Heat kernel measures on random metrics

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Random surfaces

This talk is about ‘random metrics’ on a surface M. In principle we would like to define probability measures on the infinite dimensional space $\mathcal{K}_\omega$ of metrics of fixed area 2π in a fixed conformal class $[\omega_0]$ on a Riemann surface M as limits

$$\int_{\mathcal{K}_\omega} F(g) e^{-S(g)} \mathcal{D}g := \lim_{k\to\infty} \int_{\mathcal{B}_k} F_k(g) e^{-S_k(g)} \mathcal{D}_k g \qquad (1)$$

of integrals over finite dimensional spaces $\mathcal{B}_k$ of *Bergman metrics*. In practice we study the asymptotics of the finite dimensional integrals.

In this talk, $e^{-S_k(g)} \mathcal{D}_k g$ will be a “heat kernel measure”.

Kähler manifold

For simplicity we set $M = \mathbb{CP}^1 = S^2$. We fix a Kaehler form on M, i.e an area form $\omega_0 = dA = \sqrt{g}dx$. The natural choice is the round area form.

The infinite dimensional space of Kaehler metrics in the class of ω_0 is

$$\mathcal{K}_\omega = \{\omega_\varphi = \omega_0 + i\partial\bar{\partial}\varphi > 0, \varphi \in C^\infty(M)\}.$$

It is the same as the set $Conf(g_0, A)$ of metrics of fixed area $A = \int_M \omega_0$ which are conformal to $g_0(X, Y) = \omega_0(JX, Y)$, where J is the complex structure. Fix $A = 2\pi$.

One way to parametrize metrics in $Conf(g_0, A)$ is the Liouville form $e^\sigma g_0$. The way we choose is by Kaehler potentials φ. That is, we write

$$\omega_\varphi := \omega_0 + i\partial\bar{\partial}\varphi = e^u dA$$

or

$$(1 - \Delta_0\varphi) = e^u.$$

Space $\mathcal{B}_k$ of Bergman metrics of degree k

It is difficult to define and study probability measures on an infinite dimensional space $\mathcal{K}_\omega$. Our approach is to approximate $\mathcal{K}_\omega$ by finite dimensional subspaces

$$\mathcal{B}_k \subset \mathcal{K}_\omega$$

of metrics called Bergman metrics. As k varies they become dense in $\mathcal{K}_\omega$ in a strong sense. It is analogous to approximating any smooth or continuous function by polynomials of degree k.

Space $\mathcal{B}_k$ of Bergman metrics of degree k

Bergman metrics are defined by holomorphic embeddings

$$\iota_{\mathcal{S}} : M \to \mathbb{CP}^{N_k}$$

of M into a high dimensional projective space. The metric induced on $\iota_{\mathcal{S}}(M)$ is the Bergman metric.

Holomorphic embeddings are constructed using holomorphic sections $s \in H^0(M, L^k)$ of powers $L^k \to M$ of an ample holomorphic line bundle over M. Here $c_1(L) = \mathcal{K}_\omega$

Metric-matrix correspondence

We will see that every Bergman metric corresponds to a positive Hermitian matrix and vice-versa. A positive Hermitian matrix corresponds to an inner product on $H^0(M, L^k)$ (once one basis is chosen). The space of inner products is denoted by $\mathcal{I}_k$. Thus,

$$\mathcal{B}_k \simeq \mathcal{I}_k \simeq GL(N_k, \mathbb{C})/U(N_k) = \mathcal{P}_{N_k}.$$

Heat kernel measures

The heat kernel measures are defined by

$$d\mu_k^t(P) := p_k(t, I, P)dV(P), \tag{2}$$

where $dV(P)$ is Haar measure, $p_k(t, P_1, P_2)$ is the heat kernel of the symmetric space $\mathcal{P}_{N_k}$ and I is the identity matrix. Under the matrix-metric identification $\mathcal{B}_k \simeq \mathcal{P}_{N_k}$ the identity matrix corresponds to the background metric ω_{φ_I} and the heat kernel measure is transported to $\mathcal{B}_k$, with the identity matrix identified with the background metric. The measure is invariant under the action of the unitary group $U(N_k)$, hence of the choice of the basis of sections used to identify metrics and matrices.

Brownian motion on positive Hermitian matrices

The heat kernel measure $\mu_{k,t}$ is the probability measure on $\mathcal{B}_k$ induced by Brownian motion on $\mathcal{P}_{N_k}$ starting at the identity I up to time t. The heat kernel measure is almost canonical, the only choices being the time t and the background metric ω_{φ_I} used to make the identification and to start the Brownian motion.

