School Of Statistics

Basic concepts – part 1

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Outline

Part I

Basics

- Sample measurements
- Error propagation
- Probabilities, Bayes Theorem
- Probability density function

Model testings

- p-value and test statistics
- Chi2 and KS tests
- Hypothesis testing

Part II

Parameter estimation

- Maximum likelihood method
- Least square fit
- BLUE

Introductory books (non exhaustive)

Excellent book of reference

- G. Cowan, *Statistical Data Analysis* (Oxford Science Publication)

Introduction to Bayesian analysis

- D. Sivia, *Data Analysis: A Bayesian Tutorial* (Oxford Science Publication)

Nice approach

- Louis Lyons, *Statistics for Nuclear and Particle Physicists* (Cambridge University Press)

En Français

- B. Clement, *Analyse de données en sciences expérimentales* (Dunod)

Population

- Let's consider a sample of values (e.g. experimental measurements)

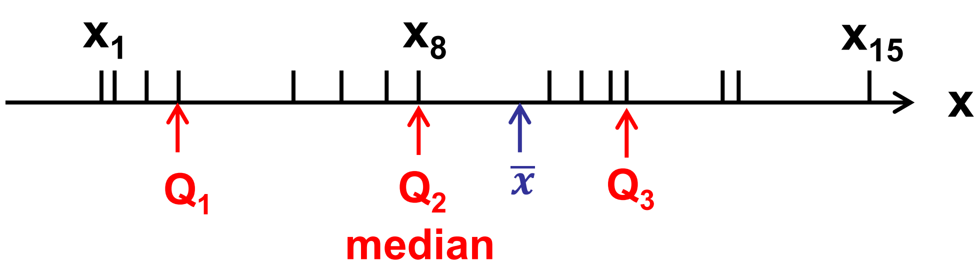
  N measurement of a variable X: $\{x_i\} = \{x_1, x_2, \ldots, x_N\}$
- There are several quantities that can be determined to characterize this population without any knowledge of the underlying model/theory

Measure of position

Arithmetic mean: $\boxed{\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i}$

Median: value that separates sample in half

Quartiles (Q_1,Q_2,Q_3): values that separates sample in four equal-size sample



Measure of dispersion

Variance: if truth sample mean μ is known
$$v = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{\mu})^2$$

But μ is in general not know and sample mean is used instead

- Sample variance (biased):
$$v = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2 = \overline{x^2} - \bar{x}^2$$

- Estimated variance (unbiased):
$$v = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2 = \frac{N}{N-1}(\overline{x^2} - \bar{x}^2)$$

→ Bias is below α if N ≥ 1/α – 1 (ex for 1% bias, N≥101)

Standard deviation (is of same unit as x): $\sigma = \sqrt{v}$

Standard deviation and error

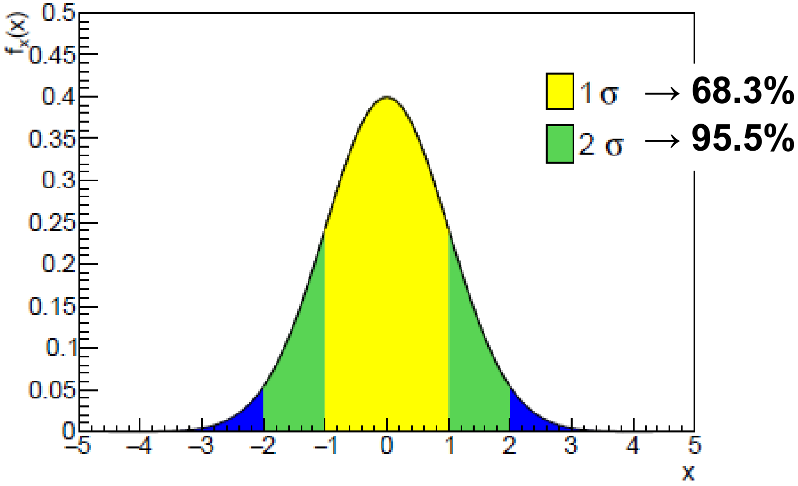
In many situations repeating an experiment a large amount of time produces a spread of results whose distribution is approximately Gaussian.

This is a consequence of the Central Limit Theorem.

Gaussian distribution

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}$$

Interval μ±σ contains 68.3% of distribution



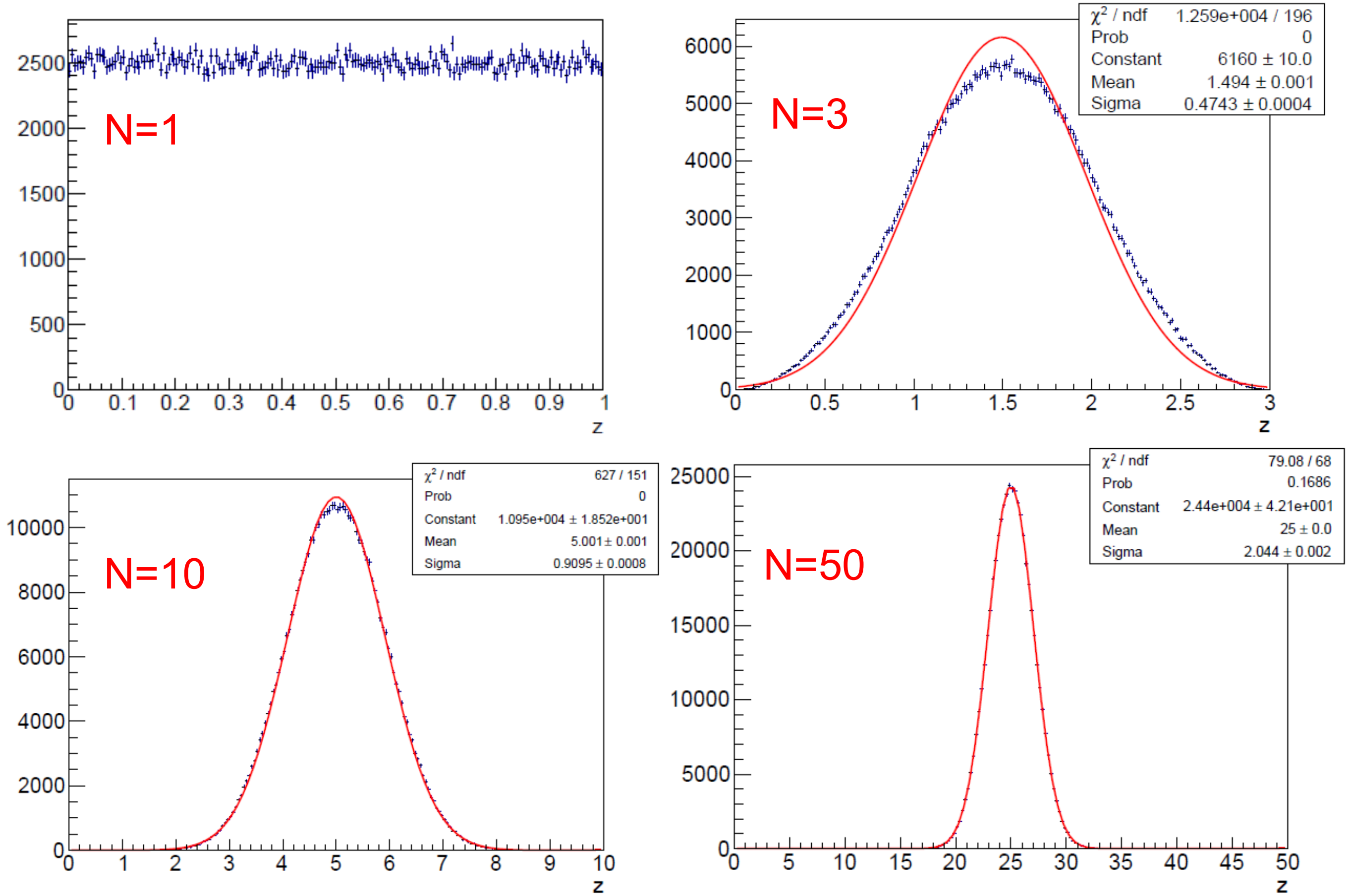
A measurement = outcome of the sum of a large number of effects.

