Aspects of conformal field theory from random loops

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Goal:

Making relations between CFT correlation functions in domain D

$$\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle_D$$

and CLE expectations in domain D

$$\mathbb{E}[X]_D$$

Difficulties:

How to go from the inherent nonlocality of CLE (loops) to locality of CFT (fields).

How to even define a field in CFT (how do we know we have constructed it).

I. Field theory of the stress-energy tensor

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Ward-Takahashi identities

Space-time symmetry transformation $x \mapsto g(x)$, action on fields $\mathcal{O}(x) \mapsto g[\mathcal{O}(x)]$:

$$\langle g[\mathcal{O}(x_1)] \cdots g[\mathcal{O}(x_n)]\rangle_{g(D)} = \langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n)\rangle_D$$

Continuous symmetry, infinitesimal (near identity): $g_\epsilon = \mathrm{id} + \epsilon\, f$

Infinitesimal transformation $g_\epsilon[\mathcal{O}(x)] = \mathcal{O}(x) + \epsilon \Delta\mathcal{O}(x) + O(\epsilon^2)$

Conserved Noether current $j^\mu(x)$:

$$\langle \partial_\mu j^\mu(x)\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle = -\mathrm{i}\sum_j \delta(x - x_j)\langle \mathcal{O}(x_1)\cdots\Delta\mathcal{O}(x_j)\cdots\mathcal{O}(x_n)\rangle$$

I. Field theory of the stress-energy tensor

Scale and Poincaré invariance

Translation invariance: generator $\Delta^\mu \mathcal{O}(x) = \partial^\mu \mathcal{O}(x)$

$\partial_\nu T^{\mu\nu}(x) = 0$ ($+$ contact terms)

Rotation invariance: scalar generator $\Delta \mathcal{O}(x) = \epsilon_{\mu\rho} x^\mu \partial^\rho \mathcal{O}(x)$

$\partial_\nu \left(\epsilon_{\mu\rho} x^\mu T^{\rho\nu}(x) \right) = 0 \;\Rightarrow\; T^{\mu\nu} = T^{\nu\mu}$ ($+$ contact terms)

Scale invariance: scalar generator $\Delta \mathcal{O}(x) = x_\mu \partial^\mu \mathcal{O}(x)$

$\partial_\nu \left(x_\mu T^{\mu\nu}(x) \right) = 0 \;\Rightarrow\; T^\mu_\mu = 0$ ($+$ contact terms)

I. Field theory of the stress-energy tensor

Holomorphicity

Use coordinates $z = \mathrm{x} + \mathrm{i}\tau$, $z = \mathrm{x} - \mathrm{i}\tau$ where x is space coordinate and τ is imaginary time. Then:

$$\partial_\nu T^{\mu\nu} = 0, \quad T^{\mu\nu} = T^{\nu\mu}, \quad T^\mu_\mu = 0$$

$$\Downarrow$$

$$T^{z\bar{z}} = T^{\bar{z}z} = 0, \quad \partial_z T^{zz} = 0, \quad \partial_{\bar{z}} T^{\bar{z}\bar{z}} = 0$$

Define $T \propto T^{\bar{z}\bar{z}}$ and $\bar{T} \propto T^{zz}$ (normalization constant: see next page). Then:

$$T = T(z) \quad \text{holomorphic}, \quad \bar{T} = \bar{T}(\bar{z}) \quad \text{anti-holomorphic}$$

I. Field theory of the stress-energy tensor

Contact terms: non-holomorphicity at fields' positions

By holomorphicity, Wilson's Operator Product Expansion must have the form

$$T(z)\mathcal{O}(x) = \sum_{n\in\mathbb{Z}} \frac{\mathcal{O}_n(x)}{(z-x)^n}, \qquad \mathcal{O}_n =: L_{n-2}\mathcal{O}$$

