Uses of Dyson-Schwinger Equations

September 16, 2016

These are lecture notes in progress: please contribute to them by sending typos or asking questions or comments. Thanks to Mihai Nica for the first version of the notes.

1 Lecture 1

1.1 Introduction

In order to calculate the magnetic momentum of an electron Feynman used diagrama and Schwinger used Green's functions. Dyson unified these two approaches. A tool is to consider Dyson-Schwinger equations. On one hand they can be thought as equations for the generating functions of the graphs that are enumerated, on the other they can be seen as an equation for the invariance of the underlying measure. A baby version of these is the charaterization of the Gaussian law $X \sim \mathcal{N}(0,1)$:

Combinatorial view point:

$$\mathbf{E}\left[X^n\right] = \#\left\{\text{pair partitions of } n \text{ points}\right\}$$

vs. invariance of the Lebesgue measure under shift, that is integration by parts:

$$\mathbf{E}\left[X f(X)\right] = \mathbf{E}\left[f'(X)\right]$$

If one applies the latter to $f(x) = x^n$ one gets

$$m_{n+1} := \mathbf{E}\left[X^{n+1}\right] = \mathbf{E}\left[nX^{n-1}\right] = nm_{n-1}$$

This last bit can also be understood as the induction relation for the number C_n of pair partitions of n points by thinking of the n ways to pair the first particle.

1.2 GUE topological expansion example

Let X be a GUE matrix, $X_{ij} = X_{ij}^{\mathbb{R}} + iX_{ij}^{i\mathbb{R}}$, each $\mathcal{N}\left(0, \frac{1}{2N}\right)$ and $X_{ii} = X_{ii}^{\mathbb{R}} \sim \mathcal{N}\left(0, \frac{1}{N}\right)$ Then the "topological expansion" is

$$\mathbf{E}\left[\frac{1}{N}\mathrm{Tr}\left[X^k\right]\right] = \sum_{g\geq 0} \frac{1}{N^{2g}} M_g(k)$$

where $M_g(k)$ is the number of rooted maps of gneus g build over a vertex of degree k. And a "map" is a connected graph properly embedded in a surface. By Euler's formula $2 - 2g = \#Vertices + \#Faces - \#Edges$, and a "root" is a distinguished edge.

Similarly, let: $W_c\left(k_1, \dots, k_p\right) = \partial_{\lambda_1} \dots \partial_{\lambda_p} \frac{1}{N^2} \log \mathbf{E}\left[\exp\left(\sum_{i=1}^k \lambda N \mathrm{Tr} X^{k_i}\right)\right] |_{\lambda=0}$ then:

$$W_c\left(k_1, \dots, k_p\right) = \sum_{g\geq 0} \frac{1}{N^{2g}} M_g\left(k_1, \dots, k_p\right)$$

where $M_g(k_1, \dots, k_p)$ is the number of maps of genus g and 1 vertex of degree k_i. This can derived using combinatorial arguments and Wick calculus to compute Gausian moments, or by the Dyson-Schwinger (DS) equation, which we do below.

1.3 Dyson-Schwinger Eqn

Let:

$$Y_k := \mathrm{Tr} X^k - \mathbf{E}\mathrm{Tr} X^k$$

Let us try to compute:

$$\mathbf{E}\left[\mathrm{Tr} X^{k_1} \prod_{i=2}^{p} Y_{k_i}\right].$$

By integration by parts, one can compute the recurence type relation

$$\mathbf{E}\left[\mathrm{Tr} X^{k_1} \prod_{i=2}^{p} Y_{k_i}\right] = \mathbf{E}\left[\frac{1}{N}\sum_{\ell=0}^{k_1-2} \mathrm{Tr} X^{\ell} \mathrm{Tr} X^{k_1-2-p} \prod_{i=2}^{p} Y_i\right] + \mathbf{E}\left[\sum_{i=2}^{p} \frac{k}{N} \mathrm{Tr} X^{k_1+k_i-2} \prod_{j=2, i\neq j}^{p} Y_j\right] \tag{1}$$

The proof of this fact goes by expanding the first term, $\mathrm{Tr} X^{k_1}$, in terms of the matrix elements and using **Gaussian integration by parts.** This is left as an exercise.

1.4 Plan of next lectures

1. Show the topological expansion for GUE can be derived from the DS eqn and Get a CLT for the centered moments $\left\{\mathrm{Tr} X^{k_i} - \mathbf{E}\mathrm{Tr} X^{k_i}\right\}$

2. Generalization of these to other β ensembles (based on joint work with Borot: http://arxiv.org/abs/1303.1045)

3. Discrete β-ensembles e.g. lozenge tilings (based on joint work with Borodin and Gorin, http://arxiv.org/abs/1505.03760)

1.5 DS equation implies genus expansion

We will show that the DS equation (1) can be used to show that:

$$\mathbf{E}\left[\frac{1}{N}\mathrm{Tr}X^k\right] = M_0(k) + \frac{1}{N^2}M_1(k) + \ldots$$

Next orders can be derived similarly.

Let:

$$m_k^N := \mathbf{E}\left[\frac{1}{N}\mathrm{Tr}X^k\right]$$

By the DS equation (with no Y terms), we have that:

$$m_k^N = \mathbf{E}\left[\sum_{\ell=0}^{k}\frac{1}{N}\mathrm{Tr}X^\ell\frac{1}{N}\mathrm{Tr}X^{k-\ell-2}\right]$$

(the second term in the DS equation is 0 here.) We now assume that we have the self-averaging property that:

$$\frac{1}{N}\mathrm{Tr}X^\ell = \mathbf{E}\left[\frac{1}{N}\mathrm{Tr}X^\ell\right] + o(1) = m_\ell^N + o(1)$$

as $N \to \infty$ (We will show this self-averaging is true later). If this is true, then the above expansion would give us:

$$m_k^N = \sum_{\ell=0}^{k-2} m_\ell^N m_{k-\ell-2}^N + o(1)$$

On the other hand, let $M_0(k)$ be the number of maps of genus 0 with k vertices. These satisfy the Catalan recurrence:

$$M_0(k) = \sum_{\ell=0}^{k-2} M_0(\ell)M_0(k-\ell-2)$$

This recurrence is shown by a Catalan-like recursion argument, which goes by dividing each map of genus 0 into two submaps (both still of genus 0) of size ℓ and $k-\ell-2$.