Haar measure on the symmetric space of positive Hermitian matrices

We refer to the matrix decomposition $P = U^\dagger e^\Lambda U$ for $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$, and $U \in U(N)$ as 'polar coordinates' on $\mathcal{P}_N$.

The CK (Cartan-Killing) metric is given by

$$ds^2 = \mathrm{Tr}(P^{-1} dP)^2 \qquad (3)$$

for $P \in SL(N, \mathbb{C})/SU(N)$. This metric is bi-invariant under the action of $SL(N, \mathbb{C})$.
The associated volume form dV on the symmetric space $SL(N, \mathbb{C})/SU(N)$ of positive Hermitian matrices with det $P = 1$ is the bi-invariant Haar measure,

$$dV = \delta\left(\sum_{j=1}^{N} \lambda_j\right) \Delta^2(e^\lambda) \prod_{j=1}^{N} d\lambda_j \cdot \frac{[dU]}{[dU_{U(1)^N}]}, \qquad (4)$$

where $[dU]$ is the standard Haar measure on unitary group.

Heat kernel measure on $SL(N,\mathbb{C})/SU(N)$

The heat kernel on $SL(N,\mathbb{C})/SU(N)$ with respect to the standard CK (Cartan-Killing) metric is given in 'polar coordinates' (λ, U) on $\mathcal{P}_N$ by

$$d\mu_t = g_t(\lambda)dV = C(t,N)\frac{\Delta(\lambda)}{\Delta(e^\lambda)}e^{-\frac{1}{4t}\sum_{j=1}^N \lambda_j^2}dV. \qquad (5)$$

Here, $\Delta(\lambda) = \prod_{i<j}(\lambda_j - \lambda_i)$ is the standard Vandermonde determinant. Recall

$$dV = \delta\left(\sum_{j=1}^N \lambda_j\right)\Delta^2(e^\lambda)\prod_{j=1}^N d\lambda_j \cdot \frac{[dU]}{[dU_{U(1)^N}]}, \qquad (6)$$

Note that one factor of $\Delta(e^\lambda)$ cancels.

Normalization constant

The normalization constant $C(t,N)$ is given by

$$C(t,N) = \frac{\sqrt{N}}{2\pi(\sqrt{4\pi t})^{N^2-1}} e^{-\frac{t}{12}N(N^2-1)}. \qquad (7)$$

The factor $e^{-\frac{t}{12}N(N^2-1)}$ is $e^{-t||\delta_N||^2}$ and arises because $||\delta_N||^2$ is the bottom of the spectrum of the Laplacian. Thus,

$$d\mu_t = \frac{\sqrt{N}e^{-\frac{t}{12}N(N^2-1)}}{2\pi(\sqrt{4\pi t})^{N^2-1}} \delta\left(\sum_{j=1}^{N} \lambda_j\right) \Delta(\lambda)\Delta(e^{\lambda})e^{-\frac{1}{4t}\sum_{j=1}^{N}\lambda_j^2} \prod_{j=1}^{N} d\lambda_j dU \quad (8)$$

for the heat kernel measure on $SL(N,\mathbb{C})/SU(N)$ with respect to the CK metric.

Observables

The main 'observables', i.e. geometric quantities, are the *area statistics*

$$X_U(\omega) = \int_U \omega \tag{9}$$

measuring the area of an open set $U \subset M$ with respect to the random area form $\omega_\varphi \in \mathcal{B}_k$.

The mean is determined by the 1-point function

$$K_{1,k}(t,z) = \int_{\mathcal{B}_k} \varphi_P(z) d\mu_{t,k}(P)$$

and the variance is determined by the two-point function

$$K_{2,k}(z_1, z_2) := \mathbf{E}_k\, \varphi_P(z_1)\varphi_P(z_2). \tag{10}$$

Calculation of the two point function

As mentioned in the introduction, the one and two-point functions are the data required to study the mean and variance of the area random variables X_U. Evidently,

$$\mathbf{E}_k X_U = \int_U \mathbf{E}_k \, \omega,$$

$$\mathbf{Var}(X_U) = \iint_U \mathbf{E}_k \left[\omega(z_1)\omega(z_2)\right] - \iint_U \mathbf{E}_k[\omega(z_1)]\mathbf{E}_k \left[\omega(z_2)\right].$$

The integrands are the one- and two-point correlation functions.