Take $g(z) = \lambda z$ (rotation + scaling). Assume field $\mathcal{O}$ transforms as

$$g[\mathcal{O}(z)] = \lambda^h \bar{\lambda}^{\tilde{h}} \mathcal{O}(\lambda z)$$

h, $\tilde{h}$ are holomorphic / anti-holomorphic dimensions of $\mathcal{O}$. Note: T has $h = 2$, $\tilde{h} = 0$. Use

$$\partial_{\bar{z}} \frac{1}{z} \propto \delta^{(2)}(z)$$

Ward-Takahishi identities are equivalent to **specifying certain terms** in the OPE:

$$T(z)\mathcal{O}(x) = \left(\cdots + \frac{h}{(z-x)^2} + \frac{\partial_x}{(z-x)} + \cdots \right) \mathcal{O}(x)$$

I. Field theory of the stress-energy tensor

Lower-boundedness of the set of dimensions and conformal Ward identities

Assume that the set of **holomorphic and anti-holomorphic dimensions** h and $\tilde{h}$ are bounded from below. Then OPE is bounded from below

$$T(z)\mathcal{O}(x) = \sum_{n\in\mathbb{Z},\ n>n_{\mathcal{O}}} \frac{L_{n-2}\mathcal{O}(x)}{(z-x)^n}.$$

Also, there must exist $\mathcal{O}$ (called primaries) such that

$$T(z)\mathcal{O}(x) = \left(\frac{h}{(z-x)^2} + \frac{\partial_x}{(z-x)} + \cdots\right)\mathcal{O}(x)$$

That is, $L_n\mathcal{O} = 0$ for all $n \geq 1$. (Note: set of all primaries and descendants under $T(z)$ (higher OPE coefficients) usually forms a closed OPE algebra).

I. Field theory of the stress-energy tensor

Consequences:

1. By Liouville's theorem and clustering $\langle T(z)\mathcal{O}(x_1)\cdots\rangle \to 0$ as $z \to \infty$, if all $\mathcal{O}$ in the correlation functions are primaries, we have the exact insertion of a holomorphic stress-energy tensor on the Riemann sphere $\hat{\mathbb{C}}$:

$$\langle T(z)\mathcal{O}(x_1)\cdots\rangle = \sum_j \left(\frac{h}{(z-x_j)^2} + \frac{\partial_{x_j}}{(z-x_j)}\right)\langle\mathcal{O}(x_1)\cdots\rangle$$

   Further, on $\mathbb{H}$, using Cardy's boundary conditions $T = \bar{T}$ (on $\mathbb{R}$) and conformal transformations of stress-energy tensor (see later), we also have exact expressions for $\langle T(z)\mathcal{O}(x_1)\ldots\rangle_D$ on any simply connected domain D.

2. There is **local conformal invariance**:

There exists Noether currents $j_n(z) := z^{n+1}T(z)$ for all $n \in \mathbb{Z}$ such that $\partial_{\bar{z}} j_n(z) = 0 +$ contact terms:

$$\partial_{\bar{z}}\Big(j_n(z)\mathcal{O}(0)\Big) \propto L_n\mathcal{O}(0)\,\delta^{(2)}(z)$$

This identifies infinitesimal symmetry transformation under $g(z) = z + \epsilon z^{n+1}$:

$$g[\mathcal{O}(0)] = \mathcal{O}(0) + \epsilon L_n\mathcal{O}(0) + \bar{\epsilon}\bar{L}_n\mathcal{O}(0) + \ldots$$

Can be exponentiated for primary fields: for any g conformal around z,

$$g[\mathcal{O}(z)] = (\partial g(z))^h\,(\bar{\partial}\bar{g}(\bar{z}))^{\tilde{h}}\mathcal{O}(g(z))$$

Extend symmetries to full groupoid of conformal maps $g : D \to g(D)$

Need regularity inside domain D: $j_n(z)$ is singular at ∞ if $n > 1$, at 0 if $n < -1$.
Regular on $\hat{\mathbb{C}}$ if and only if $n \in \{-1, 0, 1\} \Rightarrow$ Möbius transformations

I. Field theory of the stress-energy tensor

Transformation of the stress-energy tensor and the Schwarzian derivative

Assume lowest dimension is 0, and the only 0-dimensional fields are multiples of identity $\mathbb{C}\mathbf{1}$.

