RG Optimized Perturbation: some QCD results, and prospects for HVP contributions to $g_\mu - 2$

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1. Introduction/Motivations

2. (Variationally) Optimized Perturbation (OPT)
and Renormalization Group improvement of OPT (RGOPT)

3. Application: determination of $F_\pi/\Lambda_{\overline{\rm ms}}^{QCD}$ and α_S

4. Application: determination of the condensate $\langle \bar{q} q \rangle$

5. Application to $g_\mu - 2$ Hadronic vacuum polarisation contributions (prospects, preliminary)

1. Introduction/Motivations

Goal: Peculiar resummations of perturbative expansions can give approximations to some nonperturbative parameters

In a nutshell: estimate this way e.g. $F_\pi(m_q = 0)/\Lambda^{\rm QCD}_{\overline{\rm MS}}$ 'nonperturbatively',

$F_\pi \simeq 92.2\text{MeV} \to F_\pi(m_q = 0) \to \Lambda^{n_f=3}_{\overline{\rm MS}} \to \alpha_S^{\overline{\rm MS}}(\mu = m_Z)$.

How?: start from *perturbative* $F_\pi^2 \simeq m_q^2 \sum_{n,p} (\alpha_S)^n f_{np} \ln^p \frac{m_q}{\mu}$ (known at present to 4-loop order for any n_f)

Now $m_{quark} \to m$ *variational* mass (in a well-defined way), optimized consistently with RG properties$\equiv \rm RG(OPT)$.

$\Rightarrow m = \mathcal{O}(\Lambda^{QCD}) \Rightarrow F_\pi^{m_q=0}/\Lambda^{n_f=3}_{\overline{\rm MS}} \simeq 0.25 \pm .01 \to \alpha_S(m_Z) \simeq 0.1174 \pm .001 \pm .001$

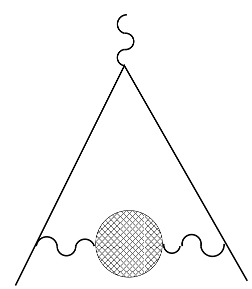
(JLK, A.Neveu, PRD88 (2013))

- applied to $\langle \bar{q}q \rangle$ at 3,4 -loops (using spectral density of Dirac operator) gives

$\langle \bar{q}q \rangle^{1/3}_{m_q=0}(2\,\rm GeV) \simeq -(0.84 \pm 0.01)\Lambda_{\overline{\rm MS}}$ (JLK, A.Neveu, PRD 92 (2015))

$g_\mu - 2$ (HVP)

Concerning $g_\mu - 2$ (Hadronic vacuum polarization contribution only):
Motivations similar to lattice: “first principle” attempt to calculate the Hadronic Vacuum Polarisation, *independently* from dispersion relations from e^+e^-, τ decay data:



• worth to test our procedure on HVP, before possibly trying on Hadronic light by light contribution: from previous cases, hope RGOPT HVP to reach $\sim 2\%$ accuracy,

light by light contribution needs less accuracy.

Chiral Symmetry Breaking (χSB) Order parameters

Conventional wisdom: hopeless from standard perturbation:

1. $\langle \bar{q}q \rangle^{1/3}$, F_π,.. $\sim \mathcal{O}(\Lambda_{QCD}) \simeq$ 300 MeV
$\to \alpha_S$ (a priori) large $\to$ invalidates pert. expansion

2. $\langle \bar{q}q \rangle$, F_π,.. perturbative series $\sim (m_q)^d \sum_{n,p} \alpha_s^n \ln^p(m_q)$
vanish for $m_q \to 0$ at any pert. order (trivial chiral limit)

seems to tell that χSB parameters are intrinsically NP

•Optimized pert. (OPT): circumvents at least 1., 2.,
and may give more clues to pert./NP bridge

2. (Variationally) Optimized Perturbation (OPT)

Trick: add and subtract a mass, consider $m\,\delta$ as interaction:

$$\mathcal{L}_{QCD}(g, m_q) \to \mathcal{L}_{QCD}(\delta\, g, m(1-\delta)) \quad (\alpha_S \equiv g/(4\pi))$$

$0 < \delta < 1$ interpolates between $\mathcal{L}_{free}$ and *massless* $\mathcal{L}_{int}$;
(quark) mass $m_q \to m$: arbitrary trial parameter

- Take any standard (renormalized) QCD pert. series, expand in δ *after:*

$$m_q \to m\,(1-\delta); \quad g \to \delta\, g$$

then take $\delta \to 1$ (to recover original massless theory):

BUT a m-dependence remains at any finite δ^k-order:
fixed typically by optimization (OPT):

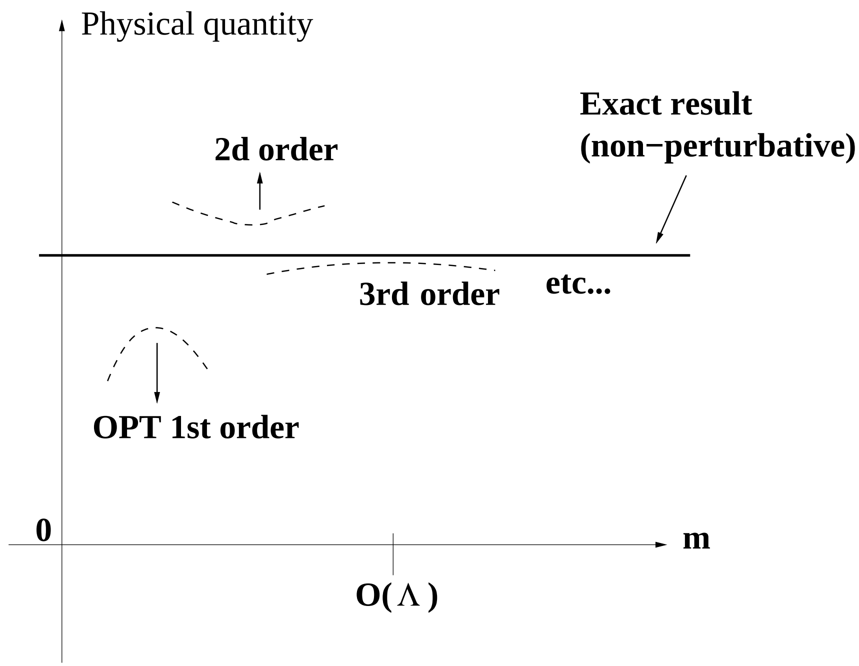
$$\frac{\partial}{\partial m}(\text{physical quantity}) = 0 \text{ for } m = \tilde{m}_{opt}(\alpha_S) \neq 0$$

Exhibit *dimensional transmutation*: $\tilde{m}_{opt} \sim \mu e^{-1/(\beta_0 g)}$

But does this 'cheap trick' always work? and why?