In general the distribution of this variable will be gaussian. The std deviation of the sample is associated to the std deviation of the Gauss distribution.

The standard deviation is then interpreted as the interval that could contain the true value with a 68.3% confidence level.

Simple illustration of CLT

- let's consider x: a random variable uniformly distributed in [0,1]
- and the distribution of N sums of x: $z = \sum_{i=1}^{N} x_i$



Uniform (N=1)

↓

Irwin-Hall

(see here)

↓

Gauss

(N>40)

Multidimensional samples

Case where N measurements are performed of M different variables

→ The sample then consists of N vectors of M measurements

$$\{\vec{x_i}\} = \{\vec{x_1}, \vec{x_2}, \ldots, \vec{x_N}\} \quad \text{with} \quad \begin{cases} \vec{x_1}: \; x_1{}^{(1)}, x_1{}^{(2)}, \ldots \; x_1{}^{(M)} \\ \qquad (\ldots) \\ \vec{x_N}: \; x_N{}^{(1)}, x_N{}^{(2)}, \ldots \; x_N{}^{(M)} \end{cases}$$

Mean and variance can be calculated for each variable $x_i^{(k)}$ but to quantify how of one variable behaves w.r.t another one uses the **covariance:**

For two variables x and y: $$\boxed{\mathrm{cov}(x, y) = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y}) = \overline{xy} - \bar{x}\bar{y}}$$

Correlation factor is defined as: $$\boxed{\rho_{xy} = \frac{\mathrm{cov}(x, y)}{\sigma_x \sigma_y}} \quad \text{with} \; -1 \leq \rho_{xy} \leq 1$$

ρ_{xy} = 1(-1) → x and y are fully (anti)correlated

ρ_{xy} = 0 → x and y are uncorrelated (≠ independent !)

Covariance matrix (aka error matrix) of sample $\{\vec{x_i}\}, \mathrm{i} = 1..\mathrm{N}$

- Real, symmetric, N×N matrix of the form:

$$C = \begin{pmatrix} \mathrm{cov}(x_1, x_1) & \cdots & \mathrm{cov}(x_1, x_N) \\ \vdots & \mathrm{cov}(x_i, x_j) & \vdots \\ \mathrm{cov}(x_N, x_1) & \dots & \mathrm{cov}(x_N, x_N) \end{pmatrix} = \begin{pmatrix} {\sigma_1}^2 & \cdots & \rho_{1N}\sigma_1\sigma_N \\ \vdots & \rho_{ij}\sigma_i\sigma_j & \vdots \\ \rho_{N1}\sigma_N\sigma_1 & \dots & {\sigma_N}^2 \end{pmatrix}$$

Correlation matrix: $\rho = \begin{pmatrix} 1 & \cdots & \rho_{1N} \\ \vdots & 1 & \vdots \\ \rho_{N1} & \dots & 1 \end{pmatrix}$

Example of usage of covariance matrix:

- Transformation of input variables
- Error propagation
- Combination of correlated measurements
- …

Decorrelation: choose a **basis** $\{\vec{y_i}\}$ where C becomes **diagonal**.

→ transformation matrix **A** such that new covariance matrix **U** is diagonal

$$y_i = \sum_{j=1}^{N} A_{ij} x_j$$

$$\boxed{Y = AX}$$

$$U_{ij} = \mathrm{cov}(y_i, y_j) = \mathrm{cov}\left(\sum_{k=1}^{N} A_{ik} x_k \sum_{l=1}^{N} A_{jl} x_l\right)$$

$$= \sum_{k,l=1}^{N} A_{ik} A_{jl} \mathrm{cov}(x_l, x_k) = \sum_{k,l=1}^{N} A_{ik} C_{kl} {A_{lj}}^{T}$$

$$\boxed{U = ACA^T}$$

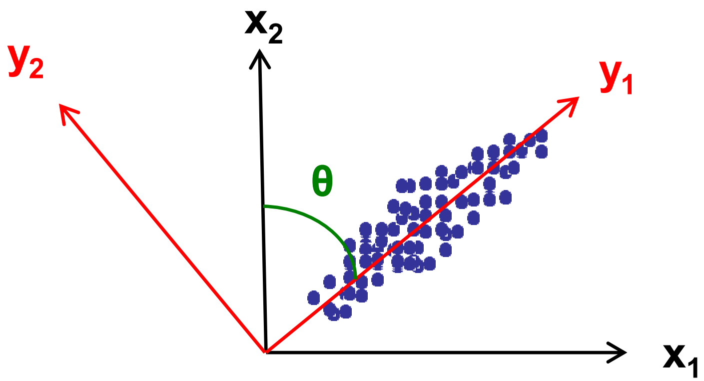
Diagonalization of C: find orthonormal eigenvectors e_j such that $\boxed{Ce_j = \lambda_j e_j}$

$$A = \begin{pmatrix} {e_1}^{(1)} & {e_1}^{(2)} & \cdots & {e_1}^{(N)} \\ & \vdots & \vdots & \\ {e_N}^{(1)} & {e_N}^{(2)} & \cdots & {e_N}^{(N)} \end{pmatrix}$$ and $$U = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix}$$

λ_i = eigenvalues of C = σ'^2_j : variance of y_i

2D example: variables x_1 and x_2 with correlation factor ρ

$$\lambda_\pm = \frac{1}{2}\left(\sigma_1{}^2 + \sigma_2{}^2 \pm \sqrt{(\sigma_1{}^2 + \sigma_2{}^2)^2 - 4(1-\rho^2)\sigma_1{}^2\sigma_2{}^2}\right)$$



$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\rho\sigma_1\sigma_2}{\sigma_1{}^2 - \sigma_2{}^2}\right)$$

Decorrelation: use cases

- Data pre-processing (for MVA): remove correlation from input variables
- Reduce dimensionality of a problem: Principal Component Analysis (PCA)

Consider only the M<N dominant eigenvalues (=variance) terms in U
→ Reduced covariance matrix C: M×M

Note: the decorrelation method is able to eliminate only **linear** correlations

Error propagation

Function ***f*** of several variables **x**=$\{x_1,\ldots,x_N\}$

- Each variable $\mathbf{x}_i$ of mean $\boldsymbol{\mu}_i$ and variance $\boldsymbol{\sigma}_i^2$
- Perform **1st order Taylor expansion** of *f* around mean value

$$f(\vec{x}) \approx f(\vec{\mu}) + \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\vec{\mu})(x_i - \mu_i)$$

$$f(\vec{x})^2 \approx f(\vec{\mu})^2 + 2f(\vec{\mu}) \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\vec{\mu})(x_i - \mu_i) + \sum_{i,j=1}^{N} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}(\vec{\mu})(x_i - \mu_i)(x_j - \mu_j)$$

Variance of ***f(x)***:

$$\boxed{\sigma_f^2 = \overline{f(\vec{x})^2} - \left(\overline{f(\vec{x})}\right)^2 \approx \sum_{i,j=1}^{N} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}(\vec{\mu}) \times \mathrm{cov}(x_i, x_j)}$$

Since $\overline{(x_i - \mu_i)} = 0$

$$\overline{(x_i - \mu_i)^2} = \sigma_i^2$$

$$\overline{(x_i - \mu_i)(x_j - \mu_j)} = \mathrm{cov}(x_i, x_j)$$

Validity: up to 2nd order, linear case, small errors

Error propagation

Example:

x and y with correlation factor ρ

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x}\sigma_x\right)^2 + \left(\frac{\partial f}{\partial y}\sigma_y\right)^2 + 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\mathrm{cov}(x,y)$$

$$f(x,y) = x + y \;\rightarrow\; \sigma_f^2 = \sigma_x^2 + \sigma_y^2 + 2\rho\sigma_x\sigma_y$$

$$f(x,y) = xy \;\rightarrow\; \sigma_f^2 = y\sigma_x^2 + x\sigma_y^2 + 2xy\rho\sigma_x\sigma_y$$