Variance of the area

Thus, $I_{2,k}(z_1, z_2)$ is the bi-potential of the variance of the area forms (or Kähler metrics in higher dimensions) relative to the exterior tensor product $\omega_0 \boxtimes \omega_0$,

$$\mathbf{Var}\big(\omega_\varphi\big) = \mathbf{E}\big(\omega_\varphi \boxtimes \omega_\varphi\big) - \mathbf{E}\big(\omega_\varphi\big) \boxtimes \mathbf{E}\big(\omega_\varphi\big) = \mathbf{E}\big(\omega_\varphi \boxtimes \omega_\varphi\big) - \omega_0 \boxtimes \omega_0, \tag{11}$$

in the sense that

$$\mathbf{Var}\big(\omega_\varphi\big) = \frac{1}{k^2}(i\partial\bar{\partial})_z\,(i\partial\bar{\partial})_w\, I_{2,k}(z,w), \;. \tag{12}$$

From eigenvalues to the Berezin kernel

It is easy to see that $\int_{\mathcal{B}_k} \varphi_P(z) d\mu_{k,t}(P) = \varphi_0(z)$, the background potential. The two-point function is much harder.
The eigenvalue density $\mathcal{F}_{\mathcal{B}_k}(e^{\lambda})$ induces a function $\mathcal{F}_{k,2}(\nu_1, \nu_2)$ on $\mathbb{R}^2_+$, so that the 2-point correlation function has the form,

$$K_{2,k}(z_1, z_2) := \mathbf{E}_k \, \varphi_P(z_1)\varphi_P(z_2) = \varphi_I(z_1)\varphi_I(z_2) + \frac{1}{k^2} I_{2,k}(\rho), \quad (13)$$

where

$$\rho(z_1, z_2) = \frac{|B_k(z_1, z_2)|^2}{B_k(z_1, z_1) B_k(z_2, z_2)} \quad (14)$$

is an important invariant of the Bergman kernel $B_k(z_1, z_2)$ of the background metric, known as the Berezin kernel.

Calculation of the two point function

THEOREM

The variance term of the two-point (or pair) correlation function on general projective Kähler manifolds is given by

$$\partial_\rho I_{2,k}(t,\rho) = \frac{2t}{\rho}$$

$$-\frac{e^{-t/2}}{\sqrt{2\pi t}}\frac{\sqrt{1-\rho}}{\rho}\int_{-\infty}^{\infty} d\lambda\,\frac{e^{-\frac{1}{2t}\lambda^2}\cosh\lambda}{\sqrt{\coth^2\lambda-\rho}}\log\frac{\sqrt{\coth^2\lambda-\rho}+\sqrt{1-\rho}}{\sqrt{\coth^2\lambda-\rho}-\sqrt{1-\rho}}.$$

We do not integrate the result because the metric is obtained by differentiating it. The RHS has no k-dependence, except for the variable ρ.

Brownian motion intuition

Heat kernel random metrics are the metrics obtained by starting at the background metric ω_{φ_I} and following a Brownian motion on $\mathcal{P}_{N_k}$ for time t.

The formula for the two-point function reflects the geometry of Brownian motion of the non-positively curved symmetric space $\mathcal{P}_{N_k}$, which is very different from that of Euclidean space. First, due to the non-isotropic nature of Haar measure, the heat measure is concentrated along the $SU(N_k)$-orbit of a distinguished element δ_{N_k}, the half-sum of the positive roots. Second, in the radial direction the heat kernel measure concentrates in a kind of annulus of radius t around the $SU(N_k)$-orbit of δ_{N_k}.

As $t \to \infty$, the heat kernel measure becomes supported on the ideal boundary $\partial_\infty \mathcal{P}_{N_k}$.

Geometry of the heat kernel and Brownian motion

In work of Anker-Setti, it is proved that the mass of the heat kernel concentrates along the exponential image of the $U(N)$-orbit of the δ_N-axis in a small annulus centered at $2|\delta_N|t$.
If we write $H = \mathrm{diag}(\lambda)$, then the Gaussian factor $t^{-(N^2-1)/2}e^{-\frac{||H||}{4t}}$ is similar to the heat kernel of Euclidean space.
But this Gaussian factor must compete with the exponential volume growth factor $\Delta(e^{\lambda})$ and the factor $e^{-\frac{t}{12}N(N^2-1)}$ due to the existence of a spectral gap for Δ. The well-known factor $\Delta(\lambda)$ pushes the eigenvalues of $\log P$ apart. The factor $J(H)$ is bounded by $e^{2\langle \delta_N, \vec{\lambda}\rangle}$ and a simplified expression for the heat kernel is $e^{-t|\delta_N|^2 + \langle \lambda, \rho_N\rangle - \frac{|\lambda|^2}{2t}}$. The maximum of the exponent occurs when $\vec{\lambda} = 2t\delta_N$.

