Universal local limits for lozenge tilings and noncolliding random walks.

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MIT (Cambridge) and IITP (Moscow)

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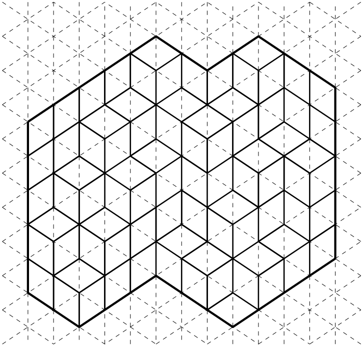
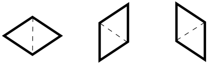
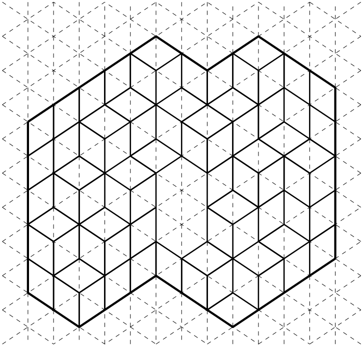
Random lozenge tilings

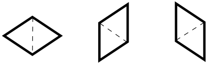




Random tilings of finite and infinite planar domains with **uniform Gibbs property**.

Random lozenge tilings





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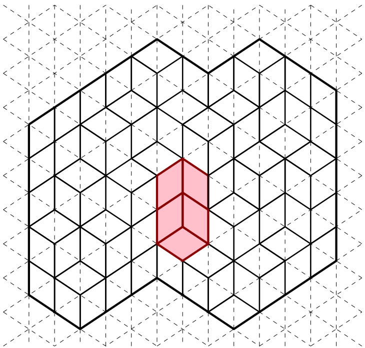

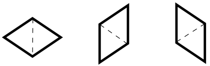

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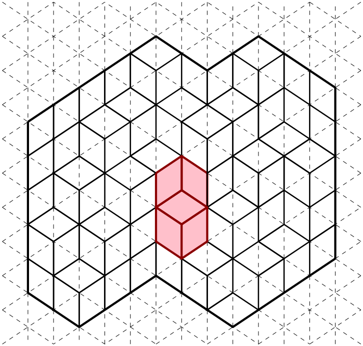

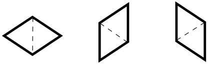

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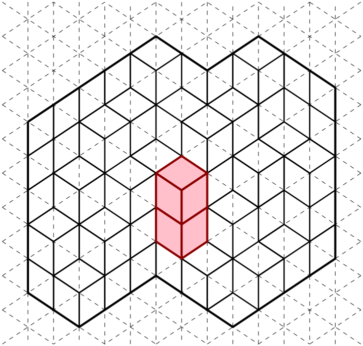
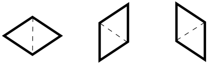
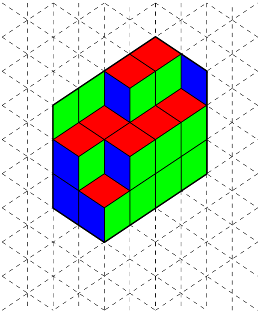
Random lozenge tilings



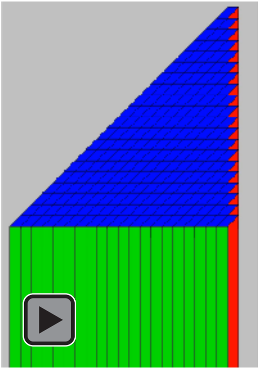


Random tilings of finite and infinite planar domains with **uniform Gibbs property**.

Random lozenge tilings: examples



1) Uniformly random tilings of a finite domain

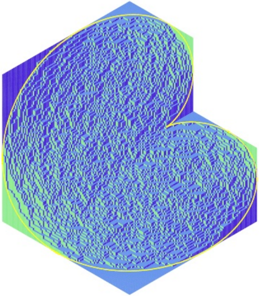


2) Surface growth (simulation of Patrik Ferrari)

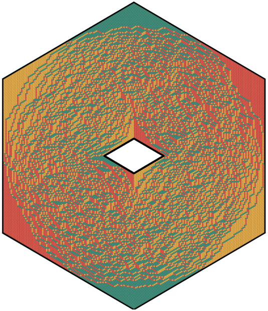
3) Path–measures in **Gelfand–Tsetlin graph** of asymptotic representation theory.

Random lozenge tilings: questions

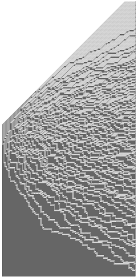
(Kenyon–Okounkov)



(Petrov)



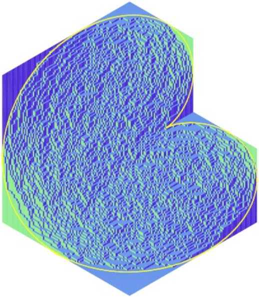
(Borodin-Ferrari)

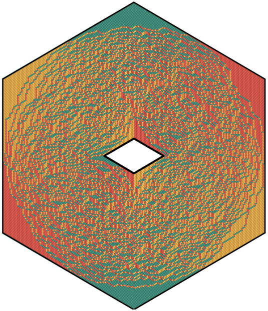


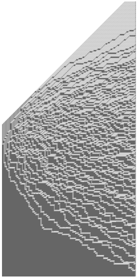
Asymptotics as mesh size $\to 0$ or size of the system $\to \infty$?

Random lozenge tilings: questions

(Kenyon–Okounkov) (Petrov) (Borodin-Ferrari)







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Universality belief:

main features do not depend on exact specifications.

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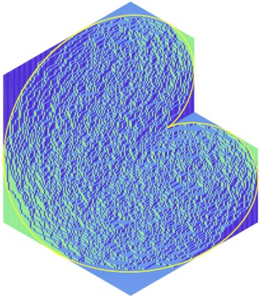
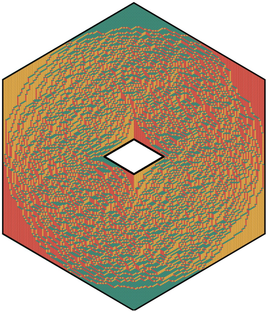
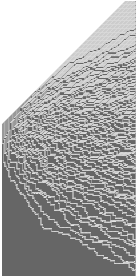






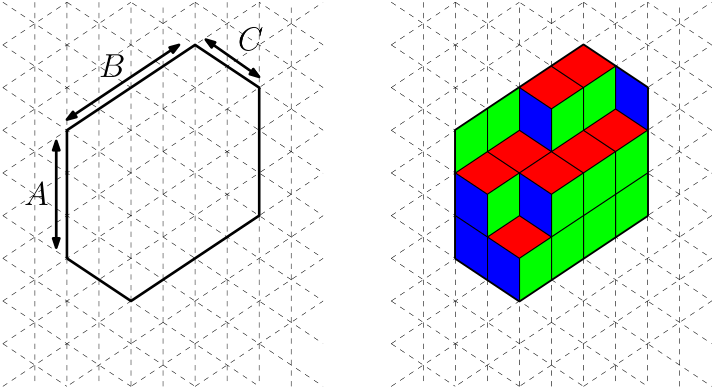
Asymptotics as mesh size $\to 0$ or size of the system $\to \infty$?