Then there must exist $c \in \mathbb{C}$ such that (using symmetry $x \leftrightarrow y$)

$$T(x)T(y) = \frac{c}{2}\frac{\mathbf{1}}{(x-y)^4} + \frac{2T(y)}{(x-y)^2} + \frac{\partial_y T(y)}{(x-y)} + \ldots.$$

That is: $L_n T = 0$ for all $n \geq 1$ except for $L_2 T = (c/2)\mathbf{1}$.

Hence we obtain $g[T(z)]$ for any infinitesimal $g(z) = z + \epsilon f(z)$ conformal around $z = 0$.

Exponentiating:

$$g[T(z)] = \frac{c}{12}\{g, z\}\mathbf{1} + (\partial g(z))^2 T(g(z))$$

where Schwarzian derivative is

$$\{g, z\} = \frac{g'''(z)}{g'(z)} - \frac{3}{2}\left(\frac{g''(z)}{g'(z)}\right)^2$$

II. The stress-energy tensor as a geometric singularity

II. The stress-energy tensor as a geometric singularity

Singular conformal transformation

Observe that

$$T = L_{-2}\mathbf{1}$$

Geometric interpretation: for $g(z) = z + \epsilon^2 z^{-1}$,

$$g[\mathbf{1}(0)] = \mathbf{1} + \epsilon^2 T(0) + \bar{\epsilon}^2 \bar{T}(0) + \dots$$

That is: $T(0)$ is result of **conformal transformation of identity field that is singular at 0**.

How to make sense of this?

Note that $g(z) = z + \epsilon^2 z^{-1}$ is conformal on $\hat{\mathbb{C}} \setminus \epsilon\mathbb{D}$.

Hence interpret $g[\mathbf{1}(0)]$ as making a hole $\epsilon\mathbb{D}$ that we deform as $g(\hat{\mathbb{C}} \setminus \epsilon\mathbb{D})$.

II. The stress-energy tensor as a geometric singularity

Use contour integral

$$\lim_{|\epsilon|\to 0} \oint \frac{\mathrm{d}\epsilon}{2\pi \mathrm{i} \epsilon^3}$$

in order to extract leading holomorphic part of $1 + \epsilon^2 T(0) + \bar{\epsilon}^2 \bar{T}(0) + \ldots$

$$\begin{aligned}
\langle T(0)\, \mathcal{O}(x_1) \cdots \rangle_{\hat{\mathbb{C}}} &= \lim_{|\epsilon|\to 0} \oint \frac{\mathrm{d}\epsilon}{2\pi \mathrm{i} \epsilon^3} \langle \mathcal{O}(x_1) \cdots \rangle_{g(\hat{\mathbb{C}} \setminus \epsilon \mathbb{D})} \\
&= \lim_{|\epsilon|\to 0} \oint \frac{\mathrm{d}\epsilon}{2\pi \mathrm{i} \epsilon^3} \langle g^{-1}[\mathcal{O}(x_1)] \cdots \rangle_{\hat{\mathbb{C}} \setminus \epsilon \mathbb{D}} \\
&= \lim_{|\epsilon|\to 0} \oint \frac{\mathrm{d}\epsilon}{2\pi \mathrm{i} \epsilon^3} \langle g^{-1}[\mathcal{O}(x_1)] \cdots \rangle_{\hat{\mathbb{C}}}
\end{aligned}$$

Check: for primary fields using $g^{-1}[\mathcal{O}(x)] = (\partial g^{-1}(x))^h (\bar{\partial} \bar{g}^{-1}(\bar{x}))^{\tilde{h}} \mathcal{O}(g^{-1}(x))$:

$$= \sum_j \left(\frac{h}{(0 - x_j)^2} + \frac{\partial_{x_j}}{(0 - x_j)} \right) \langle \mathcal{O}(x_1) \cdots \rangle_{\hat{\mathbb{C}}}$$

II. The stress-energy tensor as a geometric singularity

Stress-energy tensor from a rotating elliptical hole

May as well make hole $b\epsilon\mathbb{D}$ for any $b > 1$. Thus we have established

