^2\int_0^\infty dt\,\frac{1}{(t+Q^2)^2}\frac{1}{\pi}{\rm Im}\Pi(t)\,.$$

Comment on Lattice QCD (L-QCD) Evaluations

They use $\omega \equiv \frac{Q^2}{m_\mu^2} = \frac{x^2}{1-x}$ instead of x-Feynman (*T. Blum '03*):

$$a_\mu^{\rm HVP} = \frac{\alpha}{\pi}\int_0^\infty \frac{d\omega}{\omega}\underbrace{\frac{1}{4}\left[(2+\omega)\left(2+\omega-\sqrt{\omega}\sqrt{4+\omega}\right)-2\right]}_{G(\omega)}\underbrace{\left(-\omega\frac{d}{d\omega}\Pi\left(\omega m_\mu^2\right)\right)}_{\text{Adler Function}}$$



L-QCD evaluations -*at a few* ω *points*- need extrapolations.

Models and/or Padé Approximants

-Dispersion Relations have Factorizable Mellin Transforms-

$$\mathcal{A}(Q^2) = -m_\mu^2 \frac{\partial \Pi(Q^2)}{\partial Q^2} = m_\mu^2 \int_0^\infty \frac{dt}{t} \frac{t}{(t+Q^2)^2} \frac{1}{\pi} \mathrm{Im}\Pi(t)\,.$$

- Mellin-Barnes representation of the Base Function $\frac{1}{\left(1+\frac{Q^2}{t}\right)^2}$

$$\frac{m_\mu^2}{t} \frac{1}{\left(1+\frac{Q^2}{t}\right)^2} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \left(\frac{Q^2}{m_\mu^2}\right)^{-s} \left(\frac{m_\mu^2}{t}\right)^{1-s} \Gamma(s)\Gamma(2-s)\,.$$

- Mellin Transform of the Spectral Function

$$\mathcal{M}(s) = \int_0^\infty \frac{dt}{t} \left(\frac{m_\mu^2}{t}\right)^{1-s} \frac{1}{\pi} \mathrm{Im}\Pi(t)\,.$$

Inserting this in the Euclidean integral a_μ^{HVP} with $Q^2 = \frac{x^2}{1-x} m_\mu^2$

$$a_\mu^{\mathrm{HVP}} = \frac{\alpha}{\pi} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \int_0^1 dx\, \frac{x}{2}(2-x) \left(\frac{x^2}{1-x}\right)^{-s} \mathcal{M}(s)\, \Gamma(s)\Gamma(2-s)\,.$$

Mellin-Barnes Representation of the Muon Anomaly (EdeR, P.L.'14)

Integrating over x, i.e. $Q^2 = \frac{x^2}{1-x} m_\mu^2$ results in:

Integral Representation of $a_\mu^{\rm HVP}$ (Model Independent)

$$a_\mu^{\rm HVP} = \left(\frac{\alpha}{\pi}\right) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds\, \mathcal{F}(s)\, \underbrace{\mathcal{M}(s)}, \qquad \mathrm{Re}\, c \in]0,+1[$$

$$\mathcal{F}(s) = -\Gamma(3-2s)\Gamma(-3+s)\Gamma(1+s)$$

$$\mathcal{M}(s) = \underbrace{\int_{4m_\pi^2}^{\infty} \frac{dt}{t} \left(\frac{m_\mu^2}{t}\right)^{1-s} \frac{1}{\pi} \mathrm{Im}\Pi(t)}_{Mellin\ Transform\ of\ the\ Spectral\ Function}$$

$\mathcal{M}(s)$ is finite for $s < 1$ and singular at $s = 1$:

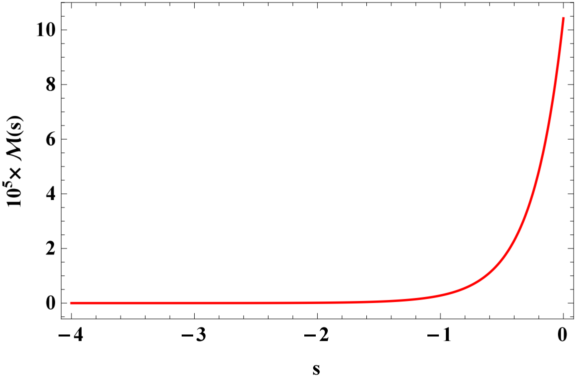
$$\mathcal{M}_{\rm pQCD}(s) \underset{s \to 1}{\sim} \left(\frac{\alpha}{\pi}\right) \left(\frac{2}{3}\right) N_c \frac{1}{3} \frac{1}{1-s}.$$

Very Useful Representation to extract Asymptotic Expansions.

Mellin Transform of Phenomenological Toy Model Spectral Function

$$\mathcal{M}(s) = \underbrace{\int_{4m_\pi^2}^{\infty} \frac{dt}{t} \left(\frac{m_\mu^2}{t}\right)^{1-s} \frac{1}{\pi}\mathrm{Im}\Pi(t)}_{\text{Mellin Transform of the Spectral Function}}$$

Shape of $\mathcal{M}(s)$ in the phenomenological toy model of *D. Bernecker et al, '11; L. Lelllouch, '14*



Integral Representation of $a_\mu^{\rm HVP}$ (Model Independent)

$$a_\mu^{\rm HVP} = \left(\frac{\alpha}{\pi}\right) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds\, \mathcal{F}(s) \underbrace{\mathcal{M}(s)}, \qquad {\rm Re}\, c \in]0, +1[$$

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Systematic Expansion in Moments Approximants.

Moment Analysis

Two types of Moments

Normal Power Moments:

$$\mathcal{M}(-n) = \int_{4m_\pi^2}^{\infty} \frac{dt}{t} \left(\frac{m_\mu^2}{t}\right)^{1+n} \frac{1}{\pi}\mathrm{Im}\Pi(t)\,, \quad n = 0, 1, 2, \ldots$$

Log Weighted Power Moments (first derivative of the Mellin transform at integer $n > 0$ values):

$$\tilde{\mathcal{M}}(-n) = \int_{4m_\pi^2}^{\infty} \frac{dt}{t} \left(\frac{m_\mu^2}{t}\right)^{1+n} \log \frac{m_\mu^2}{t} \frac{1}{\pi}\mathrm{Im}\Pi(t)\,, \quad n = 1, 2, 3, \cdots$$

Expansion in Moment Approximants has Fast Convergence

$$\begin{aligned} a_\mu^{\mathrm{HVP}} &= \left(\frac{\alpha}{\pi}\right) \left\{ \frac{1}{3}\mathcal{M}(0) + \frac{25}{12}\mathcal{M}(-1) + \tilde{\mathcal{M}}(-1) \right. \\ &+ \frac{97}{10}\mathcal{M}(-2) + 6\tilde{\mathcal{M}}(-2) \\ &+ \left. \frac{208}{5}\mathcal{M}(-3) + 28\tilde{\mathcal{M}}(-3) + \mathcal{O}\left[\tilde{\mathcal{M}}(-4)\right] \right\} \end{aligned}$$

These moments are known phenomenologically from e^+e^- data

The Moment Approximants in a Phenomenological Toy Model

$$a_\mu^{\rm HVP}(e^+e^-) = (6.923 \pm 0.042) \times 10^{-8} \quad (0.6\%)$$

M. Davier et al' 10

$$a_\mu^{\rm HVP}(\text{toy model}) = 6.936 \times 10^{-8}$$

D. Bernecker and H.B. Meyer, '11; L. Lelllouch, '14

Numerical Values of the Moment Approximants (Toy Model)

$$\left(\frac{\alpha}{\pi}\right) \frac{1}{3}\mathcal{M}(0) = 8.071 \times 10^{-8} \quad (16\%)$$

$$\left(\frac{\alpha}{\pi}\right) \left[\frac{1}{3}\mathcal{M}(0) + \frac{25}{12}\mathcal{M}(-1) + \tilde{\mathcal{M}}(-1)\right] = 7.240 \times 10^{-8} \quad (4\%)$$

$$\left(\frac{\alpha}{\pi}\right) \left[\frac{1}{3}\mathcal{M}(0) + \frac{25}{12}\mathcal{M}(-1) + \tilde{\mathcal{M}}(-1) + \frac{97}{10}\mathcal{M}(-2) + 6\tilde{\mathcal{M}}(-2)\right] = 7.022 \times 10^{-8} \quad (1\%)$$