Universality belief:

main features do not depend on exact specifications.

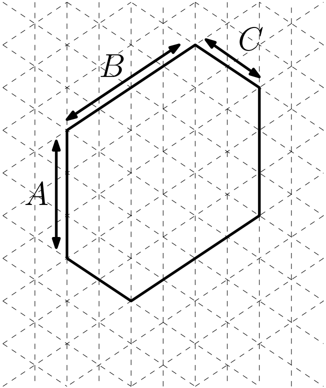
What are these features?

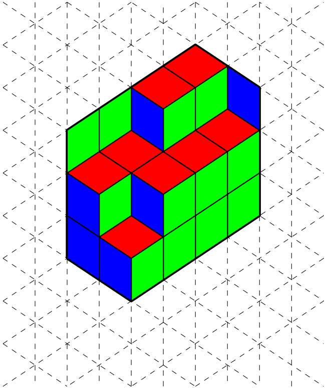
Random lozenge tilings: hexagon



Representative example: uniformly random lozenge tiling of $A \times B \times C$ hexagon.

Random lozenge tilings: hexagon





Representative example: uniformly random lozenge tiling of $A \times B \times C$ hexagon.

Equivalently: decomposition of irreducible representation of $U(B + C)$ with signature $(A^B, 0^C)$.

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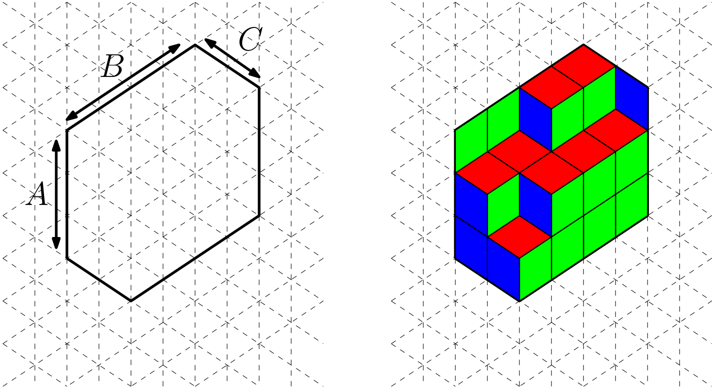
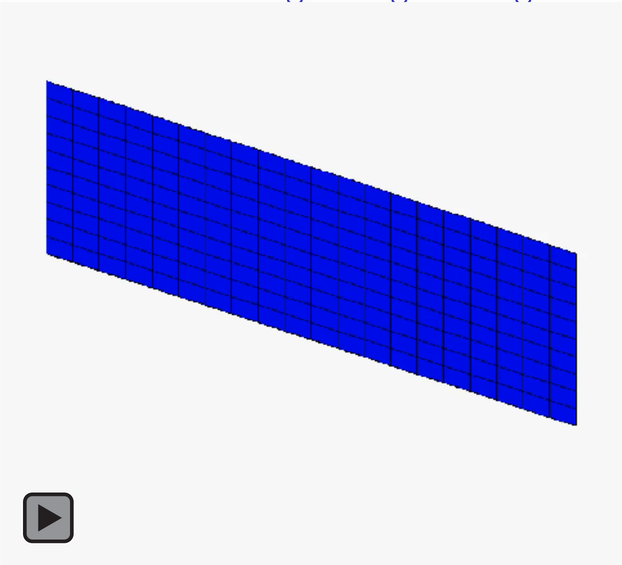


Representative example: uniformly random lozenge tiling of $A \times B \times C$ hexagon.

Equivalently: decomposition of irreducible representation of $U(B + C)$ with signature $(A^B, 0^C)$.

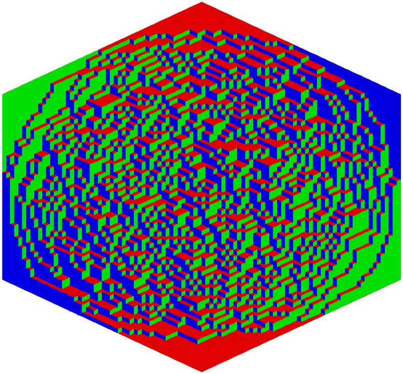
Equivalently: fixed time distribution of a $2d$-particle system.

Random lozenge tilings: hexagon



Shuffling algorithm (Borodin–Gorin)

Random lozenge tilings: features



Law of Large Numbers
(Cohn–Larsen–Propp)

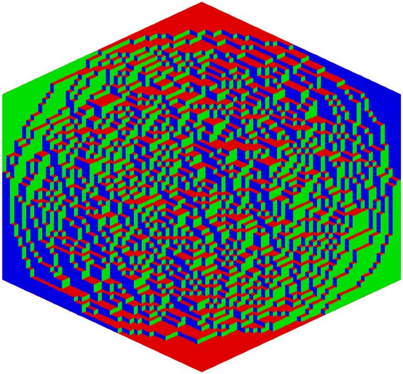
And for general domains
(Cohn–Kenyon–Propp)
(Kenyon–Okounkov)
(Bufetov–Gorin)

$$A = aL, B = bL, c = cL$$

$$L \to \infty$$

Theorem. Average proportions of three types of lozenges converge in probability to explicit **deterministic** functions of a point inside the hexagon. Equivalently, the rescaled height function $\frac{1}{L}H(Lx, Ly)$ converges to a deterministic limit shape.

Random lozenge tilings: features



Central Limit Theorem
(Kenyon), (Borodin-Ferrari), (Petrov), (Duits), (Bufetov–Gorin)

Liquid region: all types of lozenges are present

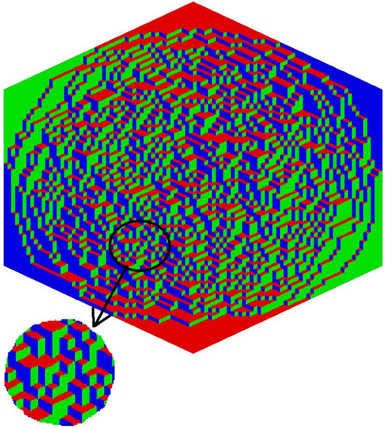
Frozen region: only one type

$$A = aL, B = bL, c = cL$$

$$L \to \infty$$

Theorem. The centered height function $H(Lx, Ly) - \mathbb{E}H(Lx, Ly)$ converges in the liquid region to a generalized Gaussian field, which can be identified with a pullback of the 2d **Gaussian Free Field**.