Expected behaviour (Ideally...)

Expect *flatter* m-dependence at increasing δ orders...



But not quite what happens.. except for $\phi^4(D=1)$ (oscillator)

Higher orders: $\rightarrow$ what about convergence?

Main pb at higher order: OPT: $\partial_m(...) = 0$ has multi-solutions (some complex!), how to choose right one??

Simpler model's support + properties

•Convergence proof of this procedure for $D = 1\ \lambda\phi^4$ oscillator (cancels large pert. order factorial divergences!) Guida et al '95

particular case of 'order-dependent mapping' Seznec+Zinn-Justin '79 (exponentially fast convergence for ground state energy $E_0 = const.\lambda^{1/3}$; good to % level at second δ-order)

•Flexible, Renormalization-compatible, gauge-invariant: applications also at finite temperature (many variants: 'screened pert.', 'hard thermal loop resummation', ...)

(NB our recent RG(OPT) version drastically improves well-known problems of unstable +badly scale-dependent thermal perturbation (JLK + M.Pinto PRL 116 (2016))

RG improved (compatible) OPT (RGOPT)

Our main additional ingredient to OPT (JLK, A. Neveu 2010):

Consider a physical quantity (i.e. perturbatively RG invariant), e.g. pole mass M (or latter will be F_π):
in addition to OPT Eq: $\frac{\partial}{\partial m} M^{(k)}(m, g, \delta = 1)|_{m \equiv \tilde{m}} \equiv 0$

Require (δ-modified!) series at order δ^k to satisfy a standard (perturbative) Renormalization Group (RG) equation:

$$\mathrm{RG}\left(M^{(k)}(m, g, \delta = 1)\right) = 0$$

with standard RG operator: ($g = 4\pi\alpha_S$)

$$\text{RG} \equiv \mu \frac{d}{d\,\mu} = \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - \gamma_m(g)\, m \frac{\partial}{\partial m}$$

$$\beta(g) \equiv -2b_0 g^2 - 2b_1 g^3 + \cdots, \quad \gamma_m(g) \equiv \gamma_0 g + \gamma_1 g^2 + \cdots$$

$\to$ Combined with OPT, RG Eq. reduces to massless form:

$$[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}]M^{(k)}(m,g,\delta=1)=0$$

Note: OPT+RG completely fix $m\equiv\tilde{m}$ and $g\equiv\tilde{g}$

• But $\Lambda_{\overline{\text{ms}}}(g)$ satisfies by def. $[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}]\,\Lambda_{\overline{\text{ms}}}\equiv 0$ consistently at a given pert. order for $\beta(g)$.

Thus equivalent to:

$$\frac{\partial}{\partial\, m}\left(\frac{M^k(m,g,\delta=1)}{\Lambda_{\overline{\text{ms}}}(g)}\right)=0\ ;\quad \frac{\partial}{\partial\, g}\left(\frac{M^k(m,g,\delta=1)}{\Lambda_{\overline{\text{ms}}}(g)}\right)=0 \text{ for } \tilde{m},\tilde{g}$$

•Sort of "virtual" (variational) fixed point (but $\beta(g)\neq 0$!)

•Optimal $\tilde{m},\tilde{g}=4\pi\tilde{\alpha}_S$ unphysical: true α_S from $\frac{F_\pi}{\Lambda_{\overline{\text{ms}}}}(\tilde{m},\tilde{g})$

•Reproduces at first order exact nonpert results in simpler (e.g. Gross-Neveu) models

OPT + RG = RGOPT main new features

•Embarrassing freedom in interpolating Lagrangian, e.g.:

$$m \to m\,(1-\delta)^{a}$$

In most previous works: linear case $a=1$ for 'simplicity'... but generally (we showed) it spoils RG invariance...

[exceptions: Bose-Einstein Condensate T_c shift, calculated from $O(2)\lambda\phi^4$, *requires* $a \neq 1$: gives real solutions +related to critical exponents (Kleinert,Kastening; JLK,Neveu,Pinto '04)

•OPT,RG Eqs: many solutions at increasing δ^k-orders

$\to$ Our approach restores RG +requires OPT, RG sol. to match standard perturbation (i.e. Asymptotic Freedom in QCD): $\alpha_S \to 0, \mu \to \infty$: $\tilde{g} = 4\pi\tilde{\alpha}_S \sim \frac{1}{2b_0 \ln \frac{\mu}{\tilde{m}}} + \cdots$

$\to$ At arbitrary order, AF-compatible RG + OPT branch, often unique, *only* appear for a critical universal a:

$$m \to m\,(1-\delta)^{\frac{\gamma_0}{2b_0}}; \qquad \text{(e.g. } \frac{\gamma_0}{2b_0}(\mathrm{QCD}, n_f=3) = \frac{4}{9}\text{)}$$