Fourth Approximation is already within 0.4% of the toy model result

The Moment Approximants in L-QCD

The Leading Moment provides a rigorous upper bound to $a_\mu^{\rm HVP}$

J.S. Bell-de Rafael '69: the operator $\partial^\lambda F^{\mu\nu}\partial_\lambda F_{\mu\nu}$ governs low-energy hadronic QED observables

$$a_\mu^{\rm HVP} < \left(\frac{\alpha}{\pi}\right)\frac{1}{3}\underbrace{\int_{4m_\pi^2}^\infty \frac{dt}{t}\,\frac{m_\mu^2}{t}\,\frac{1}{\pi}{\rm Im}\Pi(t)}_{\mathcal{M}(0)} = \left(\frac{\alpha}{\pi}\right)\frac{1}{3}\underbrace{\left(-m_\mu^2\frac{d}{dQ^2}\Pi(Q^2)\right)_{Q^2=0}}_{\rm L-QCD}$$

- The bound overestimates $a_\mu^{\rm HVP}$ by less than 18% (*not bad for a rigorous bound*)
- The slope of $\Pi\left(Q^2\right)$ at the origin (r.h.s.) should be evaluated in lattice QCD
- *It is difficult to imagine that, unless lattice QCD does better than phenomenology in this simple case, it will ever reach a competitive accuracy of the full determination of $a_\mu^{\rm HVP}$.*

$\mathcal{M}(-n)$ Moments correspond to successive derivatives of $\Pi(Q^2)$ at $Q^2=0$

$$\underbrace{\mathcal{M}(-n)}_{n=0,1,2\dots} = \int_{4m_\pi^2}^\infty \frac{dt}{t}\left(\frac{m_\mu^2}{t}\right)^{1+n}\frac{1}{\pi}{\rm Im}\Pi(t) = \frac{(-1)^{n+1}}{(n+1)!}(m_\mu^2)^{n+1}\left(\frac{\partial^{n+1}}{(\partial Q^2)^{n+1}}\Pi(Q^2)\right)_{Q^2=0}$$

These derivatives can (*perhaps?*) be determined in Lattice QCD

The Log Weighted Moments in L-QCD

$$\tilde{\mathcal{M}}(-n) = \int_{4m_\pi^2}^{\infty} \frac{dt}{t} \left(\frac{m_\mu^2}{t}\right)^n \log \frac{m_\mu^2}{t} \frac{1}{\pi} \mathrm{Im}\Pi(t)$$

They require the evaluation of integrals of the type

Integrals in the Euclidean to be evaluated in L-QCD

$$\Sigma(-n) \equiv \int_{4m_\pi^2}^{\infty} dQ^2 \left(\frac{m_\mu^2}{Q^2}\right)^{n+1} \left(-\frac{\Pi(Q^2)}{Q^2}\right) \quad n = 1, 2, 3 \ldots$$

Example:

$$\tilde{\mathcal{M}}(-1) = -\log \frac{4m_\pi^2}{m_\mu^2} \underbrace{\mathcal{M}(-1)}_{\mathrm{L-QCD}} + \underbrace{\Sigma(-1)}_{\mathrm{L-QCD}} - \frac{m_\mu^2}{4m_\pi^2} \underbrace{\mathcal{M}(0)}_{\mathrm{L-QCD}} + \mathcal{O}\left[\mathcal{M}(-2)\right]$$

- Contrary to the evaluation of a_μ^{HVP}, the Euclidean moments $\Sigma(-1)$, $\Sigma(-2)$, ... are not weighted by a heavily peaked kernel at small Q^2.
- The threshold of integration is at a rather large value $Q^2 = 4m_\pi^2$ instead of zero.
- The determination of these Euclidean moments in L-QCD and their comparison with the corresponding phenomenological expressions in terms of the hadronic spectral function, provide *valuable further tests.*

Conclusions about the HVP Moments Contribution

- Present L-QCD determinations of HVP in the Euclidean *need to be complemented* by approximation methods in order to get $a_\mu^{\rm HVP}$.