Random lozenge tilings: features



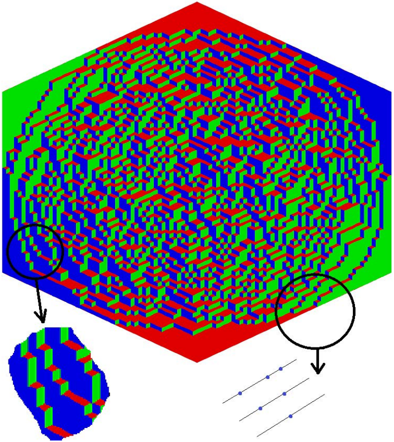
Bulk local limit
(Okounkov–Reshetikhin),
(Baik-Kriecherbauer-McLaughlin-Miller),
(Gorin), (Petrov)

$$A = aL, B = bL, c = cL$$

$$L \to \infty$$

Theorem. Near each point (xL, yL) the point process of lozenges converges to a (unique) **translation invariant ergodic Gibbs measure** on tilings of plane of the slope given by the limit shape.

Random lozenge tilings: features

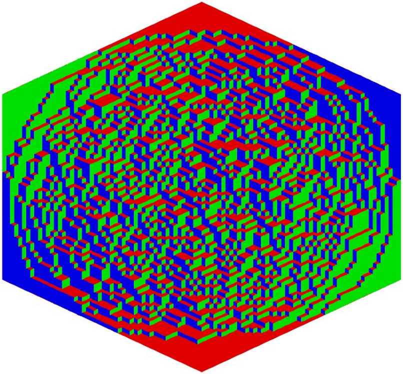


Edge local limit at a generic point
(Ferrari–Spohn),
(Baik-Kriecherbauer-McLaughlin-Miller),
(Petrov)

Edge local limit at a tangency point
(Johansson–Nordenstam),
(Okounkov–Reshetikhin),
(Gorin–Panova), (Novak)
$A = aL, B = bL, c = cL, L \to \infty$

Theorem. Near a generic (or tangency) point of the frozen boundary its fluctuations are governed by the **Airy line ensemble** (or **GUE–corners process**, respectfully)

Random lozenge tilings: features



1. Law of Large Numbers
2. Central Limit Theorem
3. Bulk local limits
4. Edge local limits at generic and tangency points

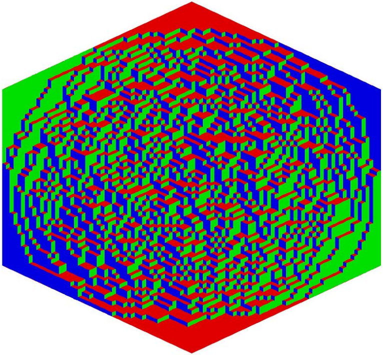
$$A = aL, B = bL, c = cL,$$

$$L \to \infty$$

Universality predicts that the same features should be present in generic random tilings models.

This is rigorously established only for the Law of Large Numbers.

Random lozenge tilings: what's new?



1. Law of Large Numbers
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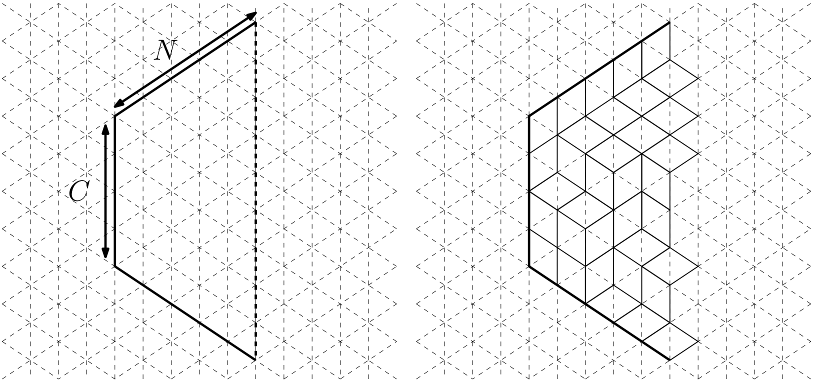
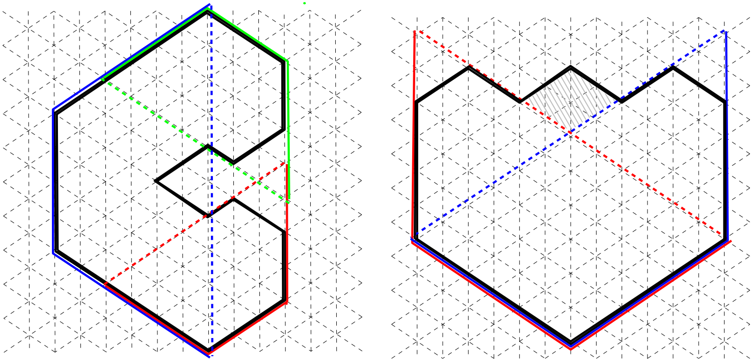
Conjecturally, should hold for generic random tilings

Today: partial universality results for bulk local limits

Trapezoids

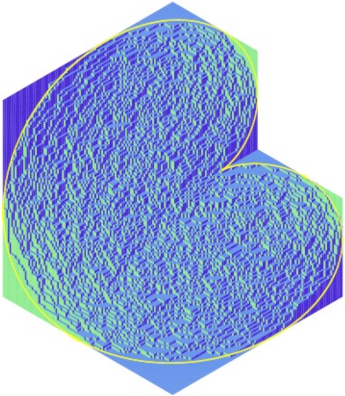


Bulk local limits: universality

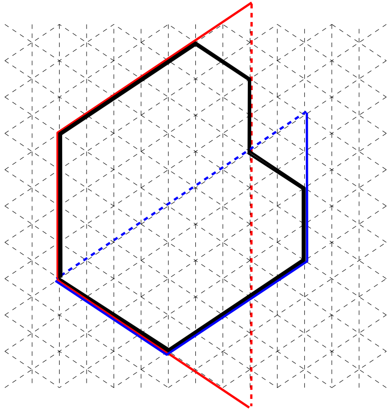


Theorem. (G.-16) Let $\Omega(L)$ be a regularly growing sequence of domains. For any part of $\Omega(L)$ covered by a trapezoid, near any point in the liquid region in this part, the uniformly random lozenge tilings of $\Omega(L)$ converge locally as $L \to \infty$ to the ergodic translation–invariant Gibbs measure of the slope given by the limit shape.