$\to$ It removes spurious solutions incompatible with AF

3. Application: Pion decay constant F_π/Λ

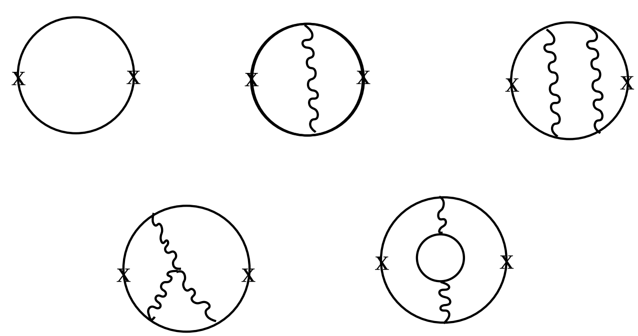
Chiral Symmetry Breaking (CSB) $SU(n_f)_L \times SU(n_f)_R \to SU(n_f)_{L+R}$
for n_f massless quarks. ($n_f = 2$, $n_f = 3$)
F_π given from (nonperturbative) definition at $p^2 \to 0$:

$$i\langle 0|T A^i_\mu(p) A^j_\nu(0)|0\rangle \equiv \delta^{ij} g_{\mu\nu} F^2_\pi + \mathcal{O}(p_\mu p_\nu)$$

where quark axial current: $A^i_\mu \equiv \bar{q}\gamma_\mu\gamma_5 \frac{\tau_i}{2}\, q$
$F_\pi \neq 0$: main (lowest order) CSB order parameter

$m_q \neq 0$: perturbative expansion known to 3,4 loops

(3-loop Chetyrkin et al ’95; 4-loop Maier et al ’08 ’09, +Maier, Marquard private comm.)



(Standard) perturbative available information

$$F_\pi^2(pert)_{\overline{\text{ms}}} = N_c \frac{m^2}{2\pi^2}\left[-L + \frac{\alpha_S}{4\pi}(8L^2 + \frac{4}{3}L + \frac{1}{6})\right.$$
$$\left.+(\frac{\alpha_S}{4\pi})^2[f_{30}(n_f)L^3 + f_{31}(n_f)L + f_{32}(n_f)L + f_{33}(n_f)] + \mathcal{O}(\alpha_S^3)\right]$$

$L \equiv \ln\frac{m}{\mu}$, $n_f = 2(3)$

Note: finite part (after mass + coupling renormalization) not separately RG-inv: (i.e. $F_\pi^2 \sim \langle 0|T A^\mu A^\nu|0\rangle$ mixes with $m^2\,1$ operator)

$\to$ (extra) renormalization by subtraction of the form:
$S(m,\alpha_S) = m^2(s_0/\alpha_S + s_1 + s_2\alpha_S + \ldots)$ where s_i fixed requiring RG-inv order by order: $s_0 = \frac{3}{16\pi^3(b_0-\gamma_0)}$, $s_1 = \ldots$

Same well-known feature for $m\langle\bar{q}q\rangle$, related to vacuum energy, needs an extra (additive) renormalization in $\overline{\text{ms}}$-scheme to be RG invariant.

Warm-up calculation: pure RG approximation

2-loop + neglecting non-RG (non-logarithmic) terms:

$$F_\pi^2(\text{RG-1}, \mathcal{O}(g)) = 3\frac{m^2}{2\pi^2}\left[-L + \frac{\alpha_S}{4\pi}(8L^2 + \frac{4}{3}L) - (\frac{1}{8\pi(b_0-\gamma_0)\,\alpha_S} - \frac{5}{12})\right]$$

$$\to F_\pi^2(m \to m(1-\delta)^{\gamma_0/(2b_0)}, \alpha_S \to \delta\alpha_S, \mathcal{O}(\delta))|_{\delta\to 1} =$$
$$3\frac{m^2}{2\pi^2}\left[-\frac{102\pi}{841\,\alpha_S} + \frac{169}{348} - \frac{5}{29}L + \frac{\alpha_S}{4\pi}(8L^2 + \frac{4}{3}L)\right]$$

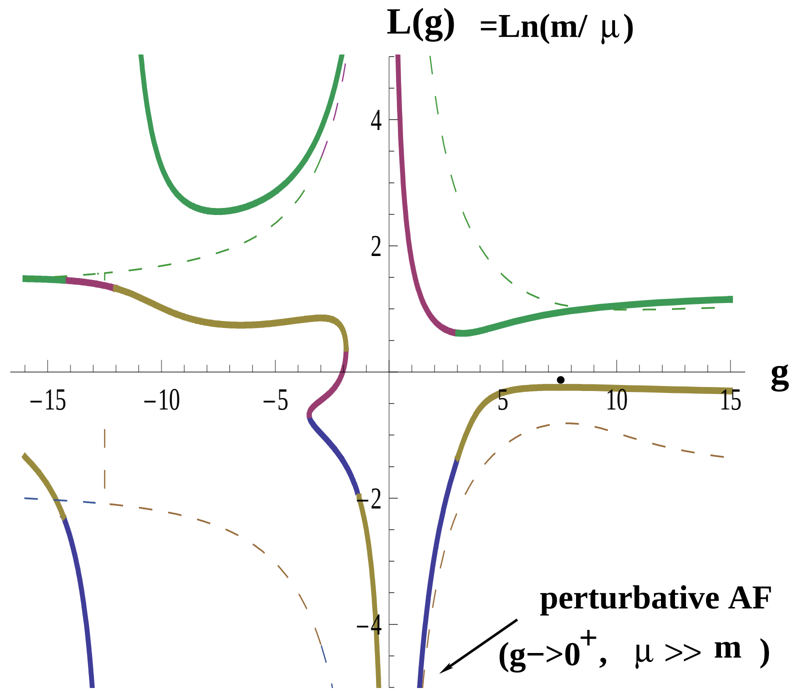
OPT+RG: $\partial_m(F_\pi^2/\Lambda^2_{\overline{\text{ms}}}), \partial_{\alpha_S}(F_\pi^2/\Lambda^2_{\overline{\text{ms}}}) \equiv 0$: have a unique AF-compatible real solution: $\tilde{L} \equiv \ln\frac{\tilde{m}}{\mu} = -\frac{\gamma_0}{2b_0}\;;\;\; \tilde{\alpha}_S = \frac{\pi}{2}$