For a set of m function $\mathbf{f_1(x)}, \ldots, \mathbf{f_m(x)}$

- **C** is the covariance of variables $\mathbf{x}=\{x_i\}$
- We can build the covariance matrix of $\{\mathbf{f_i(x)}\}$: **U**

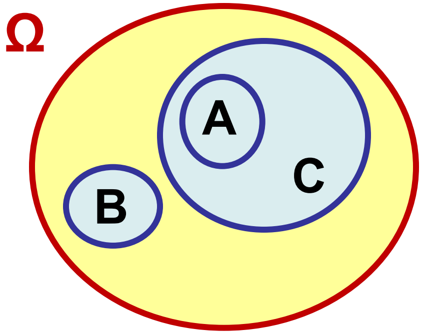
$$U_{kl} = \mathrm{cov}(f_k, f_l) = \sum_{i,j=1}^{N} \frac{\partial f_k}{\partial x_i}\frac{\partial f_l}{\partial x_j}(\vec{\mu}) \times \mathrm{cov}(x_i, x_j)$$

This can be expressed as $\boxed{U = ACA^T}$ where $\boxed{\mathrm{A_{ij}} = \frac{\partial f_i}{\partial x_j}(\vec{\mu})}$ (matrix of derivatives)

You are given a coin, you toss it and obtain “tail”.
What is the probability that both sides are “tail” ?

It depends on the **prior** that the coin is **unfair** (and on the person that gave you the coin)

Who is more likely to give a fair coin ?



Sample space: Ω

- Set of all possible results of an experiment
- Populated by events

Probability

- Frequentist: related to frequency of occurrence

$$P(A) = \frac{\text{number of time event A occurs}}{\text{number of time experience is repeated}}$$

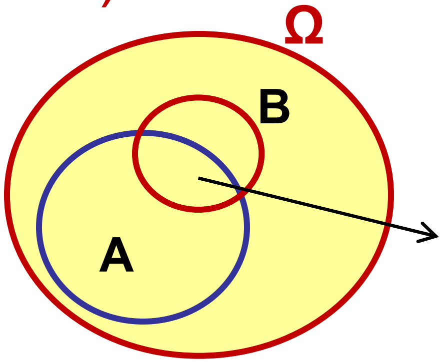
- Subjectivist (Bayesian): degree of belief that A is true
  Introduces concepts of prior and posterior probability

$$P(A|\text{data}) \propto P(\text{data}|A) \times P(A)$$

Knowledge on A increases using data

Mathematical formalization (Kolmogorov)

$$P(\Omega) = 1$$
$$0 \le P(A) \le 1$$
$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$



$P(A \cap B)$

Incompatible events: $P(A \cap B) = \emptyset \Rightarrow P(A \cup B) = P(A) + P(B)$

Conditional probability: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

Independent events: $P(A \cap B) = P(A|B)P(B) = P(A)P(B)$

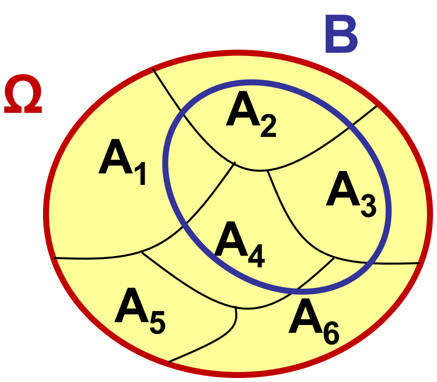
An Essay towards solving a Problem in the Doctrine of Chances. By the late Rev. Mr. Bayes, *communicated by Mr.* Price (1763)

"If there be two subsequent events, the probability of the second b/N and the probability of both together P/N, and it being first discovered that the second event has also happened, from hence I guess that the first event has also happened, the probability I am right is P/b."

http://www.stat.ucla.edu/history/essay.pdf

Thomas Bayes (?)
c. 1701 –1761

$$\boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$$



$A_i \cap A_j = \emptyset \ (i \neq j)$

If the sample space Ω can be divided in disjoint subsets A_i

$$P(B) = \sum_i P(B \cap A_i) = \sum_i P(B|A_i)P(A_i)$$

$$\boxed{P(A|B) = \frac{P(B|A)P(A)}{\sum_i P(B|A_i)P(A_i)}}$$

Bayes Theorem in everyday life

Example: 10 coins, **one** of which is **unfair** (two-sided tail): You flip a random coin and obtain **tail**. What is the probability that this is the unfair coin ?

A: event where the coin is **unfair**, **B**: event where the result is **tail**

You want **P(A|B)**: $\boldsymbol{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$

where: $P(B) = P(B \cap A) + P(B \cap \bar{A}) = P(B|A)P(A) + P(B|\bar{A})P(\bar{A})$

$$P(B|A) = 1, P(A) = \frac{1}{10}$$

$$\boldsymbol{\Rightarrow P(A|B) = \frac{1 \times \frac{1}{10}}{1 \times \frac{1}{10} + \frac{1}{2} \times \frac{9}{10}} = \frac{2}{11}}$$

In Bayesian language: P(A) is the prior probability and P(B|A) the posterior

Consequences of not knowing Bayes Th.

Simple tools for understanding risks: from innumeracy to insight (2003)

G. Gigerenzer, A. Edwards, BMJ 327, 2003 http://www.ncbi.nlm.nih.gov/pmc/articles/PMC200816/

Conditional probabilities

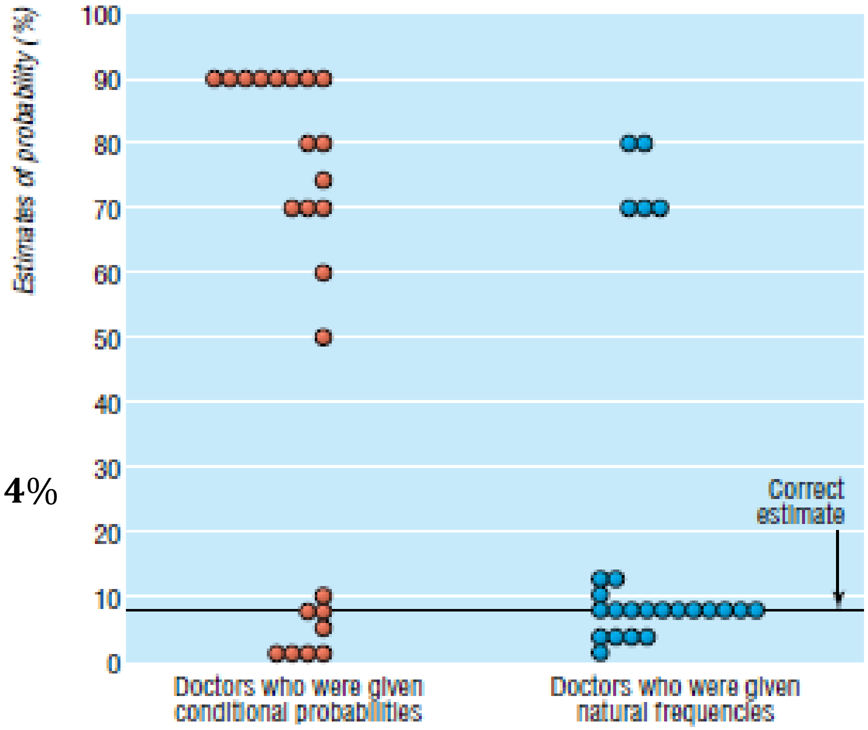
The probability that a woman has breast cancer is 0.8%. If she has breast cancer, the probability that a mammogram will show a positive result is 90%. If a woman does not have breast cancer the probability of a positive result is 7%. Take, for example, **a woman who has a positive result. What is the probability that she actually has breast cancer?**

$$P(C|+) = \frac{P(+|C)P(C)}{P(+)} = \frac{0.9 \times 0.008}{0.9 \times 0.008 + 0.07 \times 0.992} = 9.4\%$$