Since these both satisfy the same recurrence (and $M_0(0) = m_0^N = 1, M_0(1) = m_1^N = 0$), we can prove by induction (assuming the self-averaging works) that:

$$m_k^N = M_0(k) + o(1) \text{ as } N \to \infty$$

It remains to prove the self-averaging.

1.5.1 Self-Averaging

Claim There exists finite constants $C.$ and $E.$ so that for every k, if $k_1, \ldots, k_\ell$ are integers so that $\sum_{i=1}^{\ell} k_i \leq k$ then:

$$a) \quad c(k_1, \ldots, k_p) := \mathbf{E}\left[\prod_{i=1}^{\ell} Y_{k_i}\right] \text{ satisfies } |c(k_1, \ldots, k_p)| \leq C_{\sum k_i}$$

and:

$$b) \quad m_\ell^N := \mathbf{E}\left[\frac{1}{N}\mathrm{Tr}X^\ell\right] \text{ satisfies } |m_\ell^N| \leq E_\ell \text{ for all } \ell \leq k$$

The proof is by induction on k. It is clearly true for $k = 0, 1$. Suppose the induction hypothesis holds for $k-1$. To see that $b)$ holds, consider that by the DS equation, we first observe that:

$$\begin{aligned}
\mathbf{E}\left[\frac{1}{N}\mathrm{Tr}X^k\right] &= \mathbf{E}\left[\sum_{\ell \geq 2} \frac{1}{N}\mathrm{Tr}X^\ell \frac{1}{N}\mathrm{Tr}X^{k-\ell-2}\right] \\
&= \sum_{\ell=0}^{k-2}(m_\ell^N m_{k-\ell-2}^N + c(\ell, k-\ell-2)) \leq \sum (E_\ell E_{k-2-\ell} + C_{k-2}) := E_k
\end{aligned}$$

where:

$$c(k,\ell) = \mathbf{E}\left[\left(\mathrm{Tr}X^k - \mathbf{E}\mathrm{Tr}^k\right)\left(\mathrm{Tr}X^\ell - \mathbf{E}\mathrm{Tr}X^\ell\right)\right]$$

To see that $a)$ holds, consider as follows

$$\begin{aligned}
\mathbf{E}\left[Y_{k_1}\prod_{j=2}^{p} Y_{k_j}\right] &= \mathbf{E}\left[\frac{1}{N}\mathrm{Tr}X_{k_1}\prod_{j=2}^{p} Y_{k_j}\right] - \mathbf{E}\left[\frac{1}{N}\mathrm{Tr}X_{k_1}\right]\mathbf{E}\left[\prod_{j=2}^{p} Y_{k_j}\right] \\
&= \mathbf{E}\left[\frac{1}{N}\sum_{\ell}\mathrm{Tr}X^\ell \mathrm{Tr}X^{k_1-\ell-2}\prod_{j=2}^{p} Y_{k_j}\right] \\
&\quad + \mathbf{E}\left[\sum \frac{k_i}{N}\mathrm{Tr}X^{k_1+k_i-2}\prod_{j=2, j\neq i}^{p} Y_{k_j}\right] \\
&\quad - \mathbf{E}\left[\frac{1}{N}\sum_{\ell}\mathrm{Tr}X^\ell \mathrm{Tr}X^{k_1-\ell-2}\right]\mathbf{E}\left[\prod_{j=2}^{p} Y_{k_j}\right]
\end{aligned}$$

where we just used the DS equation. We next substract the last term to the first and observe that

$$\begin{aligned}
&\mathrm{Tr}X^\ell \mathrm{Tr}X^{k_1-\ell-2} - \mathbf{E}[\mathrm{Tr}X^\ell \mathrm{Tr}X^{k_1-\ell-2}] \\
=\ & NY_\ell m_{k-2-\ell}^N + NY_{k-2-\ell} m_\ell^N + c(\ell, k-2-\ell)
\end{aligned}$$

to deduce

$$
\begin{aligned}
\mathbf{E}\left[Y_{k_1}\prod_{j=2}^{p}Y_{k_j}\right] &= 2\sum_{\ell} m_\ell^N c(k-2-\ell,k_2,\ldots,k_p)\\
&+\frac{1}{N}\sum c(\ell,k-2-\ell)c(k_2,\ldots,k_p)\\
&+\sum_{i=2}^{p} k_i m^N_{k_1+k_i-2}c(k_2,..,k_{i-1},k_{i+1},.,k_p)\\
&+\frac{1}{N}\sum_{i=2}^{p} k_i c(k_1+k_i-2,k_2,..,k_{i-1},k_{i+1},.,k_p) \qquad (2)
\end{aligned}
$$

which is bounded uniformly by induction.

Remark. You can understand the same thing from the Feynman picture: the first term is somehow the interaction to other pieces and the second term is the interaction wtih "itself".

1.5.2 Second Order

The above self averaging properties prove that $m_k^N = C(k) + o(1)$. To get the next order correction you have to analyze the **limiting covariance** of the terms:

$$c(k,\ell) = \mathbf{E}\left[\left(\mathrm{Tr}X^k - \mathbf{E}\mathrm{Tr}^k\right)\left(\mathrm{Tr}X^\ell - \mathbf{E}\mathrm{Tr}X^\ell\right)\right]$$

we will show that $c(k,\ell)$ converges towards:

$M_0(k,\ell) = \#\{\text{planar maps with 1 vertex of degree } \ell \text{ and one vertex of degree } k\}$

If we can show this, then we will have:

Corollary. *$N^2(m_k^N - C(k)) = c_k^1 + o(1)$ where c_k^1 are defined recusivly:*

$$c_k^1 = 2\sum_{\ell=0}^{k-2} c_\ell^1 M_0\,(k-\ell-2) + \sum_{\ell=0}^{k-2} M_0(\ell, k-\ell-2)$$

Proof. Again we prove the result by induction over k. It is fine for $k = 0,1$ where $c_k^1 = 0$. By (2) with $p = 0$ as well as the a priori bounds on the moments (a) we proved by induction we have :

$$
\begin{aligned}
N^2(m_k^N - C(k)) &= \sum M_0(\ell)N^2\left(m^N_{k-\ell-2} - M_0(k-2-\ell)\right)\\
&+\sum N^2\left(m_\ell^N - M_0(\ell)\right)\left(m_{k-\ell-2} - M_0(k-2-\ell)\right)\\
&+\sum c(\ell,k-\ell-2)\\
&+N^2\sum_{\ell}(m_\ell^N - M_0(\ell))(m^N_{k-2-\ell} - M_0(k-2-\ell))
\end{aligned}
$$

from which the result follows by taking the large N limit on the right hand side. ∎

Exercise. Show that $c_k^1 = M_1(k)$ is the number of planar maps of genus 1.