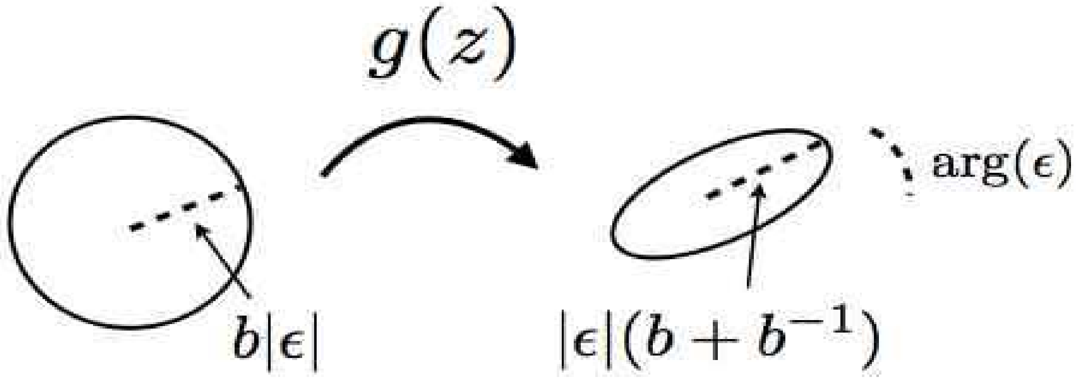
$$\langle\, T(0)\, \mathcal{O}(x_1) \cdots \rangle_{\hat{\mathbb{C}}} = \lim_{|\epsilon|\to 0} \oint \frac{\mathrm{d}\epsilon}{2\pi \mathrm{i}\epsilon^3} \langle \mathcal{O}(x_1) \cdots \rangle_{g(\hat{\mathbb{C}}\setminus b\epsilon\mathbb{D})}$$

Observe that

$$g(\hat{\mathbb{C}} \setminus b\epsilon\mathbb{D}) = \hat{\mathbb{C}} \setminus E(\epsilon, b)$$

$E(\epsilon, b) =$ **elliptical domain centered at 0 of major semi-axis $|\epsilon|(b + 1/b)$**

and angle $\arg(\epsilon)$ wrt to positive real axis



II. The stress-energy tensor as a geometric singularity

Therefore:

$$\langle T(0)\,\mathcal{O}(x_1)\cdots\rangle_{\hat{\mathbb{C}}} = \lim_{|\epsilon|\to 0}\oint \frac{\mathrm{d}\epsilon}{2\pi \mathrm{i}\epsilon^3}\langle \mathcal{O}(x_1)\cdots\rangle_{\hat{\mathbb{C}}\setminus E(\epsilon,b)}$$

II. The stress-energy tensor as a geometric singularity

Conformal Ward identities from derivatives with respect to conformal maps

Thus we have established, for $g(z) = z + \epsilon^2 z^{-1}$,

$$\langle\, T(0)\, \mathcal{O}(x_1) \cdots \rangle_{\hat{\mathbb{C}}} = \lim_{|\epsilon|\to 0} \oint \frac{\mathrm{d}\epsilon}{2\pi \mathrm{i} \epsilon^3} \langle g^{-1}[\mathcal{O}(x_1)] \cdots \rangle_{\hat{\mathbb{C}}}$$

Consider some complex function $F(g)$ on a space of conformal maps $g : A \to g(A)$ near the identity. Define derivatives as follows: for h holomorphic on a neighborhood of the closure of A,

$$(\nabla_h F)(\mathrm{id}) = \frac{\mathrm{d}}{\mathrm{d}t} F(\mathrm{id} + th)|_{t=0}, \quad \Delta_h = \frac{1}{2}(\nabla_h - \mathrm{i}\nabla_{\mathrm{i}h})$$

One can show that

$$(\Delta_h F)(\mathrm{id}) = \lim_{|\epsilon|\to 0} \oint \frac{\mathrm{d}\epsilon}{2\pi \mathrm{i} \epsilon^3} F(\mathrm{id} + \epsilon^2 h)$$