Bulk local limits: universality



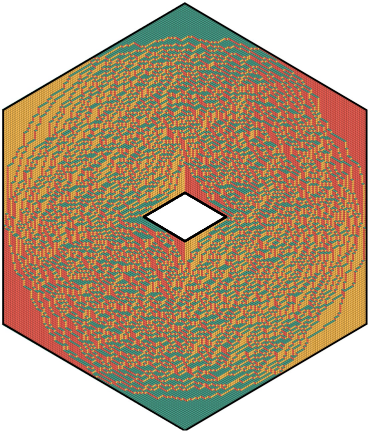
Picture from (Kenyon–Okounkov)



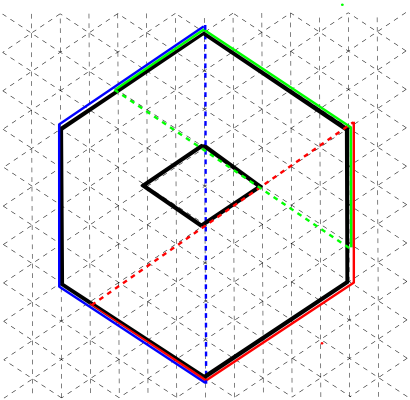
Bulk limits were not known for this domain before

Many domains are **completely covered** by trapezoids and therefore the conjectural bulk universality is now a **theorem** for them.

Bulk local limits: universality



Simulation by L. Petrov



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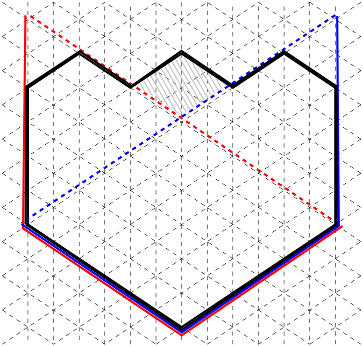

Some domains are only **partially** covered by trapezoids.

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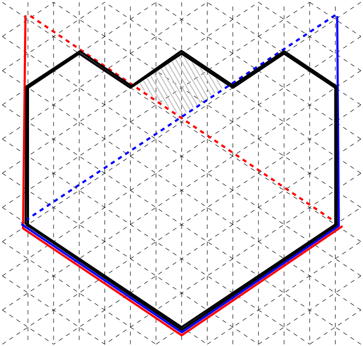

Some domains are only **partially** covered by trapezoids.

The theorem also holds for more general **Gibbs measures** on tilings covered by trapezoids ($2+1$–dimensional interacting particle systems, asymptotic representation theory).

Bulk local limits: universality

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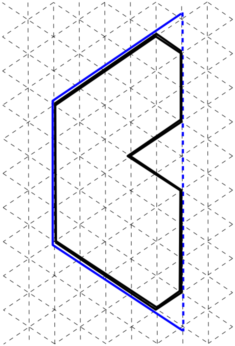
Previous results:



(Petrov-12)
Local bulk limits for **polygons** covered by **single** trapezoid.

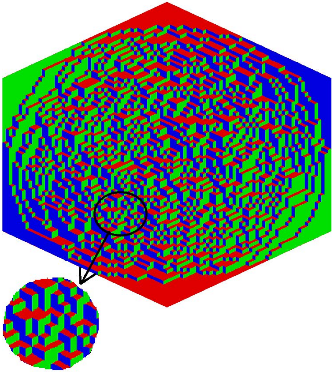
(Kenyon–04)
Local bulk limits for a class of domains with **no** straight boundaries.

(Borodin–Kuan–07)
Local bulk limits for Gibbs measures arising from **characters of $U(\infty)$**

(Okounkov–Reshetikhin–01)
Local bulk limits for Schur processes

Ergodic translation–invariant Gibbs measures

Theorem. ... near any point in the liquid region as $L \to \infty$ we observe an **ergodic translation–invariant Gibbs measure**.



Theorem. (Sheffield). For each slope, i.e. average proportions of lozenges $(p^{\square}, p^{\square}, p^{\diamond})$ there is a unique e.t.-i.G. measure.

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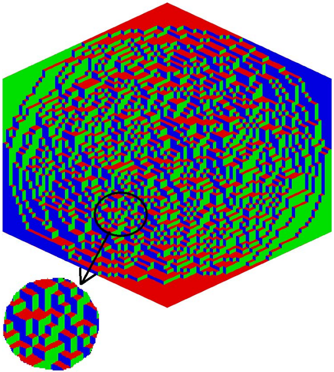

Theorem. (Sheffield). For each slope, i.e. average proportions of lozenges $(p^{▯}, p^{▱}, p^{\diamond})$ there is a unique e.t.-i.G. measure.

Description. (Cohn–Kenyon–Propp, Okounkov–Reshetikhin) Red lozenges in e.t.-i.G. measure form a **determinantal point process** with incomplete Beta kernel.

$$\rho_k((x_1, n_1), \dots, (x_k, n_k)) = \det_{i,j=1}^{n} \left[\frac{1}{2\pi \mathbf{i}} \int_{\bar{\xi}}^{\xi} w^{x_j - x_i - 1}(1-w)^{n_j - n_i} dw \right]$$

contour intersects $(0,1)$ when $n_j \geq n_i$ and $(-\infty, 0)$ otherwise.

Ergodic translation–invariant Gibbs measures

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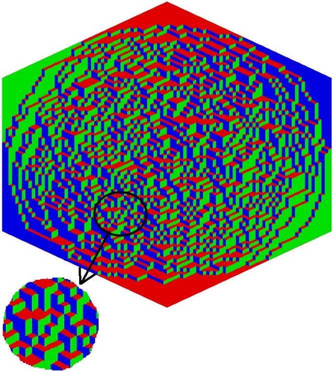

Theorem. (Sheffield). For each slope, i.e. average proportions of lozenges $(p^{\lozenge}, p^{\lozenge}, p^{\lozenge})$ there is a unique e.t.-i.G. measure.

Description. (Cohn–Kenyon–Propp, Okounkov–Reshetikhin) Red lozenges in e.t.-i.G. measure form a determinantal point process with incomplete Beta kernel.

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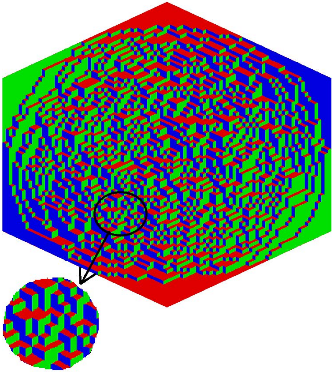
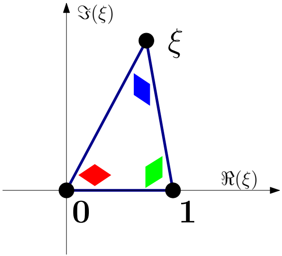
$\frac{\sin(\arg \xi (x_j - x_i))}{\pi (x_j - x_i)}$ for $n_j = n_i$. Extension of discrete sine kernel.

Ergodic translation–invariant Gibbs measures

Theorem. ... near any point in the liquid region as $L \to \infty$ we observe an ergodic translation–invariant Gibbs measure.