Remark. The proof goes again by showing $M_1(k)$ satisfies the same type of recurrence as c_k^1 by considering the matching of the root: either it cuts the map of genus 1 into a map of genus 1 and a map of genus 0, or there remains a (connected) planar maps.

Proof. (Of the $c(k,\ell)$ convergence thm) Observe that $c(k,\ell)$ converges for $K = k+\ell \leq 1$ and assume you have proven convergence towards $\sigma(k,\ell)$ up to K. Take $k+\ell = K+1$ and use (2) with $p=1$ to deduce that $c(k,\ell)$ converges towards $\sigma(k,\ell)$ which is given by the induction relation

$$\sigma(k,\ell) \quad = \quad 2\sum_{p=0}^{\ell-2} M_0(p)\sigma(k-2-p,\ell) + \ell M_0(k+\ell-2)$$

You then can check that $\sigma(k,\ell) = M_0(k,\ell)$ as they satisfy the same induction relation (check it by pairing the root).

□

2 Second Lecture

Last time X was a GUE matrix. We saw that:

$$\mathbf{E}\left[\frac{1}{N}\mathrm{Tr}X^k\right] = M_0(k) + \frac{1}{N^2}M_1(k) + o\left(\frac{1}{N^2}\right)$$

and:

$$\mathbf{E}\left[Y_k Y_\ell\right] = M_0\left(\ell, k\right) + o(1)$$

where:

$$Y_k := \mathrm{Tr}X^k - \mathbf{E}\mathrm{Tr}X^k$$

and where M_0 is the number of planar maps, M_1 is the number of maps of genus 1 and $M_0(\ell,k)$ is the number of planar maps with one vertex of degree ℓ and another vertex of degree k.

Remark. Have that if $\lambda_1,\ldots,\lambda_N$ are the eigenvalues then:

$$\mathbf{E}\left[\frac{1}{N}\sum_{i=1}^{N}\lambda_i^k\right] = M_0(k) + o(1)$$

$$\mathbf{E}\left[(\sum \lambda_i^k - \mathbf{E}\left(\sum \lambda_i^k\right))^2\right] = O(1)$$

This result is enough to conclude by Borel Cantelli Lemma that almost surely:

$$\mu = \frac{1}{N}\sum \delta_{\lambda_i} \to \sigma(\mathrm{d}x)$$

where σ is the semi-circle law : $\sigma(x^n) = M_0(n)$. Indeed, the almost sure convergence for moments follows from the summability of

$$P\left(\left|\sum \lambda_i^k - \mathbf{E}\left(\sum \lambda_i^k\right)\right| \geq N\varepsilon\right) \leq \frac{1}{\varepsilon^2 N^2},$$

Borel-Cantelli Lemma and convergence of m_k^N towards $\sigma(x^n)$. The convergence of $\frac{1}{N}\sum_{i=1}^N f(\lambda_i)$ follows then since polynomials are dense in the set of continuous functions on $[-\sqrt{2}, \sqrt{2}]$.

2.1 CLT for Y_k

Let $c(k_1, \ldots, k_p) = \mathbf{E}\left[Y_{k_1} \cdots Y_{k_p}\right]$ then we claim that as $N \to \infty$ to $G(k_1, \ldots k_p)$ given by:

$$(\dagger) \quad G(k_1, \ldots, k_p) = \sum_{i=2}^{k} M_0(k_1, k_i) G(k_2, \ldots, \hat{k}_i, \ldots, k_p)$$

where $\hat{}$ is the absentee hat. This type of moment convergence is equivalent to a Wick formula and is enough to prove (by the moment method) that $Y_{k_1}, \ldots, Y_{k_p}$ are jointly Gaussian. We will prove this by induction by using the DS equations. Now assume that $\dagger$ holds for any $k_1, \ldots, k_p$ such that $\sum_{i=1}^p k_i \leq k$. (induction hypothesis) We have by (4). Notice by the a priori bound on correlators (a) that the terms with a $1/N$ are neglectable in the right hand side and m_k^N close to $M_0(k)$, yielding

$$\begin{aligned}
\mathbf{E}\left[Y_{k_1} \prod_{j=2}^{p} Y_{k_j}\right] &= 2\sum_{\ell} m_\ell^N c(k-2-\ell, k_2, \ldots, k_p) \\
&\quad + \frac{1}{N}\sum c(\ell, k-2-\ell) c(k_2, \ldots, k_p) \\
&\quad + \sum_{i=2}^{p} k_i m_{k_1+k_i-2}^N c(k_2, .., k_{i-1}, k_{i+1}, ., k_p) \\
&\quad + \frac{1}{N}\sum_{i=2}^{p} k_i c(k_1+k_i-2, k_2, .., k_{i-1}, k_{i+1}, ., k_p) \\
&= 2\sum_{\ell} M_0(\ell) c(k-2-\ell, k_2, \ldots, k_p) \\
&\quad + \sum_{i=2}^{p} k_i M_0(k_1+k_i-2) c(k_2, .., k_{i-1}, k_{i+1}, ., k_p) + O(\frac{1}{N})
\end{aligned}$$

By using the induction hypothesis, this gives rise to:

$$\begin{aligned}
\mathbf{E}\left[\prod_{i=1}^{p} Y_{k_i}\right] &= o(1) + 2\sum M_0(\ell) G(k_1-\ell-2, k_2, \ldots, k_p) \\
&\quad + \sum k_i M_0(k_i+k_j-2) G(k_2, \ldots, \hat{k}_i, \ldots k_p)
\end{aligned}$$

It follows that

$$G(k_1,\dots,k_p) = 2\sum M_0(\ell)G(k_1-\ell-2,k_2,\dots,k_p)+\sum k_i M_0(k_i+k_j-2)G(k_2,\dots,\hat{k}_i,\dots k_p)$$

But using the induction hypothesis, we get

$$G(k_1,\dots,k_p) = \sum_{i=2}^{p}(2\sum M_0(\ell)M(k_1-\ell-2,k_i)+k_i M_0(k_i+k_j-2))G(k_2,\dots,\hat{k}_i,\dots k_p)$$

and recalling that

$$M_0(k_1,k_i) = 2\sum M_0(\ell)M(k_1-\ell-2,k_i)+k_i M_0(k_1+k_i-2)$$

proves the induction.