Natural frequencies

Eight out of every 1000 women have breast cancer. Of these eight women with breast cancer seven will have a positive result on mammography. Of the 992 women who do not have breast cancer some 70 will still have a positive mammogram. Take, for example, a sample of women who have positive mammograms. **How many of these women actually have breast cancer?**

$$P(C|+) = \frac{0.9 \times 8}{0.9 \times 8 + 0.07 \times 992} = 9.4\%$$



"Bad presentation of medical statistics such as the risks associated with a particular intervention can lead to patients making poor decisions on treatment"

Bayes Theorem and statistical inference

Statistical inference

Estimate true parameters of a theory or a model using data

- Frequentist: perform measurement (or set limits)
- Bayesian: Improve prior knowledge using data

Going Bayesian

Likelihood of observing these data given a theory

Posterior knowledge on theory

Prior knowledge on theory

$$P(\text{theory}|\text{data}) = \frac{P(\text{data}|\text{theory})\,P(\text{theory})}{P(\text{data})}$$

Usually just an irrelevant normalisation factor

Probability distribution

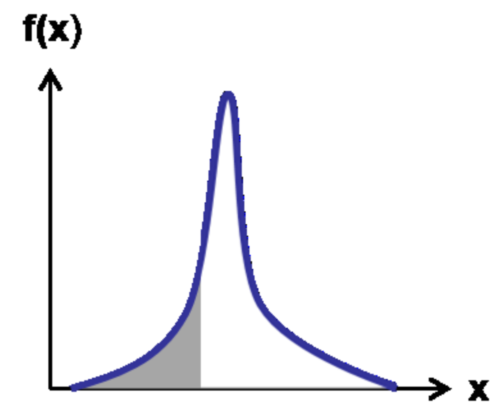
Random variable X

Discrete random variable: result (realizations) $x_i \in \Omega$ with probability $P(x_i)$

→ **P** is the **probability distribution** and $\sum_{i}^{N} P(x_i) = 1$

For continuous variable: probability of observing x in infinitesimal interval

→ Given by the **probability density function** (p.d.f) f(x)



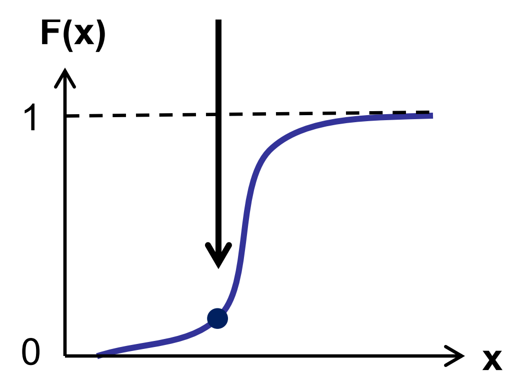
$$\boxed{\textbf{Probability of x in } [x, x+dx] = f(x)dx}$$

$$\textbf{Probability of x in } [a,b] = \int_a^b f(x)dx$$

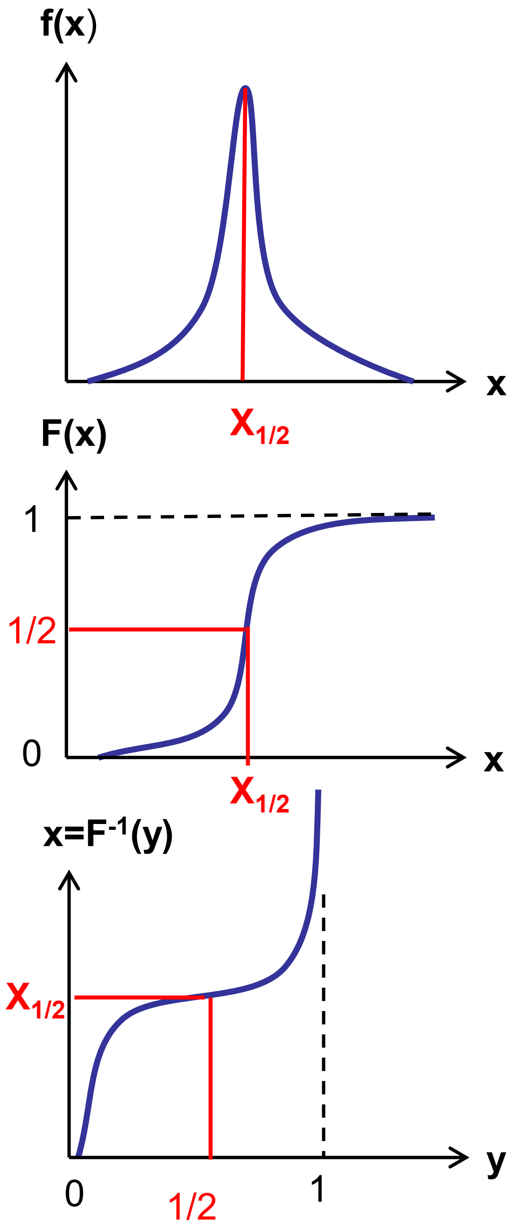
with: $\int_{\Omega} f(x)dx = 1$

→ **Cumulative distribution** F(x): $\boxed{F(x) = \int_{-\infty}^{x} f(x')dx'}$

hence: $f(x) = \frac{dF}{dx}(x)$



Quantiles



Probability density function: $f(x)$

Cumulative distribution: $F(x)=y$

Inverse cumulative distribution: $x=F^{-1}(y)$

Median: x such that $F(x)=1/2 \rightarrow x_{1/2} = F^{-1}(1/2)$

Quantile of order α: $x_\alpha = F^{-1}(\alpha)$

Expectation value of a random variable X:

For a **function** of x, **a(x)**, the expectation value is: $E[a(x)] = \int_{-\infty}^{\infty} a(x) f(x) dx$

- **mean of X:** $\boxed{E[x] = \int_{-\infty}^{\infty} x f(x) dx = \mu}$

- **nth order moment:** $E[x^n] = \int_{-\infty}^{\infty} x^n f(x) dx = \mu_n$

- **Characteristic function $\phi(t)$:**

$$\phi(t) = E\left[e^{itx}\right] = \int e^{itx} f(x) dx = \mathrm{FT}^{-1}(f) \quad \text{where } \mu_n = (-i)^n \frac{d^n \phi}{dt^n}(0)$$

- **Variance:**

$$\boxed{\begin{aligned} V[x] &= E\left[(x - E[x])^2\right] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx \\ &= E\left[x^2\right] - E[x]^2 \end{aligned}}$$

- **Standard deviation:** $\sigma = \sqrt{V[x]}$

Some common distributions

Binomial law: efficiency, trigger rates, …

$$\boldsymbol{B(k;n,p) = C_k^n p^k (1-p)^{n-k}, \mu = np, \sigma = \sqrt{np(1-p)}}$$

Poisson distribution: counting experiments, hypothesis testing

$$\boldsymbol{P(n;\lambda) = \frac{\lambda^n e^{-\lambda}}{n!}, \mu = \lambda, \sigma = \sqrt{\lambda}}$$

Gauss distribution (aka Normal): many use-case (asymptotic convergence)