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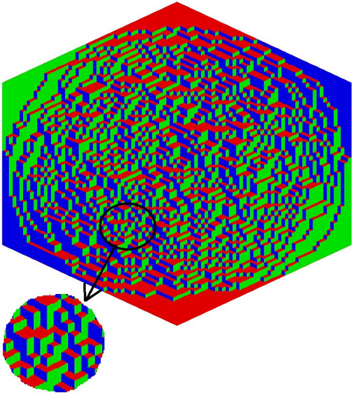
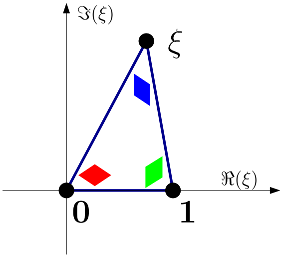
Local vs global meanings of slope $(p^{\lozenge}, p^{\lozenge}, p^{\diamond})$

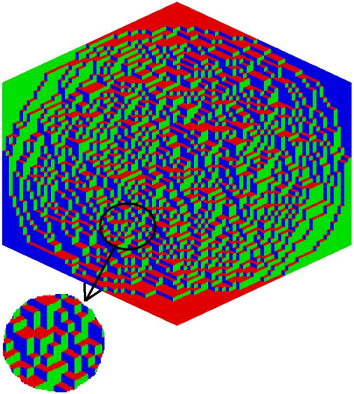


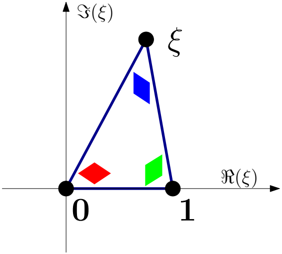


Meaning 1: It describes the **e.t.-i.G. measure** in the bulk

$$\frac{1}{2\pi \mathbf{i}} \int_{\bar{\xi}}^{\xi} w^{x_j - x_i - 1} (1 - w)^{n_j - n_i} dw$$

Local vs global meanings of slope $(p^{\Box}, p^{\Box}, p^{\Diamond})$





Meaning 1: It describes the **e.t.-i.G. measure** in the bulk

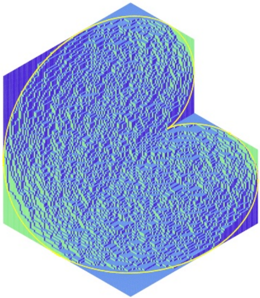
$$\frac{1}{2\pi \mathbf{i}} \int_{\bar{\xi}}^{\xi} w^{x_j - x_i - 1} (1 - w)^{n_j - n_i} dw$$

Meaning 2: Law of Large Numbers. Normalized lozenge counts inside a subdomain $\mathcal{D}$ converge to **deterministic vector**

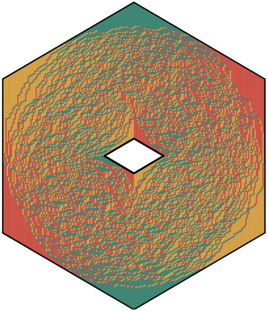
$$\left(\int_{\mathcal{D}} p^{\Box}(\mathbf{x}, \eta) d\mathbf{x} d\eta, \int_{\mathcal{D}} p^{\Box}(\mathbf{x}, \eta) d\mathbf{x} d\eta, \int_{\mathcal{D}} p^{\Diamond}(\mathbf{x}, \eta) d\mathbf{x} d\eta \right)$$

How to find slope $(p^{\lozenge}, p^{\lozenge}, p^{\diamond})$?

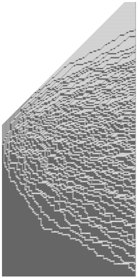
(Kenyon–Okounkov)



(Petrov)

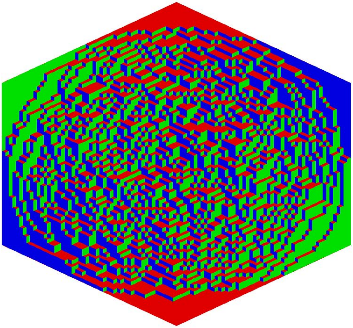


(Borodin-Ferrari)



Both local bulk limits and global law of large numbers are parameterized by the **same** position–dependent slope which one needs to find.

How to find slope $(p^{\square}, p^{\lozenge}, p^{\diamond})$?

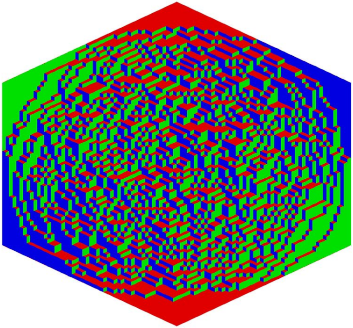


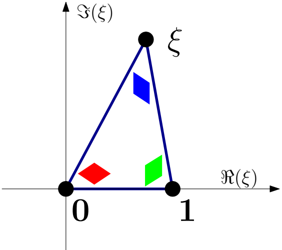
Method 1. (Cohn–Kenyon–Propp)
Solve **variational problem** for tilings of a generic domain Ω.

$$\int_{\Omega} \sigma\left(p^{\square}(\mathbf{x},\boldsymbol{\eta}), p^{\lozenge}(\mathbf{x},\boldsymbol{\eta}), p^{\diamond}(\mathbf{x},\boldsymbol{\eta})\right) d\mathbf{x} d\boldsymbol{\eta} \longrightarrow \max$$

$\sigma(\cdot,\cdot,\cdot)$ is an explicitly known **entropy** (or surface tension)

How to find slope $(p^{\lozenge}, p^{\lozenge}, p^{\lozenge})$?





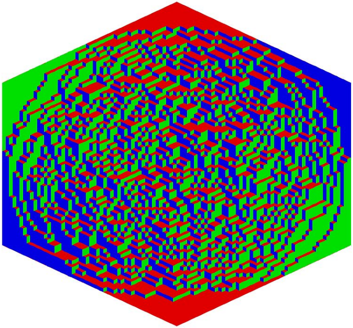
Method 1. (Cohn–Kenyon–Propp)
Solve variational problem for tilings of a generic domain.

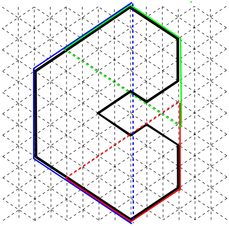
Method 2. (Kenyon–Okounkov)
For simply–connected polygons the solution is found through an algebraic procedure.

$$Q(\xi, 1-\xi) = \mathbf{x}\xi + \boldsymbol{\eta}(1-\xi)$$

Q is a polynomial uniquely defined by a set of algebraic conditions such as degree and tangency to polygon's sides.

How to find slope $(p^{\text{▯}}, p^{\text{▱}}, p^{\diamond})$?



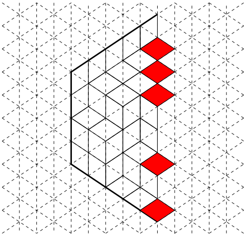


Method 1. (Cohn–Kenyon–Propp)
Solve **variational problem** for tilings of a generic domain.