2.2 β-ensembles

Consider an ensemble given by the probability measure:

$$\mathrm{d}P_N^{\beta,V}(\lambda_1,\dots,\lambda_N) = \frac{1}{Z_N^{\beta,V}}\Delta(\lambda)^\beta e^{-N\beta\sum V(\lambda_i)}\prod_{i=1}^{N}\mathrm{d}\lambda_i$$

where $\Delta(\lambda) = \prod_{i<j}|\lambda_i-\lambda_j|$.

Remark. The case $V(X) = \frac{1}{2}x^2$ and $\beta = 2$ is exactly the GUE case we were looking at in the previous lecture. (the case $\beta = 1$ corresponds to GOE and $\beta = 4$ to GSE)

Notice that we can rewrite this as:

$$\begin{aligned}\frac{\mathrm{d}P_N^{\beta,V}}{\mathrm{d}\lambda} &= \exp\left\{\frac{1}{2}\beta\sum_{i\neq j}\log|\lambda_i-\lambda_j| - \beta N\sum V(\lambda_i)\right\}\\ \text{"}=\text{"} &\quad \exp\left\{-\beta N^2\mathcal{E}\left(\hat{\mu}_N\right)\right\}\end{aligned}$$

where $\hat{\mu}_N$ is the empircal measure (total mass 1), and:

$$\mathcal{E}(\mu) = \int V(x)\mathrm{d}\mu(x) - \frac{1}{2}\int\int\ln|x-y|\,\mathrm{d}\mu(x)\mathrm{d}\mu(y)$$

(the "=" is in quotes because we have thrown out the fact that $\ln|x-y|$ is not well defined for a dirac mass on the "self-interaction" diagonal terms)

Theorem. *Assume that* $\liminf_{|x|\to\infty}\frac{V(x)}{\ln(|x|)} > 1$ *(i.e. $V(x)$ goes to infinity fast enough to be dominante the log term at infinty) and V is continuous*

We have that $\hat{\mu}_N \Rightarrow \mu_V^{eq}$ a.s where μ_V^{eq} is the equilibrium measure for V, ie. namely the minimizer of $\mathcal{E}(\mu)$ and moreover this is bounded below, i.e. $\exists C_V$ s.t.

$$V_{eff}(x) := V(x) - \int\ln|x-y|\,d\mu_V^{eq}(y) - C_V = 0 \qquad \forall x\in supp(\mu_V^{eq})$$

and $V_{eff}\geq 0$ off of $supp(\mu_V^{eq})$.

Proof. (sketch) We prove later the a.s. convergence towards μ_V^{eq} and concentrates on the existence, haracterization and uniqueness of μ_V^{eq}. First we observe that the level sets $\{\mu : \mathcal{E}(\mu) \leq M\}$ are compact(exercise). Then by compactness we can find **a** (possibly not unique) minimizer, call it μ_V^{eq}. By using that $\mathcal{E}(\mu_V^{eq}+t\nu) \geq \mathcal{E}(\mu_V^{eq})$ for t small enough and all measure ν which is non negative outside the support of μ_V^{eq} and with mass zero (so that $\mu_V^{eq}+t\nu$ is a probability measure), we find that μ_V^{eq} satisfies that V_{eff} vanishes on its support and is non negative everywhere (for some constant C_V).

To see this is unique for any other measure μ with finite $\mathcal{E}$ write:

$$\begin{aligned}\mathcal{E}(\mu) &= \mathcal{E}(\mu_V^{eq}) - \frac{1}{2}\int \ln|x-y|\, \mathrm{d}\left(\mu-\mu_V^{eq}\right)(x)\mathrm{d}\left(\mu-\mu_V^{eq}\right)(y) \\ &\quad + \int V_{eff}(x)\mathrm{d}\mu(x)\end{aligned}$$

But this is a kind of distance function: if μ,ν such that $\mathcal{E}(\mu),\mathcal{E}(\nu)$ are finite:

$$D(\mu,\nu) := -\int \log|x-y|\, \mathrm{d}\left(\mu-\nu\right)\mathrm{d}\left(\mu-\nu\right) = \int_0^\infty \frac{1}{r}\left|\widehat{\mu-\nu}_r\right|^2 \mathrm{d}r$$

as the Fourier transform of the logarithm is $1/t$. Hence, $\mathcal{E}(\mu)=\mathcal{E}(\mu_V^{eq})$ implies $D(\mu_V^{eq},\mu)=0$(as V_{eff} is non-negative) and therefore $\mu=\mu_V^{eq}$. □

2.3 Concentration of Measure

We are going to regularize $\hat{\mu}_N$ so that it has finite energy, following an idea of Maurel-Segala and Maida. First define $\tilde{\lambda}$ by $\tilde{\lambda}_1 = \lambda_1$ and $\tilde{\lambda}_i = \tilde{\lambda}_{i-1} + \max\{\sigma_N, \lambda_i - \lambda_{i-1}\}$ where σ_N will be chosen to be like N^{-p}. Remark that $\tilde{\lambda}_i - \tilde{\lambda}_{i-1} \geq \sigma_N$ whereas $|\lambda_i - \tilde{\lambda}_i| \leq N\sigma_N$. Define $\tilde{\mu}_N = \mathbf{E}_U\left[\frac{1}{N}\sum \delta_{\tilde{\lambda}_i+U_i}\right]$ where U_i are iid unif $[0,N^{-q}]$ (i.e. we smooth the measure by putting little rectangles instead of Dirac masses and make sure that the eigenvalues are at least distance N^{-p} apart). Then we claim that