$\to F_\pi(\tilde{m}, \tilde{\alpha}_S) = (\frac{5}{8\pi^2})^{1/2}\tilde{m} \simeq 0.25\Lambda_{\overline{\text{ms}}}$ (for $\Lambda^{1-loop}_{\overline{\text{ms}}} = \mu\, e^{-1/(\beta_0\alpha_S)}$)

•Includes higher orders +non-RG terms: $\tilde{m}_{opt}$ remains $\mathcal{O}(\Lambda_{\overline{\text{ms}}})$ (rather than $m \sim 0$): RG-consistent 'mass gap',

And $\tilde{\alpha}_S \simeq .5$ *stabilizes* to more perturbative values
NB $\tilde{m}, \tilde{\alpha}_S$ variational parameters (not directly physical)

Exact F_π RG+OPT solutions at 4-loops ($\overline{\text{ms}}$)



All branches of RG (thick) and OPT(dashed) solutions $Re[L \equiv \ln \frac{m}{\mu}(g)]$ to the δ-modified 3rd order (4-loop) perturbation ($g = 4\pi\alpha_S$). Unique AF compatible sol.: black dot

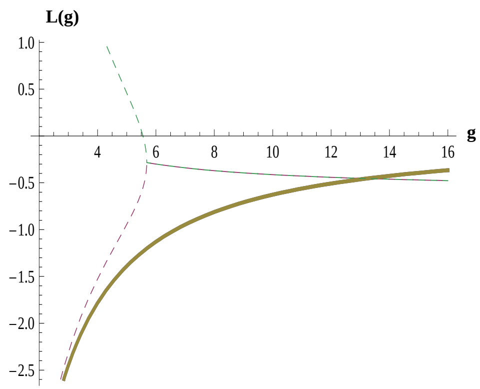
•However beyond lowest order, AF-compatibility and reality of solutions often incompatible...
But, complex solutions are artefacts of solving *exactly* the RG and OPT (polynomial in L) Eqs, in $\overline{\text{ms}}$-scheme...

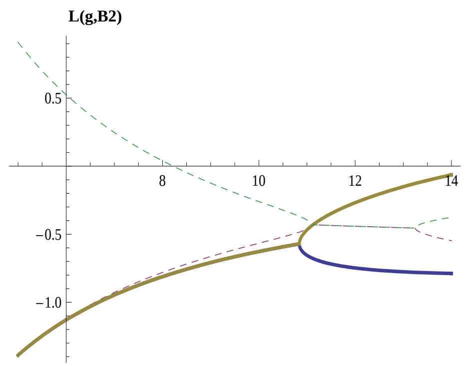
Recovering real AF-compatible solutions

Perturbative 'deformations' consistent with RG?:

Evidently: Renormalization scheme changes (RSC)

$$m \to m'(1 + B_1 g' + B_2 g'^2 + \cdots),\ g \to g'(1 + A_1 g' + A_2 g'^2 + \cdots)$$

$\mathcal{O}(\delta)$, $\overline{\text{ms}}$: 

$\rightarrow$ 

$\rightarrow$ We require *contact* solution (thus closest to $\overline{\text{MS}}$):

$$\frac{\partial}{\partial g}\text{RG}(g, L, B_i)\frac{\partial}{\partial L}\text{OPT}(g, L, B_i) - \frac{\partial}{\partial L}\text{RG}\,\frac{\partial}{\partial g}\text{OPT} \equiv 0$$

RSC affects pert. coefficients, but with property:

$$F_\pi^{\overline{MS}}(\overline{m}, g; \overline{f}_{ij}) = F'_\pi(m', g'; f'_{ij}(B_i)) + g^{k+1}\text{remnant}(B_i)$$

$\rightarrow$ differences *should* decrease with perturbative order

Results with theoretical uncertainties

Beside recovering real solution, RSC offer reasonably convincing uncertainty estimates: non-unique RSC
$\rightarrow$ we take differences between those as th. uncertainties

Table 1: Main optimized results at successive orders ($n_f = 3$)

δ^k order	nearest-to-$\overline{\text{ms}}$ RSC $\tilde{B}_i$	$\tilde{L}'$	$\tilde{\alpha}_S$	$\frac{F_0}{\Lambda_{4l}}$ (RSC uncertainties)
δ, RG-2l	$\tilde{B}_2 = 2.38\,10^{-4}$	-0.523	0.757	$0.27 - 0.34$
δ^2, RG-3l	$\tilde{B}_3 = 3.39\,10^{-5}$	-1.368	0.507	$0.236 - 0.255$
δ^3, RG-4l	$\tilde{B}_4 = 1.51\,10^{-5}$	-1.760	0.374	$0.2409 - 0.2546$

$n_f = 2$: $\frac{F}{\Lambda}(\delta^2) = 0.213 - 0.269$ ($\tilde{\alpha}_S = 0.46 - 0.64$)
$\frac{F}{\Lambda}(\delta^3) = 0.2224 - 0.2495$ ($\tilde{\alpha}_S = 0.35 - 0.42$)

• Empirical stability/convergence exhibited, with
$2b_0\tilde{g}\ln(\tilde{m}/\mu) \simeq 1$ i.e. $\tilde{m}_{opt} \simeq \mu\, e^{-1/(2b_0\tilde{g})}$ (like first RG order)

Final step: explicit symmetry breaking

•Need to account for explicit chiral symmetry breaking from genuine quark masses $m_u, m_d, m_s \neq 0$:
This relies at this stage on other (mainly lattice) results:

$\frac{F_\pi}{F} \sim 1.073 \pm 0.015$ [robust, $n_f = 2$ ChPT + lattice]

$\frac{F_\pi}{F_0} \sim 1.172(3)(43)$ (lattice MILC collaboration '10 using NNLO ChPT fits)

But there are different values by other collaborations

+ hint of slower convergence of $n_f = 3$ ChPT, e.g. Bernard, Descotes-Genon, Toucan '10

Alternative: implement explicit sym. break. within OPT

(to be less dependent of lattice/ChPT results):