Bergman metrics

We choose a basis of sections $\{s_i(z)\} = \{s_1(z), ..., s_{N_k}(z)\}$ of $H^0(M, L^k)$ which is orthonormal with respect to the reference (background) metric h_0^k on L^k and the corresponding Kähler metric $\omega_0 = -\frac{1}{k} i\partial\bar{\partial} \log h_0^k$ on M

$$\frac{1}{V} \int_M \bar{s}_i(z) s_j(z) h_0^k \frac{\omega_0^n}{n!} = \delta_{ij}, \tag{15}$$

where $n = \dim M$. The Bergman kernel of the background metric is the kernel of the orthogonal projection onto $H^0(M, L^k)$ with respect to the inner product above, and is given by

$$B_k(z_1, z_2) = \sum_{j=1}^{N_k} s_j(z_1)\overline{s_j(z_2)} \tag{16}$$

Given a positive Hermitian matrix $P = P_{ij}$ the associated Bergman metric is,

$$\omega_a(z) = \frac{1}{k}\partial\bar{\partial} \log \bar{s}_i(z) P_{ij} s_j(z). \tag{17}$$

In terms of $A \in GL(N_k, \mathbb{C})$ above, $P = A^\dagger A$.

Berezin kernel

We introduce the Bergman potential as follows

$$\varphi_P = \frac{1}{k} \log \bar{s}_i(z) P_{ij} s_j(z) = \frac{1}{k} \log |\langle e^{\Lambda} Us(z), Us(z) \rangle|^2. \tag{18}$$

The key invariant is the Berezin kernel (14), given in the above notation by

$$\rho = \frac{|\langle s(z_1), s(z_2) \rangle|^2}{|s(z_1)|^2 |s(z_2)|^2}, \tag{19}$$

or in terms of the Bergman kernel

$$\rho = P_k^2(z_1, z_2) := \frac{|B_k(z_1, z_2)|^2}{B_k(z_1, z_1) B_k(z_2, z_2)} . \tag{20}$$

Matrix-metric correspondence

The matrix-metric correspondence Eq. (17) uses a choice of basis $\{s_j\}$ of $H^0(M, L^k)$. Any natural measure on $\mathcal{B}_k$ must be independent of the choice of this basis. We pause to describe such natural measures.

Any Kähler metric $\omega = \omega_0 + i\partial\bar{\partial}\varphi$ in $\mathcal{K}_\omega$ induces an inner product $\mathrm{Hilb}_k(\varphi)$ on $H^0(M, L^k)$ by the rule

$$\langle s_1, s_2 \rangle_{\mathrm{Hilb}_k(\varphi)} = \int_M \bar{s}_1(\bar{z}) s_2(z) h^k \frac{\omega^n}{n!}.$$

Given a background inner product $G_0 = \mathrm{Hilb}_k(\varphi_0)$, any other inner product has the form $\langle s_1, s_2 \rangle_G = \langle P_G s_1, s_2 \rangle_{G_0}$ where P_G is a positive Hermitian operator on $H^0(M, L^k)$ with respect to G_0. It has a well-defined polar decomposition $e^\Lambda U$ where $U \in U(G_0)$ is unitary with respect to G_0. Its eigenvalues are encoded by the diagonal matrix Λ_G and its eigenvectors are encoded by U.

Bases of $H^0(M, L^k)$ and holomorphic embeddings

We let $\mathcal{B}H^0(M, L^k)$ denote the manifold of all bases $\underline{s} = \{s_0, \dots, s_{d_k}\}$ of $H^0(M, L^k)$. Given a basis, we define the Kodaira embedding

$$\iota_{\underline{s}} : M \to \mathbb{CP}^{d_k}, \quad z \to [s_0(z), \dots, s_{d_k}(z)]. \tag{21}$$

A Bergman metric of height k is

$$h_{\underline{s}} := (\iota_{\underline{s}}^* h_{FS})^{1/k} = \frac{h_0}{\left(\sum_{j=0}^{d_k} |s_j(z)|^2_{h_0^k}\right)^{1/k}}, \tag{22}$$

where h_{FS} is the Fubini-Study Hermitian metric on $\mathcal{O}(1) \to \mathbb{CP}^{d_k}$.