Lemma. *Assume V is C^1. For $1<p-1<q$ there exists $C_{p,q}$ finite such that*

$$P_N^{\beta,V}\left(D(\tilde{\mu}_N,\mu_V^{eq}) \geq t\right) \leq e^{C_{p,q}N\ln N - N^2t^2}$$

Corollary. *For any function φ set:*

$$|\varphi|_{\frac{1}{2}} := \left(\int_0^\infty |\hat{\varphi}_s|^2\, s ds\right)^{\frac{1}{2}}, |\varphi|_L = \sup\frac{|\varphi(x)-\varphi(y)|}{|x-y|}$$

$$\mathbf{P}\left(\left|\int \varphi d\left(\hat{\mu}_N - \mu_V^{eq}\right)\right| \geq N^{-p+1}|\varphi|_L + t\,|\varphi|_{\frac{1}{2}}\right) \leq e^{CN\ln N - N^2t^2}$$

Proof. It goes by triangle inequality basically:

$$\begin{aligned}
\int \varphi \mathrm{d}\left(\hat{\mu}_N - \mu_V^{eq}\right) &= \int \varphi \mathrm{d}\left(\hat{\mu_N} - \tilde{\mu}_N\right) + \int \varphi \mathrm{d}\left(\tilde{\mu}_N - \mu_V^{eq}\right) \\
&\leq \frac{1}{N}\sum_{i=1}^{N} \mathbf{E}_U[\varphi(\lambda_i) - \varphi(\tilde{\lambda}_i + U)] + \int \hat{\varphi}_s (\widehat{\mu_N - \tilde{\mu}_N})_s ds + \\
&\leq |\varphi|_L \left(N^{-p+1} + N^{-q}\right) + |\varphi|_{\frac{1}{2}} D\left(\tilde{\mu}_N, \mu_V^{eq}\right)
\end{aligned}$$

where we noticed that $|\lambda_i - \tilde{\lambda}_i|$ is bounded by N^{-p+1} and U by N^{-q}. Hence, the previous lemma proves the claim. □

We next prove Lemma 2.3. We first show that:

$$Z_N^{\beta,V} \geq \exp\left(-N^2\mathcal{E}(\mu_V^{eq}) + CN\ln N\right)$$

This follows since:

$$\begin{aligned}
Z_N^{\beta,V} &= \int \prod |\lambda_i - \lambda_j|^\beta \exp\left(-N\sum V(\lambda_i)\right) \mathrm{d}\lambda \\
&\geq \int_{|\lambda_i - x_i| \leq N^{-s}} \prod |\lambda_i - \lambda_j|^\beta \exp\left(-N\sum V(\lambda_i)\right) \mathrm{d}\lambda
\end{aligned}$$

Choose points x_i to be the "typical" eigenvalue locations for the equilibrium configuration i.e. so that $\mu_V^{eq}[(-\infty, x_i)] = \frac{i}{N}$. As it can be shown the density of μ_V^{eq} is bounded

$$\frac{1}{N} = \mu_V^{eq}([x_i, x_{i+1}]) \leq \|\frac{d\mu_V^{eq}}{dx}\|_\infty |x_{i+1} - x_i|$$

and we choose $s \geq 2$ so that $\lambda_{i+1} - \lambda_i \simeq (x_{i+1} - x_i)(1 + O(\frac{1}{N}))$. Then we have:

$$\begin{aligned}
Z_N^{\beta,V} &\geq (2N^{-s})^N \prod |x_i - x_j|^\beta \exp\left(-N\beta\sum V(x_i) + N^3 N^{-s}\right) \\
&\geq \exp\{-\beta N^2\mathcal{E}(\mu_V^{eq}) + CN\log N\}
\end{aligned}$$

since if $i > j$

$$\log|x_i - x_j| \geq \int_{x_i}^{x_{i+1}} \int_{x_{j-1}}^{x_j} \log|x - y| d\mu_V^{eq}(x) d\mu_V^{eq}(y)$$

and $|N^{-1}V(x_i) - \int_{x_i}^{x_{i+1}} V(x) d\mu_V^{eq}(x)| \leq C/N^2$ as V is Lipschitz.

Now consider that:

$$\begin{aligned}
Z_N^{\beta,V} \frac{\mathrm{d}P_N^{\beta,V}}{\mathrm{d}\lambda} &= \prod_{i<j} |\lambda_i - \lambda_j|^\beta \exp\left(-N\beta\sum V(\lambda_i)\right) \\
&\leq \prod_{i<j} \left|\tilde{\lambda}_i - \tilde{\lambda}_j\right|^\beta \exp\left(-N\beta\sum V(\tilde{\lambda}_i) + N^2 N^{-p}\right)
\end{aligned}$$

because the $\tilde{\lambda}$ only increased the differences and $|\lambda_i - \tilde{\lambda}_i| \leq N^{1-p}$. Now because adding the uniform variables only creates an error of order $N^2 N^{-q} N^p$ we can conclude that if $q > p + 1$,

$$\frac{\mathrm{d}P_N^{\beta,V}}{\mathrm{d}\lambda} \leq \exp\left(-N^2\beta\left(\mathcal{E}(\tilde{\mu}_N) - \mathcal{E}(\mu_V^{eq})\right) + CN\ln N\right)$$

We now use the fact that $\mathcal{E}(\tilde{\mu}_N) - \mathcal{E}(\mu_V^{eq}) = D(\tilde{\mu}_N, \mu_V^{eq})^2 + \int (V_{eff})(x))\mathrm{d}\tilde{\mu}_N(x)$ so that

$$P_N^{\beta,V}\left(D(\tilde{\mu}_N, \mu_V^{eq}) \geq t\right) \leq e^{CN\log N - \beta N^2 t^2} \left(\int e^{-NV_{eff}(x)} dx\right)^N$$

where the last integral is bounded by a constant as V_{eff} is non-negative and goes to infinity at infinity faster than logarithmically.