II. The stress-energy tensor as a geometric singularity

Now choose $A \supset \{x_i\}_i$ and define (for fixed $x_i \neq 0\ \forall i$)

$$F : g \mapsto \langle g[\mathcal{O}(x_1)] \cdots g[\mathcal{O}(x_n)] \rangle_{\hat{\mathbb{C}}}$$

Let

$$h_n(z) = (-z)^{n+1}.$$

(Infinitesimal) Möbius invariance:

$$(\Delta_{h_n} F)(\mathrm{id}) = 0 \quad \text{for } n = -1, 0, 1.$$

Conformal Ward identities in terms of “conformal derivatives”:

$$\langle\, T(0)\, \mathcal{O}(x_1) \cdots \rangle_{\hat{\mathbb{C}}} = (\Delta_{h_{-2}} F)(\mathrm{id}).$$

II. The stress-energy tensor as a geometric singularity

Conformal Ward identities on simply connected domains

Now let D be a simply connected (say Jordan) domain. For any conformal map g on ∂D, bijective on a neighborhood of ∂D, define

$$g^\sharp(\partial D) = D' \quad : \quad \begin{array}{l} D' \text{ simply connected} \\ \partial D' = g(\partial D) \\ g(z) \in D' \;\forall\; z \in D \text{ near enough to } \partial D \end{array}$$

If g is conformal on D, then $g^\sharp(D) = g(D)$.
Let $A \supset \partial D \cup \{x_i\}_i$ and define (for fixed x_is)

$$F : g \mapsto \langle g[\mathcal{O}(x_1)] \cdots g[\mathcal{O}(x_n)]\rangle_{g^\sharp(D)}$$

Then (say $0 \in D$)

$$\langle\, T(0)\, \mathcal{O}(x_1) \cdots \rangle_D - \langle T(0)\rangle_D \,\langle \mathcal{O}(x_1) \cdots \rangle_D = (\Delta_{h_{-2}} F)(\mathrm{id}).$$

Also by conformal covariance $\langle T(0)\rangle_D = (c/12)\{g, 0\}, \quad g : D \to \mathbb{D}$.

III. The stress-energy tensor in CLE

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We have found two things that may be transferred to CLE:

$T(0) =$ insertion of small spin-2 rotating elliptical hole centered at 0

and

Conformal Ward identities $=$ identification of insertion of $T(0)$ with conformal derivative

Here is how we transfer these concepts.

III. The stress-energy tensor in CLE

CLE conformal Ward identities

Consider CLE on a simply connected domain D, random variables X supported on some subset $\mathrm{supp}(X) \subset D$ and expectations $\mathbb{E}[X]_D$. Conformal invariance: for any g conformal on $\mathrm{supp}(X)$ there is an action $X \mapsto g[X]$, and for any g conformal on D we have

$$\mathbb{E}[g[X]]_{g(D)} = \mathbb{E}[X]_D.$$

Define

$$F : g \mapsto \mathbb{E}[g[X]]_{g^\sharp(D)}.$$

Then the right-hand side of the conformal Ward identities is

$$(\Delta_{h_{-2}} F)(\mathrm{id})$$

III. The stress-energy tensor in CLE

CLE stress-energy tensor

Consider elliptical domain $E(\epsilon, b)$. For small enough $\eta > 0$ consider indicator variable

$\mathrm{T}_{\epsilon,\eta}$ = indicator for event that at least one CLE loop winds around the annular domain

$$E(\epsilon, b) \setminus E(\epsilon, (1-\eta)b)$$

The following limit exists:

$$\mathrm{T}_\epsilon = \lim_{\eta \to 0} \frac{\mathrm{T}(\epsilon, \eta)}{\mathbb{E}[\mathrm{T}_{\epsilon,\eta}]_{\hat{\mathbb{C}}}}.$$

By restriction property, **this separates inside from outside of elliptical domain** $E(\epsilon, b)$.