Space of Bergman metrics

We then define

$$\mathcal{B}_k = \{h_{\underline{s}}, \;\; \underline{s} \in \mathcal{B}H^0(M, L^k)\}. \tag{23}$$

We use the same notation for the associated space of potentials φ such that $h_{\underline{s}} = e^{-\varphi} h_0$ and for the associated Kähler metrics ω_φ. Given one embedding $\iota_{\underline{s}}$, corresponding to one choice of basis of $H^0(M, L^k)$,

$$\mathcal{B}_k = \{\frac{i}{2}\partial\bar{\partial} \log |A\iota_{\underline{s}}(z)|, \;\; A \in GL(d_k + 1)\}.$$

$\mathcal{I}_k$ = symmetric space of inner products on $H^0(M, L^k)$

Since $U(d_k+1)$ is the isometry group of ω_{FS}, $\mathcal{B}_k \simeq GL(d_k+1)/U(d_k+1)$, $= \mathcal{P}_{N_k}$, the space of positive Hermitian matrices.

$\mathcal{B}_k$ may be identified with the space $\mathcal{I}_k$ of Hermitian inner products on $H^0(M, L^k)$, the correspondence being that a basis is identified with an inner product for which the basis is Hermitian orthonormal.

$Hilb_k : \mathcal{K}_\omega \to \mathcal{I}_k$

Not only is $\mathcal{B}_k \subset \mathcal{K}_\omega$ but also there exist natural maps

$$Hilb_k : \mathcal{K}_\omega \to \mathcal{I}_k,$$

by the rule that a Hermitian metric $h \in \mathcal{K}_\omega$ induces the inner products on $H^0(M, L^k)$,

$$||s||^2_{Hilb_k(h)} = R \int_M |s(z)|^2_{h^k} dV_h, \tag{24}$$

where $dV_h = \frac{\omega_h^m}{m!}$, and where $R = \frac{d_k+1}{Vol(M,dV_h)}$. Also, h^k denotes the induced metric on L^k.

$FS_k : \mathcal{I}_k \to \mathcal{B}_k$

Further, we define

$$FS_k : \mathcal{I}_k \simeq \mathcal{B}_k$$

as follows: an inner product $G = \langle \, , \, \rangle$ on $H^0(M, L^k)$ determines a G-orthonormal basis $\underline{s} = \underline{s}_G$ of $H^0(M, L^k)$ and an associated Kodaira embedding (21) and Bergman metric (26). Thus,

$$FS_k(G) = h_{\underline{s}_G}. \tag{25}$$

The right side is independent of the choice of h_0 and the choice of orthonormal basis.

$$h_{\underline{s}} := (\iota_{\underline{s}}^* h_{FS})^{1/k} = \frac{h_0}{\left(\sum_{j=0}^{d_k} |s_j(z)|^2_{h_0^k}\right)^{1/k}}, \tag{26}$$

where h_{FS} is the Fubini-Study Hermitian metric on $\mathcal{O}(1) \to \mathbb{CP}^{d_k}$.

Density of Bergman metrics (Tian-Yau-Z-(Catlin) expansion...)

A basic fact is that the union

$$\mathcal{B} = \bigcup_{k=1}^{\infty} \mathcal{B}_k$$

of Bergman metrics is dense in the C^∞-topology in the space $\mathcal{H}$. Indeed,

$$\frac{FS_k \circ Hilb_k(h)}{h} = 1 + O(k^2), \tag{27}$$

where the remainder is estimated in $C^r(M)$ for any $r > 0$; left side moreover has a complete asymptotic expansion .

Bergman geodesics

A geodesic in the space of Bergman metrics is defined by moving an embedding $\iota_{\mathcal{S}}(M)$ by a 1 PS e^{tA} of $SL(d_k + 1, \mathbb{C})$. The Bergman metric at time t is the intrinsic metric on M induced by $e^{tA}\mathcal{I}_{\mathcal{S}}(M)$. Its potential has the form

$$\varphi_t(z) = \frac{1}{k} \log |e^{tA}\iota_{\mathcal{S}}(z)|^2.$$

Concentration of mass of the heat kernel

Let $\gamma(t)$ be a positive function with $t^{\frac{1}{2}}\gamma(t) \to \infty$ as $t \to \infty$, and let $R(t)$ be a positive function such that $t^{-\frac{1}{2}}R(t) \to \infty$. Consider the annulus

$$A(2|\delta_N|t - R(t), 2|\delta_N|t + R(t)) := \{H : 2|\delta_N|t - R(t) \leq |H| \leq 2|\delta_N|t + R(t)$$

and consider the solid cone

$$\Gamma(t) = \text{solid cone around the } \delta_N \text{ axis of angle } \gamma(t),$$