3 Lecture 3

3.1 Goal and strategy

We want to show for sufficiently many functions f that

- $$\mathbf{E}[\frac{1}{N}\sum f(\lambda_i)] = \mu_V^{eq}(f) + \sum_{g=1}^{K} \frac{1}{N^g} c_g(f) + o(\frac{1}{N^K})$$
- $\sum f(\lambda_i) - \mathbf{E}[\sum f(\lambda_i)]$ converges to a centered Gaussian.

We will restrict ourselves to Stieljes transform $f(x) = (z-x)^{-1}$ for $z \in \mathbb{C}\backslash\mathbb{R}$, which in fact gives these results for all analytic function f. We will as well restrict ourselves to $K = 2$, but the strategy is similar to get higher order expansion. The strategy is similar to the case of the GUE:

- We derive a set of equations, the Dyson-Schwinger equations, for our observables (the moments of Stieljes transform): it is an infinite system of equations, a priori not closed. However, it will turn out that asymptotically it can be closed.
- We linearize the equations around the limit. We show that some terms are negligable. The concentration estimates of Lemma 2.3 implies that

  $$|\mathbf{E}[\prod_{j=1}^{p}(\sum(z_j - \lambda_i)^{-1} - \mathbf{E}[\sum(z_{j-}\lambda_i)^{-1}]| \leq (\sqrt{N}\ln N)^p / \prod |\Im z_i| \quad (3)$$

  which is not good enough for our purpose. So we improve these estimates by using the Dyson-Schwinger equations in the spirit of what we did for the GUE.

- To solve the linearized equation, we can not use induction as before, but we invert some linear operator. This shows that once smaller terms have been neglected, the system of equations can be solved (that is is closed).

- We proceed recursively to get all orders of the corrections.

3.2 Dyson-Schwinger Equation

We will look for moments of the Stieltjes transform, namely moments of:

$$G_N(z) := \frac{1}{N}\sum_{i=1}^{N}\frac{1}{z-\lambda_i}$$

For $z \in \mathbb{C}\backslash\mathbb{R}$. Let:

$$Y_z = \sum_{i=1}^{N}\frac{1}{z-\lambda_i} - \mathbf{E}\left[\sum_{i=1}^{N}\frac{1}{z-\lambda_i}\right]$$

We will assume that

Assumption. -V *is real analytic,*
-μ_V^{eq} *has a connected support* $[a,b]$,
-V_{eff} *is strictly positive outside the support of* μ_V^{eq}.

FIrst notice that:

$$\int \mathrm{d}\lambda \sum_{i=1}^{N}\partial_{\lambda_i}\left[\frac{1}{z-\lambda_i}\frac{\mathrm{d}P_N^{\beta,V}}{\mathrm{d}\lambda}\prod_{j=1}^{k}Y_{z_j}\right] = 0$$

(This follows by integration by parts formula $\int_{-\infty}^{\infty}\partial_x f(x)\mathrm{d}x = 0$) On the other hand, if we expand out this derivative we have:

$$\int \mathrm{d}\lambda \left\{\sum_{i=1}^{N}\frac{1}{(z-\lambda_i)^2} + \frac{1}{z-\lambda_i}\left\{\beta\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j} - \beta N V'(\lambda_i)\right\} + \sum_j\sum_i \frac{1}{z-\lambda_i}\frac{1}{(z_j-\lambda_i)^2}\frac{1}{Y_{z_j}}\right\}\prod_{k=1}^{p}Y_{z_k}$$

If we set $G_N(z) = \frac{1}{N}\sum\frac{1}{z-\lambda_i}$ and define $G(z) = \int\frac{1}{z-x}\mathrm{d}\mu_V^{eq}(x)$ this is:

$$\mathbf{E}\Bigg[\left(\frac{1}{N}\partial_z G_N(z)\left[-1+\frac{\beta}{2}\right] + \frac{\beta}{2}G_N(z)^2 - \frac{\beta}{N}\sum\frac{V'(\lambda_i)}{z-\lambda_i}\right)\prod_k Y_{z_k}\Bigg] \quad (4)$$
$$= \frac{1}{N}\mathbf{E}\Bigg[\sum_{j=1}^{p}\partial_{z_j}\mathbf{E}\left[\frac{G_N(z_i)-G_N(z_j)}{z-z_j}\right]\prod_{\ell\neq j}Y_{z_\ell}\Bigg]$$

We now use the fact that V is real analytic so that Cauchy formula implies that

$$\sum \frac{V(\lambda_i)}{z-\lambda_i} = -\frac{1}{2\pi\imath}\oint \frac{V(\xi)}{z-\xi}\sum \frac{1}{\xi-\lambda_i}\mathrm{d}\xi = \frac{1}{2\pi\imath}\oint \frac{V(\xi)}{z-\xi}G_N(\xi)\mathrm{d}\xi$$

where the contour encircles the λ_i's. It can be proven that when V_{eff} is positive outside the support of μ_V^{eq}, for any $\varepsilon > 0$, there exists $c(\varepsilon) > 0$ so that

$$P_N^{\beta,V}\left(\exists i : \lambda_i \in [a-\varepsilon, b+\varepsilon]^c\right) \le e^{-c(\varepsilon)N}\,.$$

This entitles us to change the probability measure to have support in $[a-\varepsilon, b+\varepsilon]$ up to exponentially small errors everywhere (that we will not discuss in what follows). We then can simply take a contour around $[a-\varepsilon, b+\varepsilon]$.

3.3 Analysis of the Dyson-Schwinger equation: heuristics

We know by Lemma 2.3 that G_N converges to G and hence (4) yields with $p = 0$ that

$$\frac{\beta}{2}G(z)^2 + \frac{1}{2\pi\imath}\oint \frac{V(\xi)}{z-\xi}G(\xi)\mathrm{d}\xi = 0\,. \tag{5}$$

We next guess the corrections to this limit.