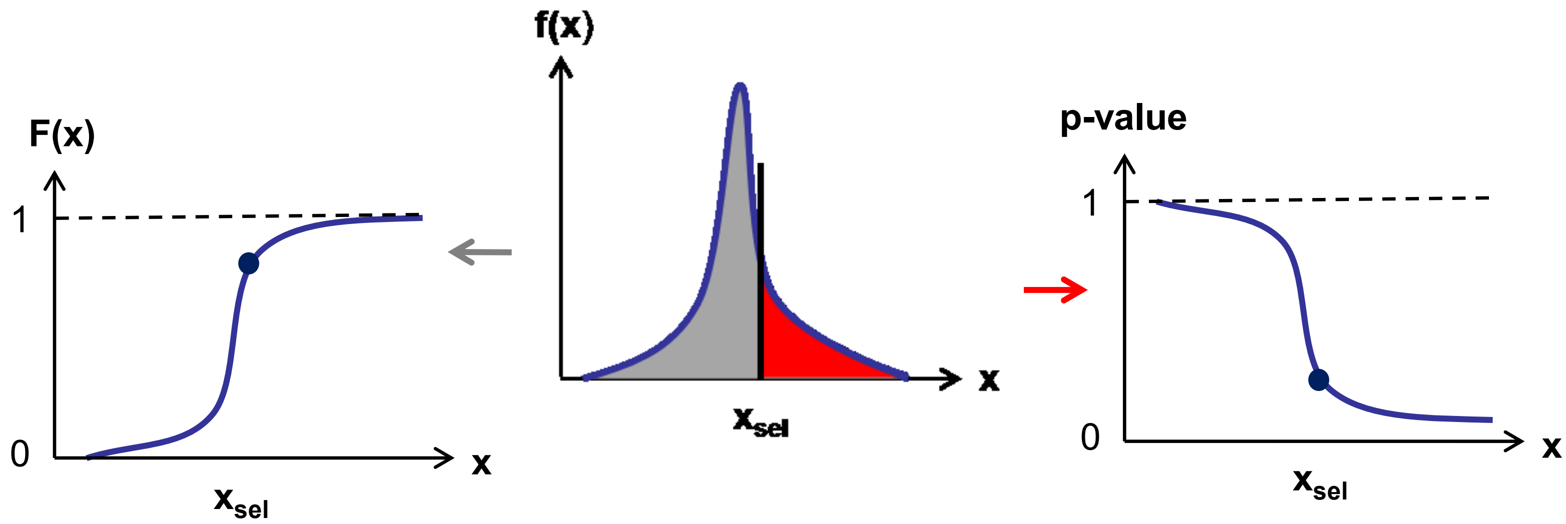
$$\boldsymbol{f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}$$

Cauchy distribution (aka Breit-Wigner): particle decay width, ...

$$\boldsymbol{f(x;x_0,\gamma) = \frac{1}{\pi\gamma\left[1+\left(\frac{x-x_0}{\gamma}\right)^2\right]}}$$

μ and σ not defined (divergent integral)

Cumulative distribution and p-value



$$F(x) = \int_{-\infty}^{x} f(x')dx'$$

$$\text{p-value} = \int_{x}^{+\infty} f(x')dx'$$

One can choose any x_{sel} to compute F(x) or p-value, that is x_{sel} does not have a preferred value: it follows the uniform distribution.

➔ The distributions of $F(x_{sel})$ and p-value are also uniform

➔ Important for MC sample generation and hypothesis testing

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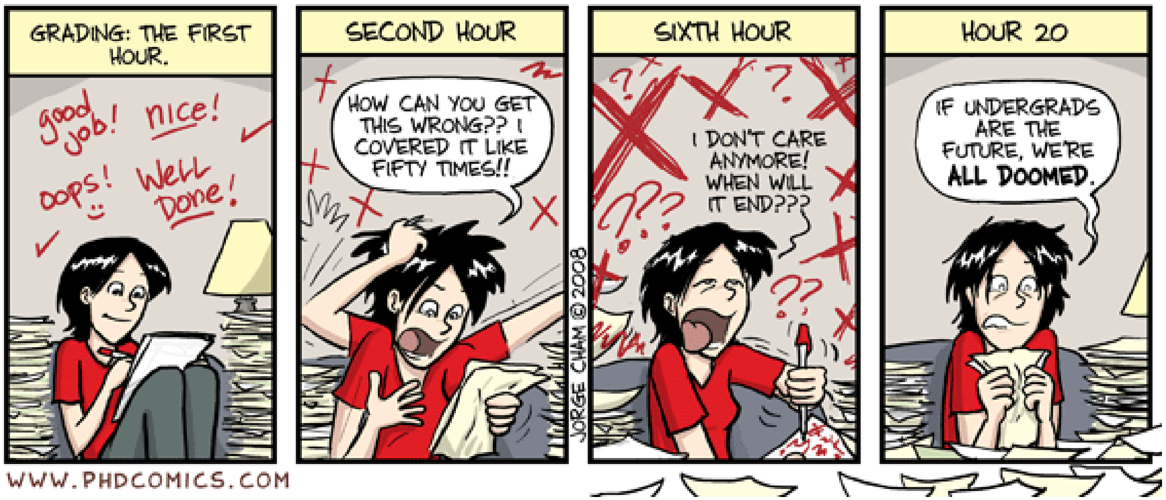


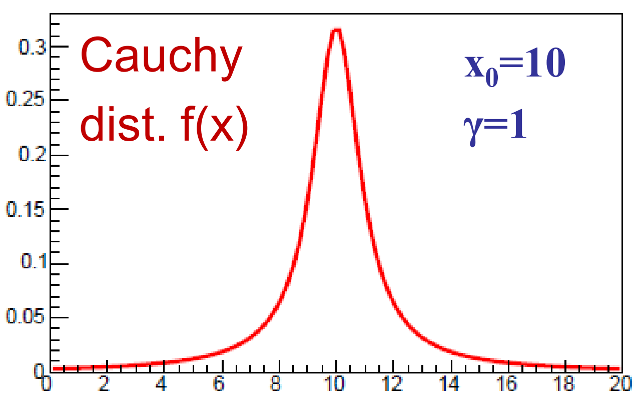
- 100 copies, grades: 0-20
- Peaked distribution at 10

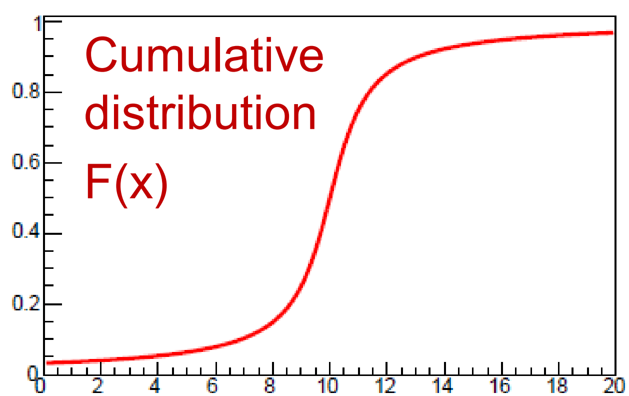
Try Cauchy distribution

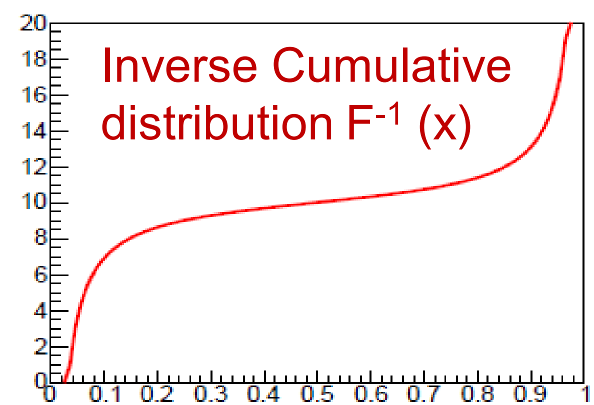
$$f(x) = \frac{1}{\pi\gamma\left[1 + \left(\frac{x - x_0}{\gamma}\right)^2\right]}$$