and let

$$\Omega(t) = A(2|\delta_N|t - R(t), 2|\rho|t + R(t)) \cap \Gamma(t).$$

Then, according to Anker-Setti,

$$\int_{U(N)\exp\Omega(t)U(N)} d\mu_t \to 1, \quad t \to \infty. \tag{28}$$

Some ideas of the calculation

We have,

$$K_{2,k}(z,w) - \varphi_I(z)\varphi_I(w)$$

$$= \frac{1}{k^2}\mathbf{E}_k\left[\log\frac{\bar{s}(z_1)U^\dagger e^\Lambda U s(z_1)}{|s(z_1)|^2}\log\frac{\bar{s}(z_2)U^\dagger e^\Lambda U s(z_2)}{|s(z_2)|^2}\right]$$

$$= \frac{1}{k^2}\lim_{\tau_1,\tau_2\to 0}\left(I_{2,k}(t,\rho,\tau_1,\tau_2) + C\right.$$

Then $I_{2,k}(t,\rho,\tau_1,\tau_2) =$

$$\iint_0^\infty x_1^{\tau_1-1}x_2^{\tau_2-1}dx_1dx_2\int_{SL(N,\mathbb{C})/SU(N)} e^{-\mathrm{Tr}\, e^\Lambda U\Phi U^\dagger}d\mu_t. \qquad (29)$$

where $\Phi_{jl} = x_1\frac{s_j(z_1)\bar{s}_l(z_1)}{|s(z_1)|^2} + x_2\frac{s_j(z_2)\bar{s}_l(z_2)}{|s(z_2)|^2}$. It has rank 2 with two non-zero eigenvalues given by

$$\varphi_{1,2} = \frac{1}{2}\left(x_1 + x_2 \pm \sqrt{(1-\rho)(x_1-x_2)^2 + \rho(x_1+x_2)^2}\right).$$

Itzykson-Zuber integral formula

The integration over the unitary group can be carried out using the standard Itzykson-Zuber formula. Namely, for any two Hermitian matrices A and B with eigenvalues a_j and b_j

$$\int_{U(N)} \frac{[dU]}{\mathrm{Vol}\, U(N)} \exp\left(\mu \mathrm{Tr} AUBU^\dagger\right)$$

$$= \left(\prod_{p=1}^{N-1} p!\right) \mu^{-N(N-1)/2} \frac{\det\left(e^{\mu a_j b_l}\right)_{1\le j,l\le N}}{\Delta(a)\Delta(b)}.$$

We end up with an integral over λ which is only non-Gaussian in two λ_j's. A somewhat lengthy and complicated calculation leads to the stated formula.

Itzykson-Zuber formula

Applying the HCIZ to the unitary integration we obtain

$$(-1)^{N(N-1)/2}\frac{N!(N-1)!}{(\varphi_1\varphi_2)^{N-2}(\varphi_1-\varphi_2)}\frac{e^{-\varphi_1 e^{\lambda_1}-\varphi_2 e^{\lambda_2}}}{\prod_{j=2}^N(e^{\lambda_1}-e^{\lambda_j})\prod_{l=3}^N(e^{\lambda_2}-e^{\lambda_l})}$$

where we used the fact that the eigenvalue measure is symmetric in λ's. Using the explicit form of the heat kernel measures μ_t:

$$I_{2,k}(t,\rho,\tau_1,\tau_2) = (-1)^{N(N-1)/2} C(t,N)\frac{\mathrm{Vol}\,U(N)}{(2\pi)^N}N!(N-1)!$$

$$\iint_0^\infty \frac{x_1^{\tau_1-1}x_2^{\tau_2-1}dx_1 dx_2}{(\varphi_1\varphi_2)^{N-2}(\varphi_1-\varphi_2)}$$

$$\int_{-\infty}^{\infty}dy\int\prod_{j=1}^N d\lambda_j\,\Delta(\lambda)\Delta_{12}(e^\lambda)$$

$$\exp\left(-\frac{1}{4t}\sum_{j=1}^N\lambda_j^2+iy\sum_{j=1}^N\lambda_j-\varphi_1 e^{\lambda_1}-\varphi_2 e^{\lambda_2}\right),$$

where $\Delta_{12}(e^\lambda)$ excludes e^{λ_1} and e^{λ_2}.