- *First order correction.* Setting $\Delta G_N := G_N - G$, (4) yields with $p = 0$ that

$$\mathbf{E}\left[\frac{1}{N}\partial_z G_N(z)\left[-1+\frac{\beta}{2}\right] + K[\Delta G_N](z) + \frac{\beta}{2}(\Delta G_N(z))^2\right] = 0 \tag{6}$$

  where

$$Kf(z) = \beta G(z)f(z) - \frac{\beta}{2\pi\imath}\oint \frac{V'(\xi)}{z-\xi}f(\xi)\mathrm{d}\xi\,.$$

  By Lemma 2.3, we know that $\mathbf{E}[(\Delta G_N(z))^2]$ is of order $(\ln N)^2/N$. Let us assume for a moment it is $o(N^{-1})$. Assume as well that K is invertible. Then we deduce from (6) that

$$\lim_{N\to\infty} N\mathbf{E}[\Delta G_N(z)] = (\frac{1}{2}-\frac{1}{\beta})K^{-1}[\partial_z G](z) =: G_1(z)\,.$$

- *Limiting covariance.* To get the limiting covariance, let us take $p = 1$. Let $c(z,z') = \mathbf{E}[Y_z Y_{z'}]$. The Dyson-Schwinger equation then reads

$$\begin{aligned} K(c(.,z'))(z) &= -\frac{\beta N}{2}\mathbf{E}[(\Delta G_N(z))^2 Y_{z'}] \\ &\quad -(\frac{\beta}{2}-1)\partial_z\mathbf{E}[G_N(z)] - \partial_{z'}\mathbf{E}[\frac{G_N(z)-G_N(z')}{z-z'}] \end{aligned} \tag{7}$$

  Assume that $\mathbf{E}[(\Delta G_N(z))^2 Y_{z'}] = o(1/N)$ even if the concentration estimates only gives that it is of order $1/\sqrt{N}$. Note that $\mathbf{E}[G_N(z)Y_{z'}] =$

$\mathbf{E}[(G_N(z) - G(z))Y_{z'}]$ grows at most logarithmically, and assume it goes to zero. As this is an analytic function, its derivative goes as well to zero. Then, we deduce from (4) that

$$\lim_{N\to\infty} \mathbf{E}[NG_N(z)Y_{z'}] = K^{-1}[\partial_{z'} \frac{G(.) - G(z')}{. - z'}](z) =: W(z, z') \,.$$

- *Second order correction.* Going back to (4) with $p = 0$ we have

$$K[N(N\Delta G_N - G_1)](z) = -\frac{\beta}{2}(\mathbf{E}[Y_z^2] + \mathbf{E}[N\Delta G_N(z)]^2) - (\frac{\beta}{2} - 1)\partial_z \mathbf{E}[N\Delta G_N(z)]$$

  and we can go to the limit $N \to \infty$ to deduce

$$\lim_{N\to\infty} K[N(N\Delta G_N - G_1)](z) = -\frac{\beta}{2}(W(z, z) + G_1(z)^2) - (\frac{\beta}{2} - 1)\partial_z G_1(z)$$

  so that taking the inverse of K yields the desired limit:

$$\lim_{N\to\infty} N(N\Delta G_N - G_1)](z) = K^{-1}(-\frac{\beta}{2}(W(.,.) + G_1(.)^2) - (\frac{\beta}{2} - 1)\partial_z G_1)(z) \,.$$

The above heuristics can be made rigorous provided we invert the operator K (and show its inverse is continuous to neglect error terms after we inverted it) and we show sufficiently strong concentration inequalities. This is what we do next.

3.4 Inverting the operator K

Observe that we want to apply K to functions which are differences of Stieljes transforms and therefore going to infinity like $1/z^2$. We therefore search for f with such a decay satisfying $g(z) = Kf(z)$ for a given g. As a consequence g goes to infinity like $1/z$ at best. We can rewrite

$$Kf(z) = \beta(G(z) - V'(z))f(z) - \beta \oint \frac{V'(\xi) - V'(z)}{2i\pi(z - \xi)} f(\xi) d\xi \,.$$

We make the following crucial assumption of off-criticality:

Assumption. *There exists a real analytic function h which does not vanish on a complex neighborhood of $[a - \varepsilon, b + \varepsilon]$ so that*

$$\frac{d\mu_V^{eq}}{dx} = h(x)\sqrt{(x - a)(b - x)} \,.$$

This implies that

$$G(z) - V'(z) = \pi\sqrt{(z - a)(b - z)}h(z)$$

with h real analytic and not vanishing on $[a - \varepsilon, b + \varepsilon]$. Indeed,(5) implies that

$$G(z)^2 - 2V'(z)G(z) + Q(z) = 0$$

with $Q(z) = 2\oint \frac{V'(\xi)-V'(z)}{2i\pi(z-\xi)} f(\xi)d\xi$. Solving this equation yields

$$G(z) = V'(z) - \sqrt{V'(z)^2 - Q(z)}$$

As V' is analytic, and $V'(z)^2 - Q(z)$ is analytic, its square root becomes as z goes to the real line the density of μ_V^{eq}. The conclusion follows.

This behaviour is essential to invert K, in the spirit of Tricomi airfol equation. Indeed, we write with $\sigma(z) = \sqrt{(z-a)(b-z)}$, for $g = Kf$

$$\begin{aligned}
\sigma(z)f(z) &= \frac{1}{2i\pi}\oint \frac{1}{z-\xi}\sigma(\xi)f(\xi)d\xi \\
&= \frac{1}{2i\pi}\oint \frac{1}{z-\xi}\frac{1}{\beta S(\xi)}(g(\xi) + \frac{1}{2}Q(\xi))d\xi \\
&= \frac{1}{2i\pi}\oint \frac{1}{z-\xi}\frac{1}{\beta S(\xi)}(g(\xi))d\xi
\end{aligned}$$

where in the first line we took a contour around z and used Cauchy formula, in the second line we passed the contour around $[a, b]$ and used the definition of $Kf = g$, using that the residue at infinity vanishes because $\sigma(z)f(z)$ goes like $1/z$, and in the last line we used that Q/S is analytic. Hence, we deduce that

$$K^{-1}g(z) = \frac{1}{\sigma(z)}\frac{1}{2i\pi}\oint \frac{1}{z-\xi}\frac{1}{\beta S(\xi)}(g(\xi))d\xi\,,$$

where the contour surrounds $[a-\varepsilon, b+\varepsilon]$. We note that away from $[a, b]$, K^{-1} is bounded. Also it maps holomorphic to holomorphic functions so that bounds on functions translate into bounds on its derivatives up to take slightly smaller imaginary part of the argument.