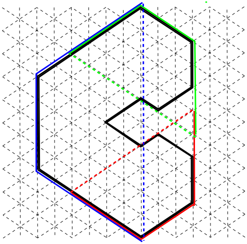
Method 2. (Kenyon–Okounkov)
For **simply–connected polygons** the solution is found through an algebraic procedure.

Method 3. (Bufetov–Gorin-13)
For **trapezoids** the solution is found through a quantization of the Voiculescu R–transform from free probability.

Slope $(p^{\lozenge}, p^{\lozenge}, p^{\diamond})$ for trapezoids.



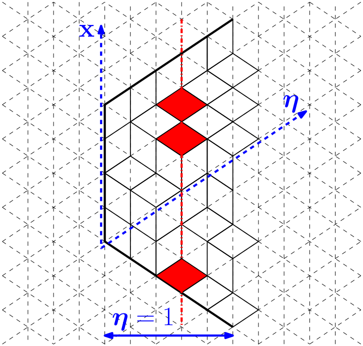
Various origins for the measure on tilings of trapezoid, e.g.:



Setup. We know the asymptotic profile of $p^{\diamond}$ along **the right boundary** of a trapezoid. The distribution of tilings of trapezoid is conditionally uniform given the right boundary (which might be random).

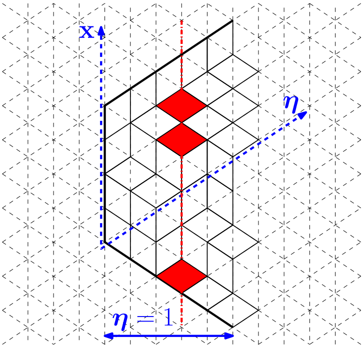
Question. How to find $(p^{\lozenge}, p^{\lozenge}, p^{\diamond})$ inside the trapezoid?

Slope $(p^{\lozenge}, p^{\lozenge}, p^{\diamond})$ for trapezoids.



$\mu[\boldsymbol{\eta}]$, $0 < \boldsymbol{\eta} \leq 1$ is a probability measure on $\mathbb{R}$ with density at a point $\mathbf{x}$ equal to $p^{\diamond}(\boldsymbol{\eta}\mathbf{x} - \boldsymbol{\eta}, \boldsymbol{\eta})$

Slope $(p^{\lozenge}, p^{\lozenge}, p^{\lozenge})$ for trapezoids.



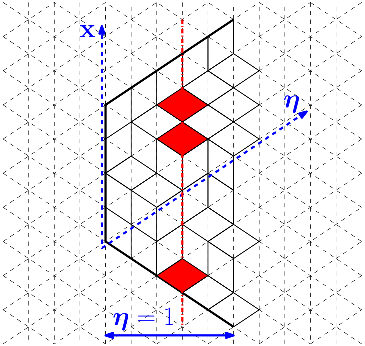
$\mu[\boldsymbol{\eta}]$, $0 < \boldsymbol{\eta} \leq 1$ is a **probability** measure on $\mathbb{R}$ with density at a point $\mathbf{x}$ equal to $p^{\lozenge}(\boldsymbol{\eta}\mathbf{x} - \boldsymbol{\eta}, \boldsymbol{\eta})$

$$E_\mu(z) = \exp\left(\int_{\mathbb{R}} \frac{1}{z-x}\mu(dx)\right).$$

$$R_\mu(z) = E_\mu^{(-1)}(z) - \frac{z}{z-1},$$

Deformation (quantization) of the **Voiculescu R transform** from the free probability theory

Slope $(p^{\lozenge}, p^{\lozenge}, p^{\diamond})$ for trapezoids.



$\mu[\eta]$, $0 < \eta \leq 1$ is a probability measure on $\mathbb{R}$ with density at a point $\mathbf{x}$ equal to $p^{\diamond}(\eta\mathbf{x} - \eta, \eta)$

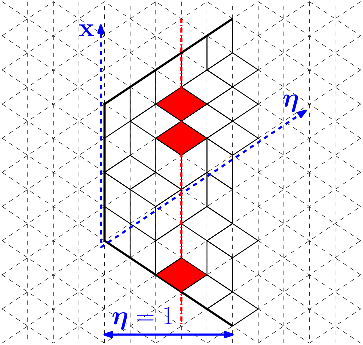
$$E_\mu(z) = \exp\left(\int_{\mathbb{R}} \frac{1}{z-x}\mu(dx)\right).$$

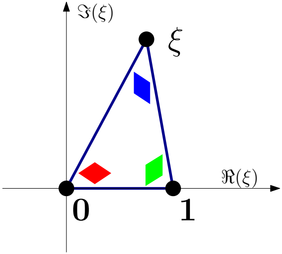
$$R_\mu(z) = E_\mu^{(-1)}(z) - \frac{z}{z-1},$$

Theorem. (Bufetov–Gorin-13) If $(p^{\lozenge}, p^{\lozenge}, p^{\diamond})$ describes the Law of Large Numbers for Gibbs measures on tilings of trapezoids, then

$$R_{\mu[\eta]}(z) = \frac{1}{\eta} R_{\mu[1]}(z).$$

Slope $(p^{\lozenge}, p^{\lozenge}, p^{\diamond})$ for trapezoids.





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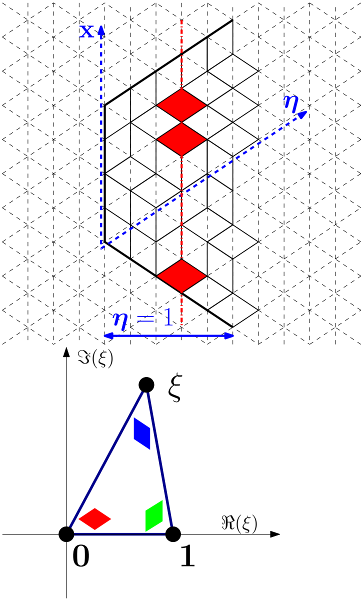
$$E_\mu(z) = \exp\left(\int_{\mathbb{R}} \frac{1}{z - x}\mu(dx)\right).$$

Corollary. (Bufetov–Gorin-13) For tilings of trapezoids also

$$\xi(\boldsymbol{\eta}\mathbf{x} - \boldsymbol{\eta}, \boldsymbol{\eta}) = E_{\mu[\boldsymbol{\eta}]}\left(\mathbf{x} - 0\mathbf{i}\right)$$

Slope $(p^{\lozenge}, p^{\lozenge}, p^{\lozenge})$ for trapezoids.