$m \to m^{true}_{u,d,s} + m(1-\delta)^{\gamma_0/(2b_0)}$: looks promising but involved RG+OPT Eqs... (work in progress)

Combined results with theoretical uncertainties:

Average different RSC +average δ^2 and δ^3 results:

$\overline{\Lambda}^{n_f=2}_{4-loop} \simeq 359^{+38}_{-26}|_{(\mathrm{rgopt\ th})} \pm 5|_{(F_\pi/F)}$ MeV

$\overline{\Lambda}^{n_f=3}_{4-loop} \simeq 317^{+14}_{-7}|_{(\mathrm{rgopt\ th})} \pm 13|_{(F_\pi/F_0)}$ MeV

To be compared with some recent lattice results, e.g.:

•'Schrödinger functional scheme' (ALPHA coll. Della Morte et al '12):

$\Lambda_{\overline{\mathsf{ms}}}(n_f = 2) = 310 \pm 30$ MeV

•Twisted fermions (+NP power corrections) (Blossier et al '10):

$\Lambda_{\overline{\mathsf{ms}}}(n_f = 2) = 330 \pm 23 \pm 22_{-33}$ MeV

•static potential (Karbstein et al '14): $\Lambda_{\overline{\mathsf{ms}}}(n_f = 2) = 331 \pm 21$ MeV

Extrapolation to α_S at high (perturbative) q^2

Use only $\Lambda^{n_f=3}_{\overline{\rm ms}}$ result, perform standard (perturbative 4-loop) evolution

$\Lambda_{\overline{\rm ms}} \ll m_{charm} \ll m_{bottom} \cdots$

•In $\overline{\rm ms}$-scheme non-trivial decoupling/matching: standard perturbative extrapolation (3,4-loop with m_c, m_b thresholds, Chetyrkin et al '06):

$$\alpha_S^{n_f+1}(\mu) = \alpha_S^{n_f}(\mu)\left(1 - \frac{11}{72}(\frac{\alpha_S}{\pi})^2 + (-0.972057 + .0846515 n_f)(\frac{\alpha_S}{\pi})^3\right)$$

$$\rightarrow \overline{\alpha}_S(m_Z) = 0.1174^{+.0010}_{-.0005}(\text{rgopt th}) \pm .0010|_{(F_\pi/F_0)} \pm .0005_{evol}$$

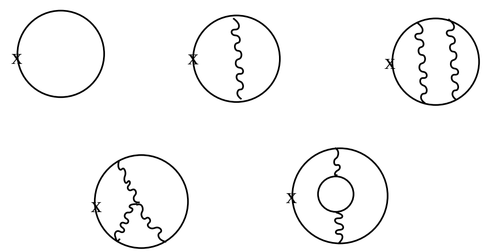
$$\overline{\alpha}_S^{n_f=3}(m_\tau) = 0.308^{+.007}_{-.004} \pm .007 \pm .002_{evol}$$

Compare to 2013 (2015) world averages:
$\alpha_S(m_Z) = 0.1185 \pm 0.0006$ ($\alpha_S(m_Z) = 0.1177 \pm 0.0013$)

4. QCD chiral condensate

Perturbative quark condensate: for n_f massive quarks ($n_f = 2, 3$)

exact result known to 3 loops (Chetyrkin et al ’94; Chetyrkin +Maier, private comm.)



$$m\,\langle\bar{q}q\rangle(m,g)_{\overline{\text{MS}}} = 3\frac{m^4}{2\pi^2}\left[\frac{1}{2} - L_m + \frac{g}{\pi^2}(L_m^2 - \frac{5}{6}L_m + \frac{5}{12})\right.$$
$$\left. + (\frac{g}{16\pi^2})^2[f_{30}(n_f)L_m^3 + f_{31}(n_f)L_m^2 + f_{32}(n_f)L_m + f_{33}(n_f)]\right]$$

($L_m \equiv \ln\frac{m}{\mu}$, $g = 4\pi\alpha_S(\mu)$)

NB: finite part (after mass + coupling renormalization) not separately RG-inv: (i.e. $m\langle\bar{q}q\rangle$ mixes with $m^4\,1$ operator: related to vacuum energy anomalous dimension

Direct RGOPT of $m\langle\bar{q}q\rangle$?

RGOPT procedure directly on the (RG-invariant) $m\langle\bar{q}q\rangle$:
first order: wrong (positive) sign of (one-loop) $\langle\bar{q}q\rangle$
Higher orders: complex $\overline{\text{MS}}$ solutions, with large imaginary parts: no pert. RSC real solutions... no stability trend.

Problem traced to strong sensitivity to (vacuum energy) anomalous dimensions, related to original quadratic divergence of the condensate

NB one-loop cutoff quadratic divergence has correct (negative) sign (success of Nambu-Jona-Lasinio model)
but sign changes in dimensional regularization $+\overline{MS}$

$\rightarrow$ Like with other variational methods, sensible to start from a more suitable basic quantity to optimize: here the spectral density of the Dirac operator, related to $\langle\bar{q}q\rangle$

Spectral density $\rho(\lambda)$ and $\langle\bar{q}q\rangle$

Euclidean Dirac operator: $i\,\not{D}\,u_n(x) = \lambda_n\,u_n(x)$; $\quad \not{D} \equiv \not{\partial} + g\,\not{A}$

On a lattice: $\rho(\lambda) \equiv \frac{1}{V}\langle\sum_n \delta(\lambda - \lambda_n^{[A]})\rangle$

$V \to \infty$: dense spectrum, and $\langle\bar{q}q\rangle_{V\to\infty} \equiv -2m\int_0^\infty d\lambda \frac{\rho(\lambda)}{\lambda^2+m^2}$

$\rho(\lambda)$: spectral density of the (euclidean) Dirac operator.