$$F(x) = \frac{1}{\pi}\arctan\left(\frac{x - x_0}{\gamma}\right) + \frac{1}{2}$$

$$F^{-1}(y) = x = \gamma\tan\left(\pi\left(y - \frac{1}{2}\right)\right) + x_0$$







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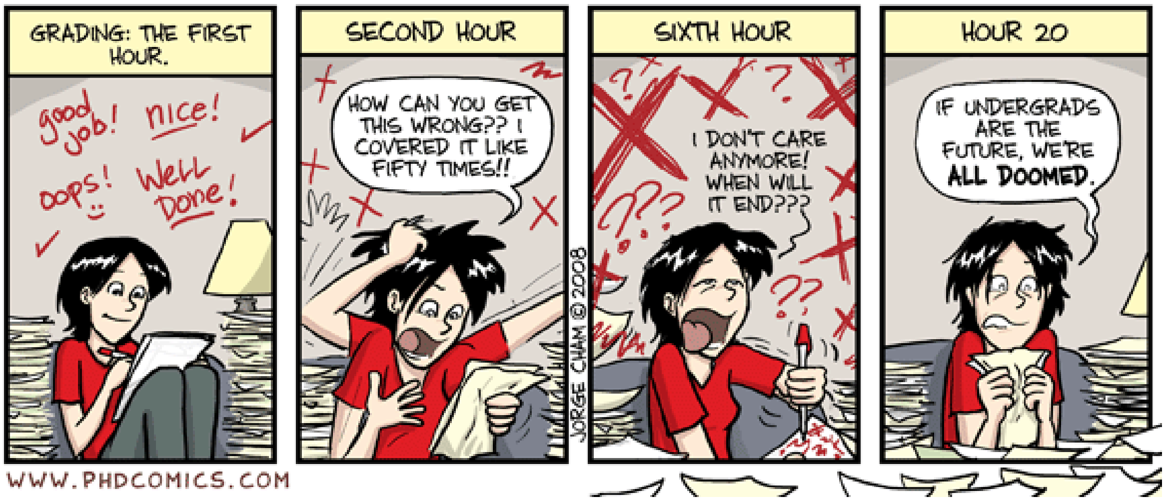

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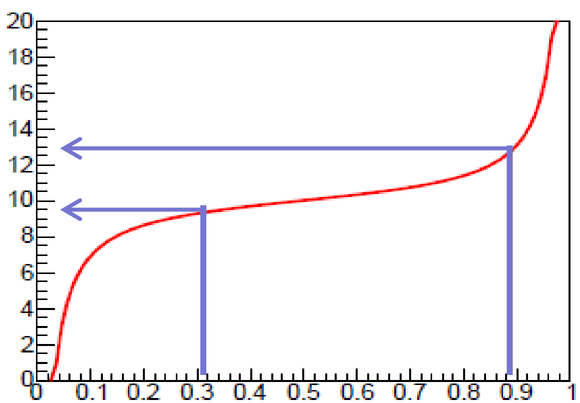
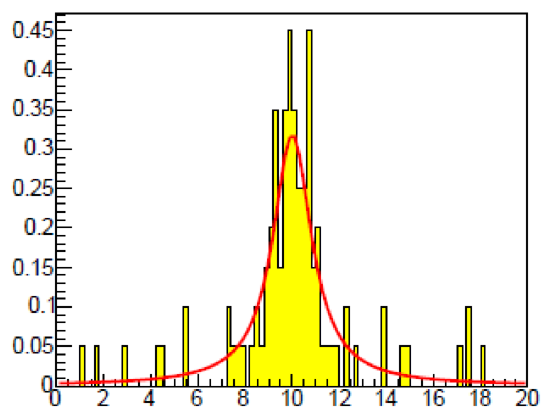




An experiment can perform a set of measurement

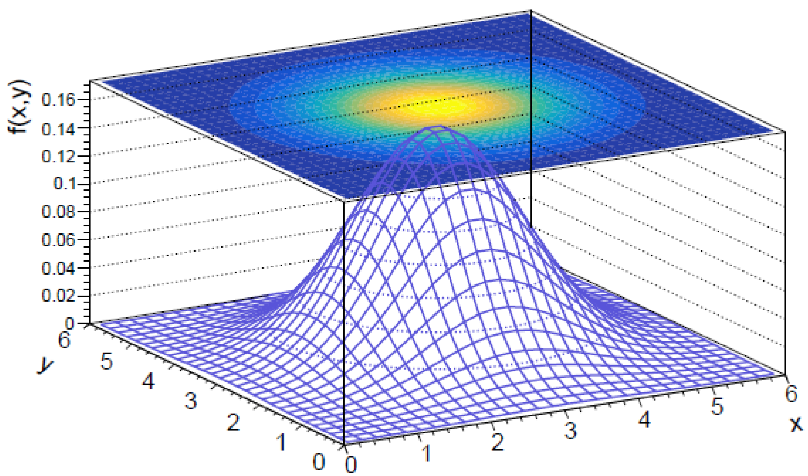
→ Vector of N measurements $\vec{x} = \{x_1, x_2, \ldots, x_N\}$

Probability of observing $\vec{x}$ in infinitesimal interval $\vec{x} + d\vec{x}$ given by joint p.d.f

$$f(\vec{x})d\vec{x} = f(x_1, \ldots, x_N)dx_1 \ldots dx_N$$

Ex: for a measurement of 2 values x and y

Probability of x in $[x, x+dx]$ and y in $[y, y+dy]$ is $f(x,y)dxdy$

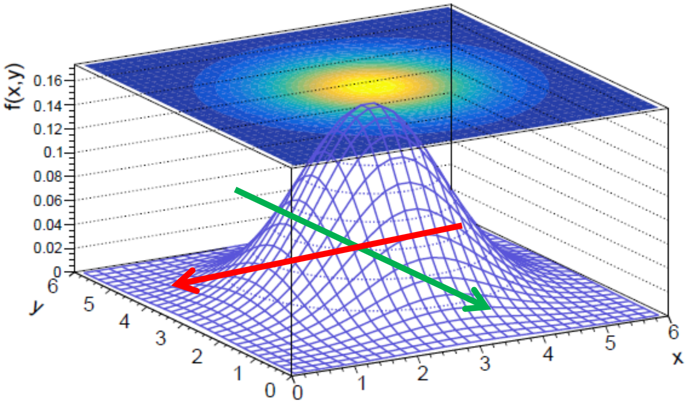


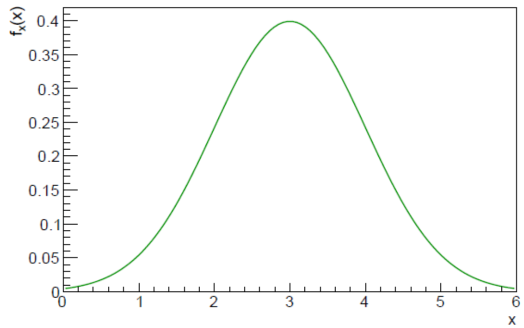
$$\iint_{\Omega} f(x,y)dxdy = 1$$

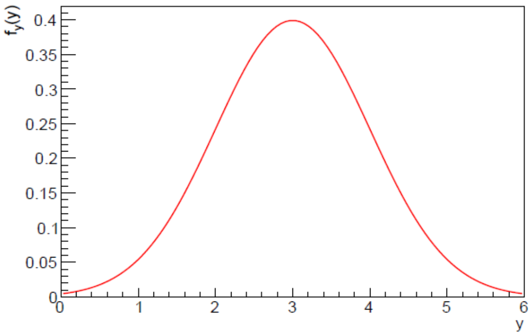
Marginal and conditional p.d.f

Marginal distribution: p.d.f of one variable regardless of the others

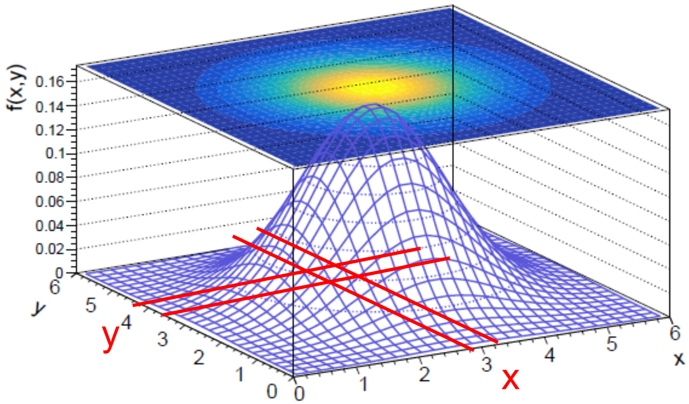
$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \qquad f_y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$