Conclusion of calculation

Except for the variables λ_1, λ_2 the integral is Gaussian (with a polynomial amplitude). A series of tricks leads to the explicit calculation. The conclusion is:

$$\partial_\rho I_{2,k}(t,\rho) = \frac{2t}{\rho}$$

$$-\frac{e^{-t/2}}{\sqrt{2\pi t}} \frac{\sqrt{1-\rho}}{\rho} \int_{-\infty}^{\infty} d\lambda \, \frac{e^{-\frac{1}{2t}\lambda^2} \cosh \lambda}{\sqrt{\coth^2 \lambda - \rho}} \log \frac{\sqrt{\coth^2 \lambda - \rho} + \sqrt{1-\rho}}{\sqrt{\coth^2 \lambda - \rho} - \sqrt{1-\rho}}.$$

Scaling asymptotics

In all of the regimes, the key to finding the scaling asymptotics is to work out the behavior of

$$A(t,\rho) := \frac{1}{\sqrt{\coth^2 t - \rho}} \log \frac{\sqrt{\coth^2 t - \rho} + \sqrt{1-\rho}}{\sqrt{\coth^2 t - \rho} - \sqrt{1-\rho}} \qquad (30)$$

as $k \to \infty$ where t may depend on k. As will be seen below, the factors of $(2\pi t)^{-\frac{1}{2}} e^{-t/2}$ in front of the integral are always cancelled, leaving the prefactor $\frac{\sqrt{1-\rho}}{\rho}$ in front of the integral and the first term $\frac{2t}{\rho}$.

Large k limit for fixed t

Note that $\rho \to 0$ as $k \to \infty$ off the diagonal. Expanding at small ρ, we get

$$\frac{1}{\sqrt{2\pi t}} e^{-t/2} \int_{-\infty}^{\infty} d\lambda\, e^{-\frac{1}{2t}\lambda^2} \frac{\cosh \lambda}{\sqrt{\coth^2 \lambda - \rho}} \log \frac{\sqrt{\coth^2 \lambda - \rho} + \sqrt{1-\rho}}{\sqrt{\coth^2 \lambda - \rho} - \sqrt{1-\rho}}$$

$$= 2t + \mathcal{O}(\rho),$$

and the first term here cancels the first term. Thus in the regime when we hold (z_1, z_2) fixed then $\rho \to 0$, we get

$$I_{2,k}(t,\rho) \simeq a_0(t) + a_1(t)\rho + \cdots ,$$

where $a_0, a_1, \ldots$ are constants independent of ρ.

2-point function

To obtain the 2-point correlation function of the Kähler metric, we then take three more derivatives. The constant a_0 does not contribute to the answer and we see that the two point correlation function is the free background term $\omega_{\varphi_I}(z_1)\omega_{\varphi_I}(z_2)$ plus a term exponentially decaying off the diagonal like $C_2(t)e^{-kD_I(z_1,z_2)}$.

The limit as $t \to \infty$ for fixed k

Now we apply steepest descent to the second integral as $t \to \infty$, and keeping k, N_k fixed. We obtain

$$\partial_\rho I_{2,k}(\infty, \rho) := \lim_{t\to\infty} \partial_\rho I_{2,k}(t, \rho) = \lim_{t\to\infty} \frac{2t}{\rho} - \frac{1}{\rho}(2t + \log(1-\rho) + \mathcal{O}(1/t$$

Thus we have,

$$I_{2,k}(\infty, \rho) = \mathrm{Li}_2(\rho) + \mathrm{const.} \tag{31}$$

This is the same as the two point correlation function between zeros of Gaussian random holomorphic sections.

The metric scaling limit with $t \to t\epsilon_k^{-2}$

The goal now is to evaluate $I_{2,k}(\epsilon_k^{-2}t, \rho)$ asympotically as $k \to \infty$. This scaling keeps the d_k-balls of uniform size as $k \to \infty$ with respect to the limit Mabuchi metric. Thus, as k changes the Brownian motion with respect to g_k probes distances of size t from the initial metric ω_0 for all k.

We get,

$$\partial_\rho I_{2,k}(\epsilon_k^{-2}t, \rho) \simeq -\frac{\log(1-\rho)}{\rho}.$$

The first term cancels the singularity and the second term has a log singularity.

Continuation of the calcuation

The scaling asymptotics of this kernel near the diagonal may be described as follows. Let $z \in M$. Then

$$P_k\left(z + \frac{u}{\sqrt{k}}, z + \frac{v}{\sqrt{k}}\right) = e^{-\frac{1}{2}|u-v|^2}[1 + R_k(u,v)]\ , \qquad (32)$$

It follows that the scaling asymptotics of the variance term for random metrics is given by

$$I_{2,k}(\epsilon_k^{-2} t, P_k(z, z + u/\sqrt{k})) \simeq \log(1 - e^{-|u|^2}), \qquad (33)$$

just as in the limit as $t \to \infty$ first.

Random zeros as the boundary of $\mathcal{B}_k$

The result of is essentially the same as the theory of zeros of random holomorphic sections. As $t \to \infty$, the mass of the heat kernel gets concentrated on a part of the boundary of $\mathcal{P}_{N_k}$ corresponding to 'singular metrics' given by zero sets of holomorphic sections.