$\mu[\boldsymbol{\eta}]$, $0 < \boldsymbol{\eta} \leq 1$ is a probability measure on $\mathbb{R}$ with density at a point $\mathbf{x}$ equal to $p^{\lozenge}(\boldsymbol{\eta}\mathbf{x} - \boldsymbol{\eta}, \boldsymbol{\eta})$

$$E_\mu(z) = \exp\left(\int_{\mathbb{R}} \frac{1}{z - x}\mu(dx)\right).$$

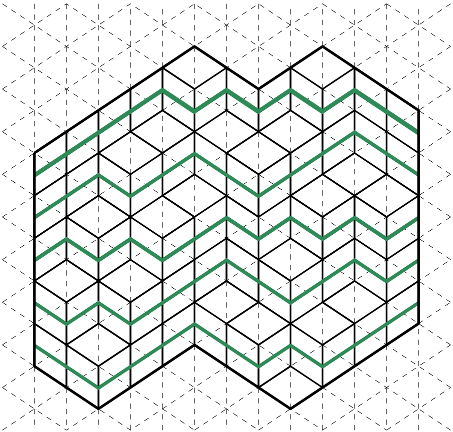
Corollary. (Bufetov–Gorin-13)
For tilings of trapezoids also

$$\xi(\boldsymbol{\eta}\mathbf{x} - \boldsymbol{\eta}, \boldsymbol{\eta}) = E_{\mu[\boldsymbol{\eta}]}(\mathbf{x} - 0\mathbf{i})$$

Angle of red lozenge is clear.
Others are **very mysterious**.

Noncolliding random walks

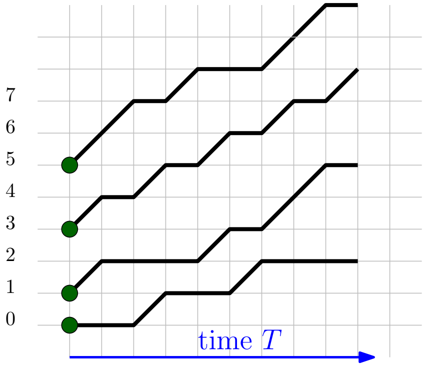
Natural ways to produce lozenge tilings of **infinite** domains?



One of them is to interpret tiling as a collection of paths.

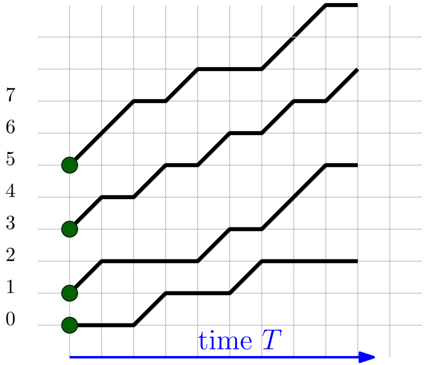
Noncolliding random walks

We drop out the domain constraints.



- N independent simple random walks
- probability of jump p
- started at *arbitrary* lattice points
- conditioned **never to collide**

Noncolliding random walks



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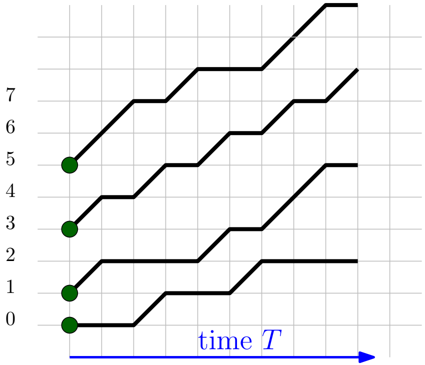
The previous discussion **predicts** bulk universality as $N \to \infty$.

Noncolliding random walks



- N independent simple random walks
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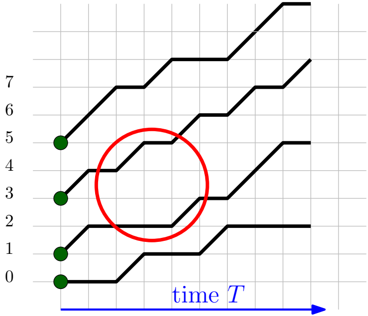
The previous discussion **predicts** bulk universality as $N \to \infty$.

Yet for finite T this **can not hold**. How large should T be?

Noncolliding random walks



- N independent simple random walks
- probability of jump p
- started at *arbitrary* lattice points
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Theorem. (Gorin–Petrov-16) Suppose that as $N \to \infty$ in the initial configuration $a_i(N)$, near point $x \cdot N$, the density of particles is bounded away from 0 and 1, and the configuration is *balanced*. Then for $T \ll N$, $T \to \infty$, the point process of lozenges near (xN, T) converges to a **translation invariant ergodic Gibbs measure** on tilings of plane of an explicit slope.

Noncolliding random walks

More details for $x = 0$.

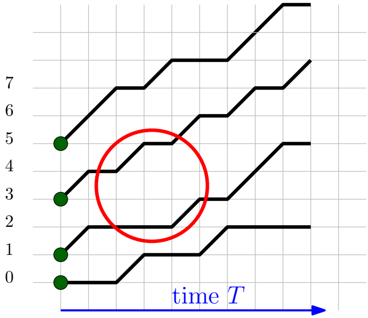
Assumption 1. There exist scales $\mathrm{D} = \mathrm{D}(N)$ satisfying $\mathrm{D}(N) \ll T(N)$ and $\mathrm{Q} = \mathrm{Q}(N)$ satisfying $T(N) \ll \mathrm{Q}(N) \ll N$, and absolute constants $0 < \rho_\bullet, \rho^\bullet < 1$, such that in every segment of length $\mathrm{D}(N)$ inside $[-\mathrm{Q}(N), \mathrm{Q}(N)]$ there are at least $\rho_\bullet \mathrm{D}(N)$ and at most $\rho^\bullet \mathrm{D}(N)$ points of the initial configuration $\mathfrak{A}(N)$.

Assumption 2. For $\delta > 0$, $\mathrm{R} > 0$ and all N large enough one has

$$\left| \sum_{i:\ \mathrm{R}\cdot T(N) \le |a_i(N)| \le \delta \cdot N} \frac{1}{a_i(N)} \right| \le A_{\mathrm{R},\delta},$$

Theorem. (Gorin–Petrov-16) Then for $T(N) \ll N$, $T(N), N \to \infty$, the point process of lozenges near $(0, T(N))$ converges to a translation invariant ergodic Gibbs measure on tilings of plane of an explicit slope.

Noncolliding random walks



- N independent simple random walks
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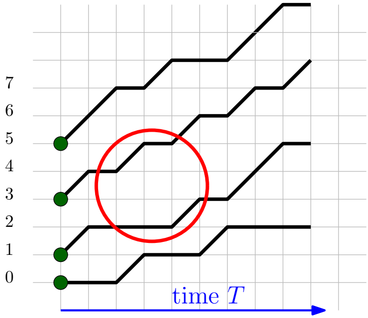
Theorem. (Gorin–Petrov-16) Suppose..., then for $T \ll N$, the lozenges near (xN, T) converge to a **translation invariant ergodic Gibbs measure** on tilings of plane.