Then define the **spin-2 rotating-ellipse variable**

$$\mathrm{T} = \lim_{|\epsilon| \to 0} \oint \frac{\mathrm{d}\epsilon}{2\pi \mathrm{i} \epsilon^3} \mathrm{T}_\epsilon$$

By translation, similarly define $\mathrm{T}(z)$. This a local at z (supported on z).

III. The stress-energy tensor in CLE

Results [BD 2013]

Using CLE conformal restriction and conformal invariance, but under some yet-unproven assumptions (existence of conformal derivatives and of certain limits).

- For D simply connected and X supported in D,

$$\mathbb{E}[\mathrm{T}(z)X]_D \text{ is holomorphic on } z \in D \setminus \mathrm{supp}(X)$$

- Conformal Ward identities (say with $z = 0$)

$$\mathbb{E}[\mathrm{T}(0)X]_D - \mathbb{E}[T(0)]_D\,\mathbb{E}[X]_D = (\Delta_{h_{-2}}F)(\mathrm{id}), \quad F : g \mapsto \mathbb{E}[g[X]]_{g^\sharp(D)}$$

III. The stress-energy tensor in CLE

- Transformation property and one-point function:

  CLE probabilities are conformally invariant under action

$$g[\mathrm{T}(z)] = \frac{c}{12}\{g,z\} + (\partial g(z))^2 \mathrm{T}(z)$$

  In particular

$$\mathbb{E}[T(z)]_D = \frac{c}{12}\{g,z\}, \quad g : D \to \mathbb{D}$$

IV. Some (formal) consequences

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Universality: any spin-2 local variable that transforms like the stress-energy tensor satisfies conformal Ward identities

Assume $\mathrm{T}'(z)$ transforms like the stress-energy tensor. Since it is spin-2, $\mathbb{E}[\mathrm{T}'(0)]_{\mathbb{D}} = 0$. Hence by covariance, $\mathbb{E}[\mathrm{T}'(z)]_D = \mathbb{E}[\mathrm{T}(z)]_D$ for any simply connected D and $z \in D$. Therefore, for $z \cap \mathrm{supp}(X) = \emptyset$,

$$
\begin{aligned}
\mathbb{E}[\mathrm{T}'(z)X]_D &= \int d\gamma\, \mathbb{E}[\mathrm{T}'(z)X|\gamma] \\
&= \int d\gamma\, \mathbb{E}[X|\gamma]\, \mathbb{E}[\mathrm{T}'(z)]_{D_\gamma} \\
&= \int d\gamma\, \mathbb{E}[X|\gamma]\, \mathbb{E}[\mathrm{T}(z)]_{D_\gamma} \\
&= \mathbb{E}[\mathrm{T}(z)X]_D
\end{aligned}
$$

IV. Some (formal) consequences

Null-vectors: CLE expectations of a spin-0 local variable that possesses a null Virasoro descendant satisfy CFT null-vector equations

Say the local CLE variable $\mathcal{O}$ has a null Virasoro descendant. For instance at level 2, $\varphi = (L_{-2} + aL_{-1}^2)\,\mathcal{O}$,

$$L_n \varphi = 0 \quad \forall n \geq 1$$

Since $\mathrm{T}(z)$ is local (supported at z), then $\varphi(z)$ also is local.

Since $L_n\varphi = 0\ \forall n \geq 1$, then

$$g[\varphi(z)] = (\partial g(z))^{h+2}(\bar{\partial}\bar{g}(\bar{z}))^h \varphi(g(z))$$

for some h. Therefore $\mathbb{E}[\varphi(z)]_D = 0$ for every simply connected D.

Hence by similar restriction arguments:

$$\mathbb{E}[\varphi(z)X]_D = 0$$

for all $z \in D \setminus \mathrm{supp}(X)$.

Conclusions

This can be generalized to all stress-energy tensor descendants (with $\Delta_{h_{-n}}$) and many insertions of stress tensors (higher powers of conformal derivatives), and one recover the full **Virasoro vertex operator algebra**.

Generalization to other symmetry currents?

Proof of assumptions / rigorization of formal arguments? Kemppainen and Werner (2014) proved important $z \mapsto 1/z$ conformal invariance of CLE on $\hat{\mathbb{C}}$.