Banks-Casher relation: $\langle\bar{q}q\rangle(m \to 0) \equiv -\pi\rho(0)$

'Washes out' large λ problems (quadratic UV divergences)

Conversely: $-\rho(\lambda) = \frac{1}{2\pi}\left(\langle\bar{q}q\rangle(i\lambda+\epsilon) - \langle\bar{q}q\rangle(i\lambda-\epsilon)\right)|_{\epsilon\to 0}$

i.e. $\rho(\lambda)$ determined by discontinuities of $\langle\bar{q}q\rangle(m)$ across imaginary axis.

Perturbative expansion: $\to \ln(m \to i\lambda)$ discontinuities

$\to$ no contributions from non-log terms (like anom. dim.)

OPT and RG adapted to spectral density

Perturbative logarithmic discontinuities from

$$\ln^n\left(\frac{m}{\mu}\right) \to \frac{1}{2i\pi}\left[\left(\ln\frac{|\lambda|}{\mu} + i\frac{\pi}{2}\right)^n - \left(\ln\frac{|\lambda|}{\mu} - i\frac{\pi}{2}\right)^n\right]$$

i.e.:

$$\ln\left(\frac{m}{\mu}\right) \to 1/2; \quad \ln^2\left(\frac{m}{\mu}\right) \to \ln\frac{|\lambda|}{\mu}; \quad \ln^3\left(\frac{m}{\mu}\right) \to \frac{3}{2}\ln^2\frac{|\lambda|}{\mu} - \frac{\pi^2}{8}$$

Modified perturbation: intuitively λ plays the role of m, so:

$$\lambda \to \lambda(1-\delta)^{\frac{4}{3}\frac{\gamma_0}{2b_0}} \;; \quad g \to \delta g$$

$\to$ OPT Eq.: $\frac{\partial}{\partial\lambda}\rho(g,\lambda) = 0$ for $\lambda = \tilde{\lambda}_{opt}(g) \neq 0$

- Using $\frac{\partial}{\partial m}\frac{m}{\lambda^2+m^2} = -\frac{\partial}{\partial\lambda}\frac{\lambda}{\lambda^2+m^2}$, one finds $\rho(\lambda)$ obeys RG eq.:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \gamma_m(g)\,\lambda\frac{\partial}{\partial\lambda} - \gamma_m(g)\right]\rho(g,\lambda) = 0$$

RGOPT 2,3,4-loop results for $\langle \bar{q}q \rangle$ $(n_f = 2, 3)$

Real AF-compatible solutions obtained:

δ^k, RG order	$\ln \frac{\tilde{\lambda}}{\mu}$	$\tilde{\alpha}_S$	$\frac{-\langle \bar{q}q \rangle^{1/3}}{\bar{\Lambda}_2}(\tilde{\mu})$	$\frac{\tilde{\mu}}{\bar{\Lambda}_2}$	$\frac{-\langle \bar{q}q \rangle^{1/3}_{RGI}}{\bar{\Lambda}_2}$
δ, RG 2-loop	-0.45	0.480	0.822	2.8	0.821
δ^2, RG 3-loop	-0.703	0.430	0.794	3.104	0.783
δ^3, RG 4-loop	-0.820	0.391	0.796	3.446	0.773

δ^k order	$\ln \frac{\tilde{\lambda}}{\mu}$	$\tilde{\alpha}_S$	$\frac{-\langle \bar{q}q \rangle^{1/3}}{\bar{\Lambda}_3}(\tilde{\mu})$	$\frac{\tilde{\mu}}{\bar{\Lambda}_3}$	$\frac{-\langle \bar{q}q \rangle^{1/3}_{RGI}}{\bar{\Lambda}_3}$
δ, RG 2-loop	-0.56	0.474	0.799	3.06	0.789
δ^2, RG 3-loop	-0.788	0.444	0.780	3.273	0.766
δ^3, RG 4-loop	-0.958	0.400	0.773	3.700	0.744

NB: $\langle \bar{q}q \rangle_{RGI} = \langle \bar{q}q \rangle(\mu)\,(2b_0\,g)^{\frac{\gamma_0}{2b_0}} \left(1 + (\frac{\gamma_1}{2b_0} - \frac{\gamma_0\,b_1}{2b_0^2})\,g + \cdots\right)$

- stability/convergence seen;

already realistic at first nontrivial (2-loop) order

Evolution to $\mu = 2$ GeV and comparison

$$\langle \bar{q}q\rangle(\mu' = 2\mathrm{GeV}) = \langle \bar{\mathrm{q}}\mathrm{q}\rangle(\tilde{\mu})\,\exp[\int_{\mathrm{g}(\tilde{\mu})}^{\mathrm{g}(2\mathrm{GeV})} \mathrm{dg}\,\frac{\gamma_{\mathrm{m}}(\mathrm{g})}{\beta(\mathrm{g})}]$$

(equivalently extract from $\langle \bar{q}q\rangle_{RGI}$ with $\alpha_S(2\mathrm{GeV}) \simeq 0.305 \pm 0.004$)

(NB for $n_f = 3$ account for $\alpha_S(\mu \sim m_c)$ threshold effects)

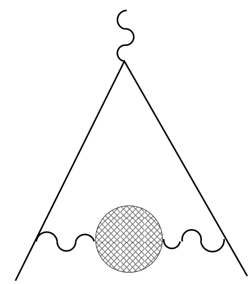
$$-\langle \bar{q}q\rangle^{1/3}_{n_f=2}(2\mathrm{GeV}) = (0.833_{(4-loop)} - 0.845_{(3-loop)})\bar{\Lambda}_2$$

$$-\langle \bar{q}q\rangle^{1/3}_{n_f=3}(2\mathrm{GeV}) = (0.814_{(4-loop)} - 0.838_{(3-loop)})\bar{\Lambda}_3$$

• Discrepancy between 3- and 4-loop results define our 'intrinsical' (RGOPT) theoretical error, $\sim 1 - 2\%$

5. Application to $g_\mu - 2$ (HVP contribution)

Motivations similar to lattice: "first principle" attempt to calculate the Hadronic Vacuum Polarisation, *independent* from dispersion relations from e^+e^-, τ decay data:



- worth to test our procedure on HVP, before possibly trying on Hadronic light by light contribution: from previous cases, hope RGOPT HVP to reach $\sim 2\%$ accuracy, light by light contribution needs less accuracy.