Conditional distribution: p.d.f of one variable given a constant other



$$k(y|x) = \frac{f(x,y)}{f_x(x)} = \frac{f(x,y)}{\int f(x,y')dy'}$$

$$g(x|y) = \frac{f(x,y)}{f_y(y)} = \frac{f(x,y)}{\int f(x',y)dx'}$$

Note: k and g are both functions of x and y

Marginal and conditional p.d.f

Bayes theorem for continuous variables

$$f(x,y) = g(x|y)f_y(y) = k(y|x)f_x(x) \rightarrow \boxed{g(x|y) = \frac{k(y|x)f_x(x)}{f_y(y)}}$$

Marginal p.d.f can also be expressed with conditional probabilities:

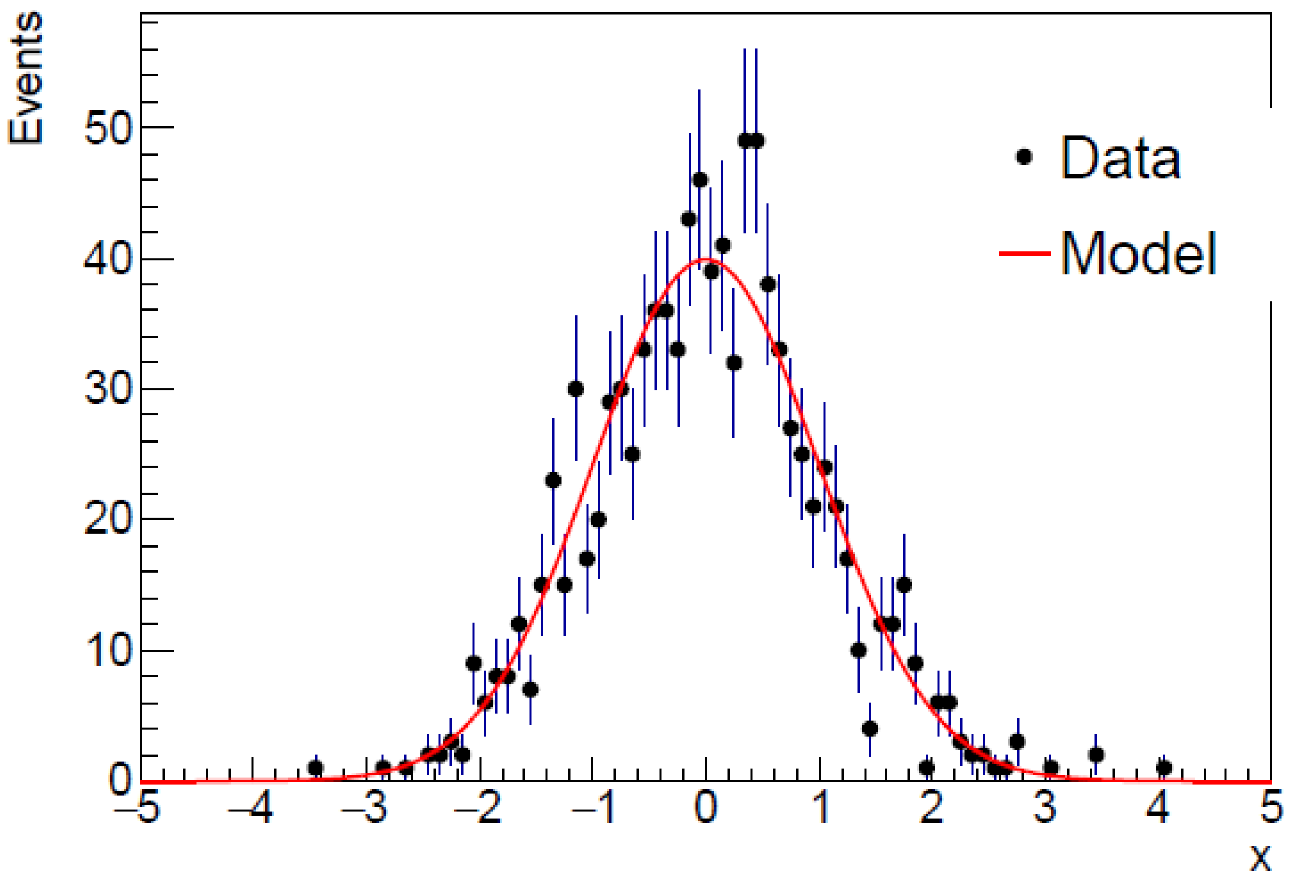
$$f_x(x) = \int_{-\infty}^{\infty} g(x|y)f_y(y)\,dy \qquad f_y(y) = \int_{-\infty}^{\infty} k(y|x)f_x(x)\,dx$$

Note: this is a generalization of the relation $P(B) = \sum_i P(B|A_i)P(A_i)$ to continuous variables

Independent variables: if x and y are independent $f(x,y) = f_y(y)f_x(x)$

Ex: 2D Gaussian function with uncorrelated variables

$$\mathrm{Gaus}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y}\exp\left(\frac{-(x-\mu_x)^2}{2\sigma_x{}^2}\right)\exp\left(\frac{-(y-\mu_y)^2}{2\sigma_y{}^2}\right)$$



What's wrong ?

Test hypothesis

Testing compatibility of observed data against a model

- **model** = background predictions (for simplicity)
  → n_b events: follows **Poisson** distribution of mean ν_b
  → n_{obs} **observed** events

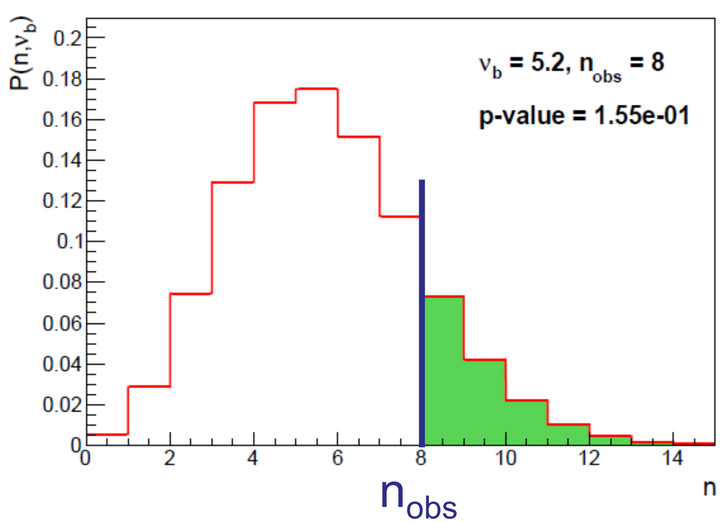
To quantify **degree of compatibility** of n_{obs} with the background-only hypothesis we calculate how likely it is to find n_{obs} or more events of background

p-value: probability that the expected number of event (background) is at least as high as the number of observed data