- The *moment analysis* approach may gradually lead to an accurate determination of $a_\mu^{\rm HVP}$, providing at the same time many tests of *L-QCD evaluations* to be confronted with phenomenological determinations using experimental data.

They approximate $\Pi(Q^2)$ with rational functions of Q^2 (*S. Peris et al' 12-16*)

$$\Pi(Q^2)_{[N,D]} = Q^2 \left(a_0 + \sum_{r=1}^{N} \frac{a_r}{Q^2 + b_r} \right) .$$

Padé's imply *Spectral Functions with N-poles*, e.g. for $a_0 = 0$:

$$\frac{1}{\pi} \mathrm{Im}\Pi(t) = \sum_{r=1}^{N} a_r \delta(t - b_r) .$$

They are not, however, Large-N_c approximations
BUT a finite *N* cannot reproduce the pQCD behaviour !

* * *

We need Approximants
with an *Infinite Number of Delta-Functions*
to reproduce the *pQCD behaviour*

Hurwitz-like Approximants (*D. Greynat-E . de Rafael, in progress*)

Inspired by previous work on Large-$\mathrm{N_c}$ QCD, BUT does not require Large-$\mathrm{N_c}$

We want functions which may approximate the QCD $\mathcal{M}(s)$ behaviour and propose

The *Hurwitz function* defined by the *Dirichlet Series*

$$\zeta(s,v)=\sum_{n=0}^{\infty}\frac{1}{(n+v)^s}\,,\quad \mathrm{Re}s>1\quad\mathrm{and}\quad \mathrm{Re(v)}\neq-\mathrm{n}\,,$$

which can be analytically continued to a meromorphic function in the entire complex s-plane.

It has only a single pole at $s=1$ with residue 1.

This fixes the pQCD requirement

Hurwitz Approximants to the Mellin Transform of the Spectral Function

- First Approximant:

$$\frac{1}{\pi}\mathrm{Im}\Pi(t)=A\sigma^2\sum_{n=0}^{\infty}\delta(t-M^2-n\sigma^2)\,,\qquad A=\left(\frac{\alpha}{\pi}\right)\left(\frac{2}{3}\right)\frac{N_c}{3}$$

$$\mathcal{M}(s)=\int_{M^2}^{\infty}\frac{dt}{t}\left(\frac{m_\mu^2}{t}\right)^{1-s}\frac{1}{\pi}\mathrm{Im}\Pi(t)\Rightarrow A\left(\frac{m_\mu^2}{\sigma^2}\right)^{1-s}\zeta\left(2-s,\frac{M^2}{\sigma^2}\right)$$

First Hurwitz Approximant to $\mathcal{M}(s)$ and $a_\mu^{\rm HVP}$

$$\mathcal{M}^{\rm first}(s)=\left(\frac{\alpha}{\pi}\right)\left(\frac{2}{3}\right)\frac{N_c}{3}\left(\frac{m_\mu^2}{\sigma^2}\right)^{1-s}\zeta\left(2-s,\frac{M^2}{\sigma^2}\right)$$

The ratio $v\equiv\frac{M^2}{\sigma^2}$ can be fixed demanding no $\frac{1}{Q^2}$ term in the OPE of $\Pi(Q^2)$

$$v\equiv\frac{M^2}{\sigma^2}=\frac{1}{2}$$ Only One Parameter Left !