The boundary relevant to the heat kernel measures and their $t \to \infty$ limit is best stated in terms of the Bergman metrics themselves and their limits along geodesic rays of $\mathcal{P}_{N_k}$.

The weak* compactification of $\mathcal{B}_k$ is $\mathcal{B}_k \cup \partial\mathcal{B}_k$ where $\partial\mathcal{B}_k$ is the set of limit points (i.e. endpoints) of the Bergman metrics along Bergman geodesic rays $\omega_k(s)$.

Bergman geodesic rays to the boundary

A geodesic ray in the space of Bergman potentials is a one-parameter family of metrics whose potentials have the form,

$$\beta_t = \frac{1}{k} \log \sum_j e^{t\lambda_j} |s_j^U(z)|^2,$$

where in $SL(N_k, \mathbb{C})/SU(N_k)$ the ray starts at the origin and has initial vector (U, Λ).
We note that

$$\frac{1}{k} \log \sum_j e^{t\lambda_j} |s_j^U(z)|^2 = \frac{t\lambda_{\max}}{k}$$

$$+\frac{1}{k} \log \left(|s_{\max}^U(z)|^2 + \sum_{j \neq \max} e^{t(\lambda_j - \lambda_{\max})} |s_j^U(z)|^2 \right).$$

Here, $\lambda_{\max}$ is the largest of the λ_j and $s_{\max}^U$ is the corresponding section.

Continuation of the calcuation

For any Bergman geodesic ray,

$$\lim_{t\to\infty} \left\| \frac{1}{k} \log \left(|s^U_{\max}(z)|^2 + \sum_{\lambda_j \neq \max} e^{t(\lambda_j - \lambda_{\max})} |s^U_j(z)|^2 \right) \right.$$

$$\left. - \frac{1}{k} \log |s_{\max}(z)|^2 \right\|_{L^1(M)} \to 0.$$

Continuation of the calcuation

PROPOSITION

The weak limits of Bergman metrics along geodesic rays are generically given by the normalized zero distributions of holomorphic sections of L^k, i.e. as $t \to \infty$,

$$\frac{1}{k}\partial\bar{\partial}\log\sum_{j} e^{t\lambda_j}|s_j^U(z)|^2 \to \frac{1}{k}\partial\bar{\partial}\log|s|^2.$$

If the highest weight has multiplicity r, then the limit is $\frac{1}{k}\partial\bar{\partial}\log\sum_{j=1}^{r}|s_j|^2$ where $1 \leq r \leq n$ and $\{s_j\}_{j=1}^{r}$ is any set of sections in $H^0(M, L^k)$.

Relation to zeros of holomorphic sections

The resulting two point function is the same as for the pair correlation function of zeros of Gaussian random holomorphic sections

$$d\gamma(s) = \frac{1}{\pi^m} e^{-|c|^2} dc \,, \qquad s = \sum_{j=1}^{n} c_j s_j \,, \tag{34}$$

on L^k, where $\{s_j\}$ is an orthonormal basis and dc is $2n$-dimensional Lebesgue measure. This Gaussian is characterized by the property that the $2n$ real variables $\Re c_j, \Im c_j$ $(j = 1, \ldots, n)$ are independent Gaussian random variables with mean 0 and variance $\frac{1}{2}$; i.e.,

$$\mathbf{E} c_j = 0, \quad \mathbf{E} c_j c_k = 0, \quad \mathbf{E} c_j \bar{c}_k = \delta_{jk} \,.$$

The current of integration Z_s over the zeros of one section is given by the Poincaré-Lelong formula

$$Z_s = \frac{\sqrt{-1}}{\pi} \partial\bar{\partial} \log |f| = \frac{\sqrt{-1}}{\pi} \partial\bar{\partial} \log \left\| s^N \right\|_h + \varphi_I(z) \,, \tag{35}$$

Relation to random zeros

As mentioned above, the limit as $t \to \infty$ of random Bergman metrics along the rays above must be random singular metrics, and we claim that the limit ensemble is equivalent to the Gaussian one. Indeed, the limit measure is $U(N_k)$-invariant and there exists just one such measure up to equivalence, namely the Gaussian measure above. This explains why the bipotential Q_k for random zeros is the same as $I_{2,k}(\rho)$.

This is precisely the formula for $I_{2,k}(\infty, \rho)$ for the limit 2 point function of the heat kernel ensemble as $t \to \infty$, corroborating that this limit ensemble is that of random zeros of sections.