- More challenging (technically): condensate $\langle \bar{q}q \rangle$: no p^2 dependence (tadpole); $F_\pi/\Lambda_{\overline{\rm ms}}$: relies only on $\Pi_A(p^2=0)$.

But $g-2$ HVP involves $\Pi_V(p^2)$; crucial p^2 dependence.

- Could give at least some (analytical) handle on the challenging low-Q^2 region.

Recipe: main steps

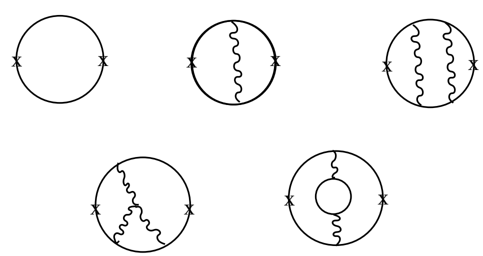
Quite similarly to $\Pi_A(p^2=0,m,\alpha_S)$ relevant to $F_\pi/\Lambda_{\overline{\rm MS}}$, we seek a RGOPT approximation of $\tilde\Pi_V(p^2,\tilde m,\tilde\alpha_S)$, now if possible for “any” (at least low) p^2.

More appropriate to work in Euclidean (cf Lattice):

$$a_\mu^{HVP} = -(\frac{\alpha}{\pi})^2\frac{1}{2}\int_0^\infty d\omega K_E(\omega)\Pi_{V,E}(\omega m_\mu^2)$$

$$K_E(\omega) = \frac{\sqrt{\frac{\omega}{4+\omega}}}{\omega}\left(\frac{\sqrt{4+\omega}-\sqrt{\omega}}{\sqrt{4+\omega}+\sqrt{\omega}}\right)^2$$

Again start from purely perturbative $\Pi_V(Q^2,m_q\neq 0,\alpha_S)$: exactly known to 2 loops approximate (low or high $p^2=-Q^2$ expansions) at 3-loop (even 4-loop): Chetyrkin et al



Note: it is known that the simple one-loop, with *constituent* quark mass $M_{cons}\sim 250$ MeV gives roughly right order of HVP contribution (De Rafael, Greynat JHEP2012)

Our construction gives a RG consistent, fully determined mass, at arbitrary orders

some peculiarities/subtleties

• Relevant quantity in $\int d\omega$ integrand is $\Pi_V(Q^2) - \Pi_V(0)$: but $\Pi_V^{\overline{\text{ms}}}(0) = \frac{1}{4\pi^2} \ln \frac{\mu^2}{m^2}$ important, both for consistent RG properties, and for non-trivial optimized solution.

• Indeed non-trivial (RG-consistent) OPT ($\partial_m(\cdots) = 0$) solutions only if terms $\propto m^2 \ln \frac{\mu^2}{m^2}$ are present.

However, $\Pi_V(Q^2, m^2) \simeq const. + \mathcal{O}(Q^2/m^2)$ (i.e. is dimensionless)

$\rightarrow$ Consider (and optimize) rather $m^2 \Pi_V(Q^2)$

Then well-defined recipe:

1) determine subtraction:
$P(Q^2, m^2) = m^2 \Pi(Q^2, m^2) - m^2(s_0(b_0, \gamma_0)/g + s_1(b_1, \gamma_1) + \cdots)$
to recover (standard) perturbative RG invariance (NB s_i should be independent of Q^2).

2) Apply OPT: $P(Q^2, m^2(1-\delta)^{\gamma_0 b_0}, \delta g)$, expand to $\mathcal{O}(\delta^k)$, then put $\delta \rightarrow 1$.
$\rightarrow$ asymptotic freedom consistently for RG solution: $g(Q^2 \gg \mu^2) \sim \frac{1}{b_0 \ \ln \frac{Q^2}{\mu^2}}$

Determine optimal $\tilde{m}^2(g, Q^2)$ from $\partial_{m^2} P(Q^2, m^2) = 0$, put in
$P(Q^2, \tilde{m}^2)/\tilde{m}^2 \equiv \tilde{\Pi}(Q^2, \tilde{m}^2)$.

NB Q^2 plays the role of 'external' parameter (with respect to RG properties),

$\tilde{m}(Q^2)$ "running" mass.

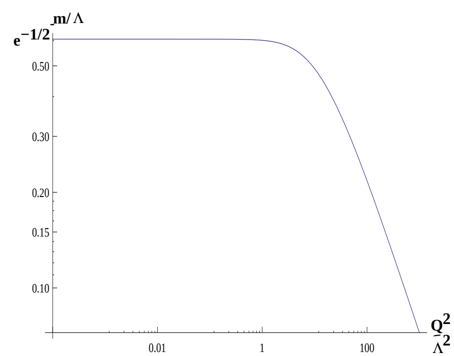
Results at one-loop (preliminary!)

$$\frac{16\pi^2}{3}\tilde{\Pi}^{(1)}(Q^2,\tilde{m}^2) = \frac{8\tilde{m}^2}{Q^2}\left(1+\frac{2\tilde{m}^2}{Q^2\sqrt{1+4\tilde{m}^2/Q^2}}\ln[1+\frac{Q^2}{2\tilde{m}2}(1-\sqrt{1+4\tilde{m}^2/Q^2})]\right)-4/3$$

- not very different from standard one-loop expression, BUT “non-perturbative” $\tilde{m}(g,Q^2)$ with highly non-trivial Q^2 dependence:

$$\tilde{m}(Q^2\sim 0)\sim \Lambda_{\overline{\rm ms}}\, e^{-\frac{1}{2}}\ \exp[-\frac{3}{280}e^2(\frac{Q^2}{\Lambda^2_{\overline{\rm ms}}})^2]$$

$$\tilde{m}^2(Q^2\to\infty)\sim \Lambda^2_{\overline{\rm ms}}\, e^{\frac{5}{3}}\ \frac{\Lambda^2_{\overline{\rm ms}}}{Q^2}$$



- The $g=4\pi\alpha_S$ dependence only enters in $\tilde{m}(\Lambda_{\overline{\rm ms}}\equiv \mu e^{-1/(2b_0 g)})$.