MATCHING the Moments $\mathcal{M}^{\rm first}(0)$ and $\mathcal{M}^{\rm exp}(0)$ fixes

$$M=687\ \mathrm{MeV}$$

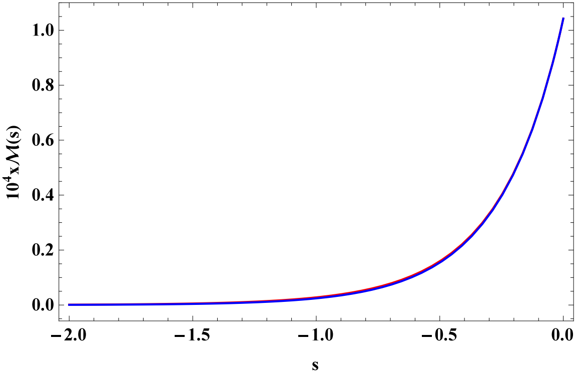
This results in the prediction

$$\begin{aligned} a_\mu^{\rm HVP}(\mathrm{first}) &= \left(\frac{\alpha}{\pi}\right)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\ \mathcal{F}(s)\times\mathcal{M}^{\rm first}(s)\\ &= \left(\frac{\alpha}{\pi}\right)^2\frac{v}{3}\frac{m_\mu^2}{M^2}\int_0^1 dx x(2-x)\zeta\left(2,\left[1+\frac{x^2}{1+x}\frac{m_\mu^2}{M^2}\right]v\right)=6.97\times10^{-8}, \end{aligned}$$

which represents an accuracy of 0.6% !!!

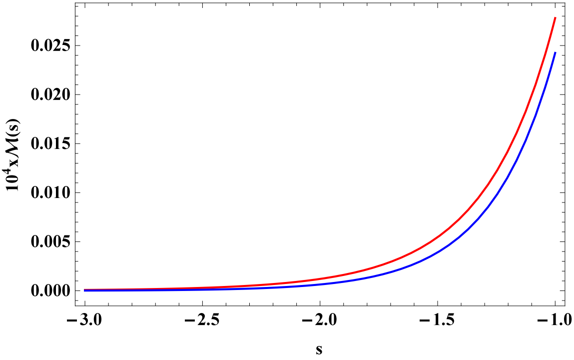
Comparison of Mellin Transforms

The Mellin Transform of the Toy Model (Red curve)
and of the First Hurwitz Approximant (Blue curve)



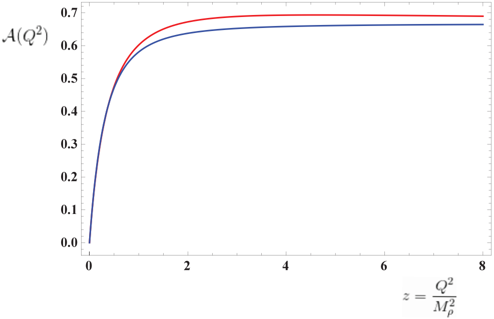
Detailed Comparison of Mellin Transforms

The Mellin Transform of the Toy Model (Red curve)
and of the First Hurwitz Approximant (Blue curve)



$$\mathcal{A}(Q^2) = -Q^2 \frac{\partial \Pi(Q^2)}{\partial Q^2} = \underbrace{A \frac{Q^2}{\sigma^2} \zeta\left(2, \frac{Q^2 + M^2}{\sigma^2}\right)}_{\text{First Hurwitz Approximant}}$$

$$A = \left(\frac{\alpha}{\pi}\right) \frac{2}{3} \frac{N_c}{3}, \quad \frac{M^2}{\sigma^2} = 1/2, \quad M = 687 \text{ MeV}$$



The Red Curve is the Phenomenological Fit. The Blue Curve the First Hurwitz Approximant

Comments and Conclusions

- **The First Hurwitz Approximant does already an excellent job! The only required input was $\mathcal{M}(0)$ (which L-QCD could get...).**
- **This Approximant can be improved, if necessary, by taking superpositions of Hurwitz functions, with input from L-QCD.**
- **Fixing the Hurwitz parameters with $\mathcal{M}(0)$ is not necessary. Input from L-QCD determination of $\Pi(Q^2)$ at values of Q^2 conveniently chosen could instead be used.**
- **No need of Padé's. No need of cutting integrals in short and long distance components.**
- **Applications to other Green's functions are underway.**