E.g. for initial configuration $a_i(N)$:

- $a_i(N) = \lfloor N * f(i/N) \rfloor$, smooth f with $f' > 1$, $T = N^\gamma$,

Noncolliding random walks



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E.g. for initial configuration $a_i(N)$:

- $a_i(N) = \lfloor N * f(i/N) \rfloor$, smooth f with $f' > 1$, $T = N^\gamma$, or
- Particles/holes form i.i.d. Bernoulli sequence of parameter q.

Dyson Brownian Motion

Noncolliding random walks have a famous continuous analogue.

Dyson Brownian Motion

Noncolliding random walks have a famous **continuous** analogue.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

Let X be $N \times N$ matrix of i.i.d. **complex** Brownian motions.

Theorem. (Dyson-62 and others) The eigenvalues of $\frac{X+X^*}{2}$ form a Markov process known as **Dyson Brownian Motion**:
N independent Brownian motions conditioned to never collide.

Dyson Brownian Motion

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(Dyson-62) predicted that local statistics in DBM become universal after very short times (before global limit shape changes).

Described by continuous sine–process (like discrete sine in tilings)

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Complete proof only recently (Landon–Yau-15, Erdos–Schnelli-15).

Critical ingredient for proofs of **universality in Random Matrices**.

Dyson Brownian Motion

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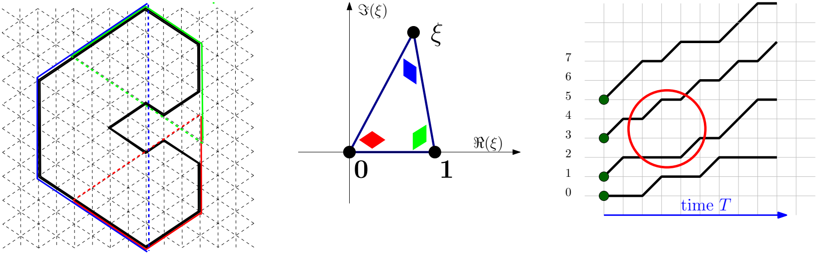
Critical ingredient for proofs of **universality in Random Matrices**.

Our result is a **discrete analogue** of Dyson's conjecture.

Ingredients of proofs

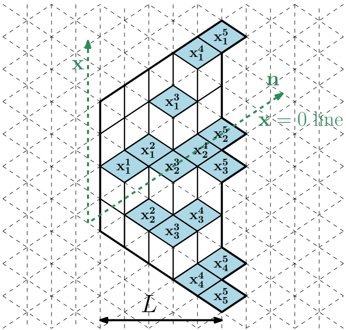


Partial universality result for bulk local limits

- Lozenge tilings of domains covered by trapezoids.
- Non-colliding random walks at short times

The results are based on determinantal structure.

Ingredients of proofs



For L–tuple $(\mathbf{t}_1 > \mathbf{t}_2 > \dots \mathbf{t}_L)$, let $\{x_i^j\}$, $1 \le i \le j \le L$ be horizontal lozenges of uniformly random lozenge tiling with positions $\mathbf{t}$ on the right boundary

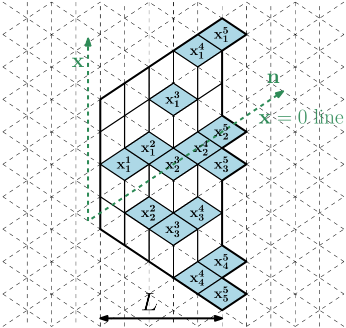
Theorem. (Petrov-2012) For any collection of distinct pairs $(x(1), n(1)), \dots, (x(k), n(k))$

$$\mathrm{P}\left[x(i) \in \{x_1^{n(i)}, x_2^{n(i)}, \dots, x_j^{n(i)}\},\ i = 1, \dots, k\right] = \det_{i,j=1}^{k} \left[K(x(i), n(i); x(j), n(j))\right]$$

$$K(x_1, n_1; x_2, n_2) = -\mathbf{1}_{n_2 < n_1} \mathbf{1}_{x_2 \le x_1} \frac{(x_1 - x_2 + 1)_{n_1 - n_2 - 1}}{(n_1 - n_2 - 1)!} + \frac{(L - n_1)!}{(L - n_2 - 1)!}$$

$$\times \frac{1}{(2\pi \mathbf{i})^2} \oint_{\mathcal{C}(x_2, \dots, \mathbf{t}_1 - 1)} dz \oint_{\mathcal{C}(\infty)} dw \frac{(z - x_2 + 1)_{L - n_2 - 1}}{(w - x_1)_{L - n_1 + 1}} \frac{1}{w - z} \prod_{r=1}^{L} \frac{w - \mathbf{t}_r}{z - \mathbf{t}_r},$$

Ingredients of proofs



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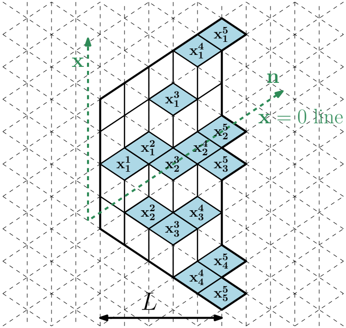
Observation. (G.-16) the bulk limit of $K(\cdot)$ depends only on the asymptotic limit shape of $\mathbf{t}$. This allows to pass from deterministic to random $\mathbf{t}$ and prove bulk universality.

Ingredients of proofs



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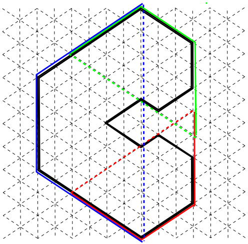
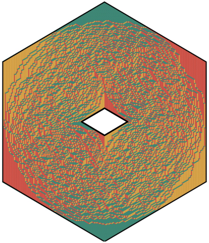
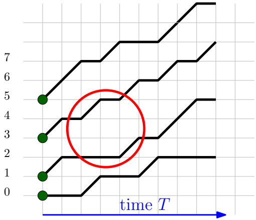
Observation. (Gorin-Petrov-16) There is a limit from trapezoids to **noncolliding Bernoulli random walks** with arbitrary initial conditions. This paves a way for the analysis of the latter.

Summary





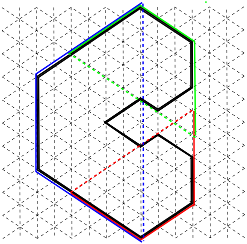


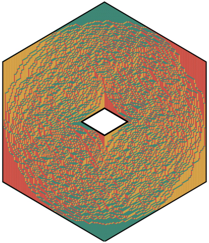
Universal bulk local limits:

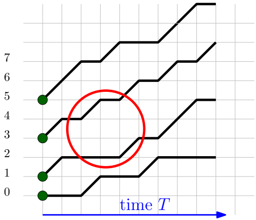
- For lozenge tilings “near” straight boundaries of domains
- For noncolliding Bernoulli random walks at short times [proving a discrete analogue of the Dyson’s (ex-)conjecture]

Key tool: double contour integral for the correlation kernel.

Summary







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How do we extend **universality** further?