$\to$ No free parameters: fully determined in terms of $\Lambda_{\overline{\rm ms}}$, as previously.

It gives $a_\mu^{HVP}(1-loop)\simeq 510\,10^{-10}$ (crude, taking constant $\tilde{m}=e^{-1/2}\Lambda_{\overline{\rm ms}}$);
$a_\mu^{HVP}(1-loop)\simeq 815\,10^{-10}$ (more correctly accounting for $\tilde{m}(Q^2)$);

(respectively $\sim 20\%$ too low (high) compared with latest $a_\mu^{HVP}\simeq(692\pm4)\,10^{-10}$)

Beyond one-loop: perspectives (very sketchy...)

From the behaviour of previous RGOPT results (F_π, $\langle\bar{q}\rangle, \cdots$), expect no drastic changes, but smooth 'incorporation of higher non-trivial α_S, Q^2 dependence.

$\Pi_V(Q^2, m^2)$ exactly known at two-loops, but will give very involved optimized mass:
$\tilde{m}^2 = f(Li_2(Q^2/\tilde{m}^2), Li_3(Q^2/\tilde{m}^2), \cdots)$

$\rightarrow$ Better attack first with simpler low Q^2 expansions (kown to high Q^2/m^2 order).

No concrete results yet, but roughly expect
$\tilde{\Pi}^{1-loop} \rightarrow \tilde{\Pi}^{1-loop}\left(1 + \tilde{\alpha}_S\, f_{2-loop}(\frac{Q^2}{m^2}) + \mathcal{O}(\alpha_S^2)\right)$

Crucial point: both optimized mass $\tilde{m}^2(Q^2, \alpha_S)$ and optimized $\tilde{\alpha}_S$ will be fully determined (pure numbers), like for F_π, $\langle\bar{q}\rangle$: no adjustable parameters (except $\Lambda_{\overline{\text{MS}}}$).

Summary and Outlook

•OPT gives a simple procedure to resum perturbative expansions, using only perturbative information.

•Our RGOPT version includes 2 major differences w.r.t. most previous OPT approaches:

OPT+ RG optimization fix $\tilde{m}$ *and* $\tilde{g} = 4\pi\tilde{\alpha}_S$

Requiring RG invariance or AF-compatible solutions after interpolation uniquely fixes the latter $m \to m(1-\delta)^{\gamma_0/(2b_0)}$: discards spurious solutions and accelerates convergence.

($\mathcal{O}(10\%)$ accuracy at 1-2-loops, expect $\sim 2\%$ accuracy + stability at 3-loop)

$g-2$ HVP contributions: technically more challenging (highly non trivial Q^2 dependence): one-loop results look encouraging ($\sim 20\%$ overestimate).

Backup: Pre-QCD guidance: Gross-Neveu model

• $D = 2$ $O(2N)$ GN model shares many properties with QCD (asymptotic freedom, (discrete) chiral sym., mass gap,..)

$$\mathcal{L}_{GN} = \bar{\Psi} i \not\partial \Psi + \frac{g_0}{2N} (\sum_1^N \bar{\Psi}\Psi)^2 \text{ (\textit{massless})}$$

Standard mass-gap (massless, large N approx.):

consider $V_{eff}(\sigma)$, obtained from $\int d\bar{\Psi} d\Psi e^{i\mathcal{L}_{GN}}$:

$$\frac{\partial V_{eff}(\sigma)}{\partial \sigma} = 0 \to \sigma(\sim \langle \bar{\Psi}\Psi \rangle) \equiv M = \mu e^{-\frac{2\pi}{g(\mu)}} \equiv \Lambda_{\overline{\text{ms}}}$$

• Mass gap also known exactly for any N:

$$\frac{M_{exact}(N)}{\Lambda_{\overline{\text{ms}}}} = \frac{(4e)^{\frac{1}{2N-2}}}{\Gamma[1 - \frac{1}{2N-2}]}$$

(From $D = 2$ integrability: Bethe Ansatz) Forgacs et al '91

Link to massive GN model

Now consider *massive* case (still large N):

$M(m,g) \equiv m(1+g\ln\frac{M}{\mu})^{-1}$: Resummed mass ($g/(2\pi) \to g$)

$= m(1-g\ln\frac{m}{\mu}+g^2(\ln\frac{m}{\mu}+\ln^2\frac{m}{\mu})+\cdots)$ (pert. re-expanded)

$M \equiv \Lambda_{\overline{\text{ms}}}$ never seen in standard pert.: $M_{pert}(m \to 0) \to 0$

- Only fully summed $M(m,g)$ gives right result, upon:

-identify $\Lambda_{\overline{\text{ms}}} \equiv \mu e^{-1/g}$; $\to M(m,g) = \frac{m}{g\ln\frac{M}{\Lambda_{\overline{\text{ms}}}}} \equiv \frac{\hat{m}}{\ln\frac{M}{\Lambda_{\overline{\text{ms}}}}}$;

-take *reciprocal*: $\hat{m}(F \equiv \ln\frac{M}{\Lambda}) = F\,e^F\,\Lambda \sim F\Lambda$ for $\hat{m} \to 0$;

$$\to M(\hat{m} \to 0) \sim \frac{\hat{m}}{\hat{m}/\Lambda + \mathcal{O}(\hat{m}^2)} = \Lambda_{\overline{\text{ms}}}$$

- But (RG)OPT gives $M = \Lambda_{\overline{\text{ms}}}$ at *first* (and any) δ-order

(at any order, OPT sol.: $\ln\frac{m}{\mu} = -\frac{1}{g}$, RG sol.: $g = 1$)

- At δ^2-order (2-loop), RGOPT $\sim 1-2\%$ from M_{exact}(any$N$)