3.5 Improving concentration estimates

To this end we use the Dyson-Schwinger equations. By using the concentration estimate, the right hand side of (2) implies we see that the right hand side is bounded by $(\ln N)^3\sqrt{N}$. This yields an a priori bound on c of order $(\ln N)^3\sqrt{N}$ as K^{-1} is bounded. This is better than the a priori estimate $N(\ln N)^2$. This already shows that $\mathbf{E}[|N\Delta G_N(z)|^2]$ is at most of order $(\ln N)^3\sqrt{N} = o(N)$, justifying our computation of the first order correction on $\mathbf{E}[G_N - G]$. To go farther, observe that by Lemma 2.3 and restricting ourselves to the event $|\Delta G_N| \le \ln N/\sqrt{N}$, we find

$$\begin{aligned}
|\mathbf{E}[(\Delta G_N(z))^2 Y_{z'}]| &\le \frac{\ln N}{N\sqrt{N}}\mathbf{E}[|N\Delta G_N(z)|^2]^{1/2}\mathbf{E}[|Y_{z'}|^2]^{1/2} + \frac{e^{-N(\ln N)^2}N}{|(\Im z)\Im z'|} \\
&\le (\ln N)^{5/2}/N
\end{aligned}$$

by the previous bound on the covariance. Going back to (2), we deduce that c is at most of order $(\ln N)^{5/2}$, and by the above bound that

$$|\mathbf{E}[(\Delta G_N(z))^2 Y_{z'}]| \le \frac{(\ln N)^{7/2}}{N\sqrt{N}}$$

We can then make the argument concerning the convergence of c rigorous. We then get the second order correction as announced.

3.6 Central limit theorem

To prove the central limit theorem we show by induction over p that

$$\lim_{N\to\infty}\mathbf{E}[\prod_{i=1}^{p}Y_{z_i}]=\sum_{i=2}^{p}W(z_1,z_j)\lim_{N\to\infty}\mathbf{E}[\prod_{\ell\neq j}Y_{z_\ell}]$$

This is true for $p=1,2$. We then use

$$\mathbf{E}\left[\left(\partial_z G_N(z)\left[-1+\frac{\beta}{2}\right]+N\beta K(G_N-G)+\frac{\beta N}{2}(G_N(z)-G(z))^2\right)\prod_{k=1}^{p}Y_{z_k}\right]$$
$$=\mathbf{E}\left[\sum_{j=1}^{p}\partial_{z_j}\mathbf{E}\left[\frac{G_N(z_i)-G_N(z_j)}{z-z_j}\right]\prod_{\ell\neq j}Y_{z_\ell}\right] \quad (8)$$

We then use that by concentration inequality Lemma 2.3 and the induction bound on $\mathbf{E}[\prod_{j=1}^{p}|Y_{z_j}|]$ if p is even or $\mathbf{E}[\prod_{j=1}^{p-1}|Y_{z_j}|]$ if p is odd which is finite

$$|\mathbf{E}[(G_N(z)-G(z))^2\prod_{k=1}^{p}Y_{z_k}]|\le(\frac{(\ln N)}{\sqrt{N}})^2(\sqrt{N}\ln N)^n+\frac{1}{(\Im z)^2\prod|\Im z_{k|}}e^{-N\ln N}$$

where $n=1$ if p is odd and 0 if p even. Let us consider the case p odd. Plugging back this estimate and inverting K (recall as well the first term $\beta K(G_N-G)+\partial_z G_N(z)\left[-1+\frac{\beta}{2}\right]=\beta K(G_N-\mathbf{E}[G_N]))$ yields that $|\mathbf{E}[N(G_N(z)-G(z))\prod_{k=1}^{p}Y_{z_k}]|$ is at most of order $\ln N\sqrt{N}$. $p+1$ is this time even so we can insert the absolute value, and deduce by concentration inequality that $\mathbf{E}[(N(G_N(z)-G(z))^2\prod_{k=1}^{p}Y_{z_k}]|$ is at most of order $(\ln N)^2$. Hence we may neglect this term and conclude as before.

3.7 Bibliography

B. Eynard and his collaborators used a lot DS equations to get large N expansions on a formal level, and study the "topological expansion" that relates the coefficients of these expansion. The idea to use DS equations to prove CLT rigorously was first followed by Johansson (97) under some assumption of convexity on the potential. Note that an expansion for linear statistics yields after interpolation expansions for the free energy as if $V_t=tV+(1-t)x^2$

$$\frac{1}{N^2}\ln\frac{Z_N^{\beta,V}}{Z_N^{\beta,x^2}}=-\int_0^1\mathbf{E}_{P_N^{\beta,V_t}}[\frac{1}{N}\sum_{i=1}^{N}\partial_t V_t(\lambda_i)]dt$$

so that if we have expansions at each t of the interpolation, we are done.This was done with Borot It was pursued by Shcherbina and G-Borot to consider the case were μ_V^{eq} has a disconnected support: in this case the number of eigenvalues lying in each connected part fluctuates as a discrete Gaussian whose mean and covariance may only converge along subsequences, so that the usual CLT is not valid anymore. More general potential(in particular none linear in the spectral measure) were considered with Borot and Kozlowski, as well as the sinsh model were the Coulomb potential is replaced by a sinsh potential. Several matrix models were considered with E. Maurel Segala, however, the inversion of the operator in general requires the potential to be small. This approach was also generalized to orthogoal and unitary matrices with Collins and Maurel Segala, and then with Novak. A related research was conducted by Chatterjee. In the next lecture, we investigate the discrete case. Then, large deviation and concentration inequalities are very similar. However, the expected Dyson -Schwinger equation given by (discrete) integration by parts is not easily tractable. Indeed the induced change of measure is not a nice function of the empirical measure $L_{N=}\frac{1}{N}\sum\delta_{\ell_{i/N}}$ as it depends on

$$\prod_{i<j}(1+\frac{1}{\ell_i-\ell_j})$$

which depends on L_N and N in a non trivial way.We shall present recent equations introduced by Nekrasov that will be convenient for asymptotic analysis.