$$\boxed{\text{p-value} = P(n \geq n_{obs}) = 1 - P(n < n_{obs})}$$

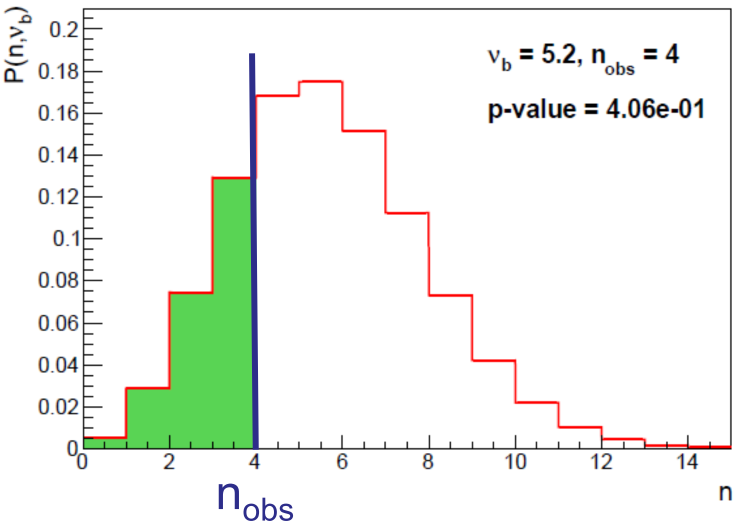
$$= \sum_{n=n_{obs}}^{+\infty} \frac{e^{-\nu_b}\nu_b^n}{n!} = 1 - \sum_{n=0}^{n_{obs}-1} \frac{e^{-\nu_b}\nu_b^n}{n!}$$

$[\text{for } \nu_b < n_{obs}]$



For the case where $\boldsymbol{\nu_b}$ **>** $\mathbf{n_{obs}}$ one can define:

$$\text{p-value} = \sum_{n=0}^{n_{obs}} \frac{e^{-\nu_b} \nu_b^n}{n!}$$



The previous sums can be **simplified** using incomplete **Gamma** functions:

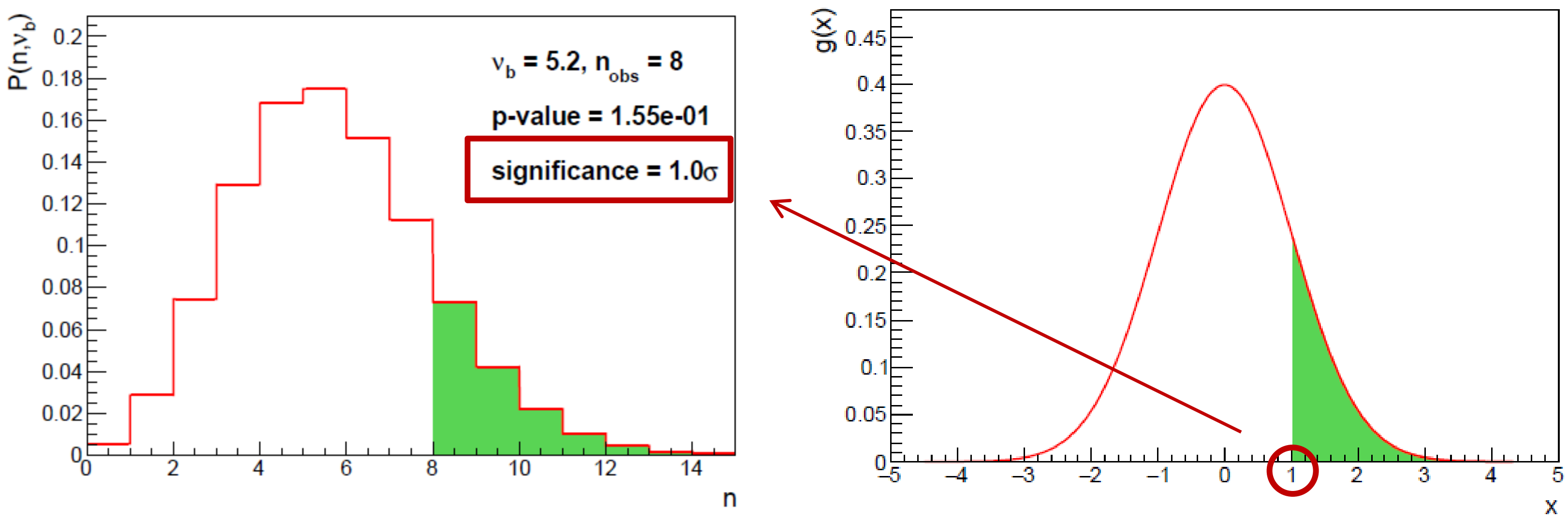
$$\boxed{\sum_{n=n_{obs}}^{+\infty} \frac{e^{-\nu_b} \nu_b^n}{n!} = \frac{1}{\Gamma(n_{obs})} \int_0^{\nu_b} t^{n_{obs}-1} e^{-t} dt = \Gamma(\nu_b, n_{obs})}$$

$$\text{with } \Gamma(n_{obs}) = \int_0^{\infty} t^{n_{obs}-1} e^{-t} dt = (n_{obs} - 1)! \quad (\text{if } n_{obs} \text{ integer})$$

It is customary to transform the p-value into a **Z-value** using the integral of the Gaussian distribution:

$$\int_{-\infty}^{Z} \mathrm{Gaus}(x, \mu = 0, \sigma = 1) dx = \int_{-\infty}^{Z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \; = 1 - \mathrm{pvalue}$$

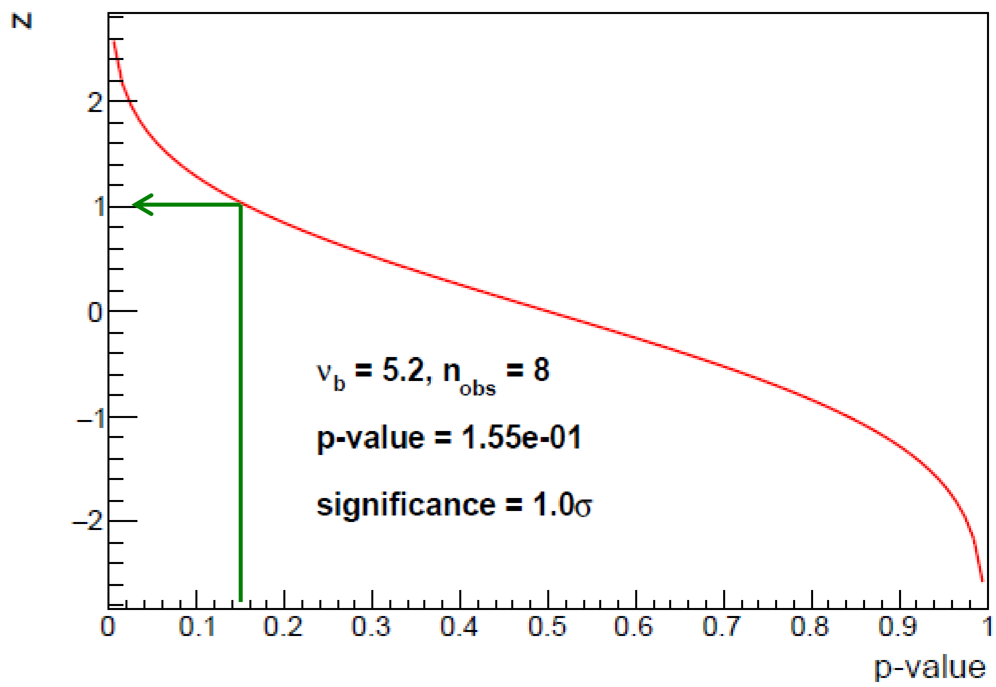
Z-value = number of standard deviation, used as a measure of the **significance** of an excess (or a deficit) w.r.t the (background) hypothesis.



Significance

In practice one uses the **inverse cumulative distribution function** of the Gaussian distribution to compute the significance:

$$Z = \sqrt{2}\mathrm{Erf}^{-1}(1 - 2 \times \mathrm{p{-}value})$$



p-value	Z
0.159	1σ
2.28×10^{-2}	2σ
1.35×10^{-3}	3σ
3.15×10^{-5}	4σ
2.85×10^{-7}	5σ