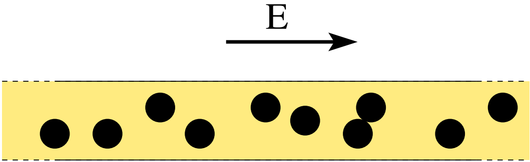
Inhomogeneous Multispecies TASEP on a ring

Luigi Cantini

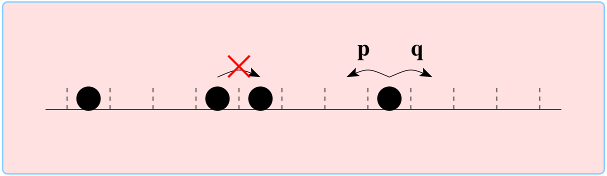
Quantum integrable systems,
conformal field theories and stochastic processes
Cargèse 2016

The Asymmetric Simple exclusion Process (ASEP)

Particles propagating under the effect of an external field

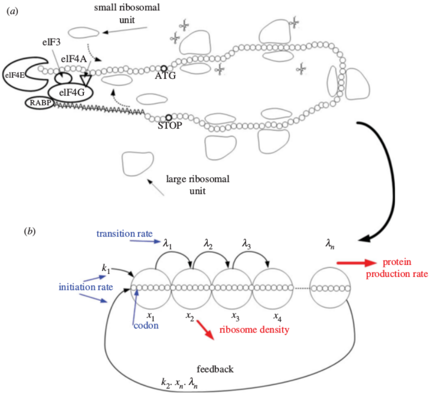


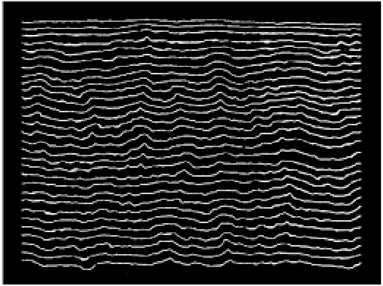
No detailed balance: Macroscopic particle current

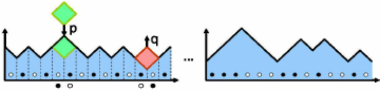


- One dimensional lattice
- Exclusion: at most one particle per site
- Asymmetric: jump rate to the right q, to the left p

Applications

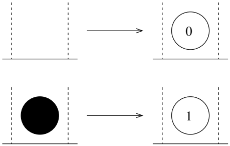




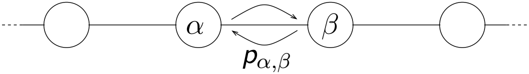


Multispecies generalization: M-ASEP

One can think at empty spaces and particles as two species of particles (0 and 1) that exchange their positions



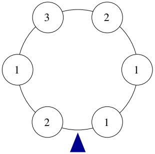
It is then natural to allow any integer label α for different species of particles and assume that the rates $p_{\alpha,\beta}$ for a local exchange $\alpha \leftrightarrow \beta$ depends on the species involved.



Multispecies ASEP on a ring

If we put the M-ASEP on a ring $\mathbb{Z}/L\mathbb{Z}$, a state of this system is just a periodic word w of length $L(w) = L$, $w_i = w_{i+L}$.

$$w = \{1, 1, 2, 3, 1, 2\}$$



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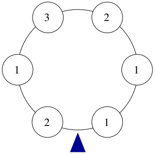

The dynamics conserves the total number of particles of a given species. We denote the *species content* of a configuration w by

$$\mathbf{m}(w) = \{\dots, m_\alpha(w), m_{\alpha+1}(w), \dots\} \in \mathbb{N}^{\mathbb{Z}}$$

which means that we have $m_\alpha(w)$ particles of species α

$$\sum_{\alpha=\mathbb{Z}} m_\alpha(w) = L(w)$$

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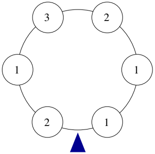

Up to equivalence we can assume $m_i \geq 1$ for $1 \leq i \leq r$ and zero otherwise.
For the example above we have

$$\mathbf{m}(w) = \{m_1 = 3, m_2 = 2, m_3 = 1\}, \qquad L = 6$$

Master equation

The master equation for the time evolution of the probability of a configuration is

$$\frac{d}{dt}P_w(t) = \sum_{w' | w' \to w} \mathcal{M}_{w,w'} P_w(t) - \sum_{w' | w \to w'} \mathcal{M}_{w',w} P_w(t)$$

$$\frac{d}{dt}P(t) = \mathcal{M} P(t)$$

An important remark here is that *the Markov matrix $\mathcal{M}$ is the sum of local terms* acting on $V_{\mathbf{m}}$, the vector space with a basis labeled by configurations of content $\mathbf{m}$

$$\mathcal{M} = \sum_{i=1}^{L} M^{(i)}, \qquad M^{(i)} = \sum_{1 \leq \alpha \neq \beta \leq N} p_{\alpha,\beta} M^{(i)}_{\alpha,\beta}$$

In this talk I will focus on the stationary probability $\mathcal{M}P = 0$

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M-TASEP: positivity conjectures

The case that we are interested in is

$$p_{\alpha,\beta} = \begin{cases} 0 & \text{for} \quad \alpha \geq \beta \\ \tau_\alpha + \nu_\beta & \text{for} \quad \alpha < \beta \end{cases}$$

We'll see later where this choice comes from.

For some content $\mathbf{m}$, call w^* the weackly increasing word

$$w_i \leq w_{i+1}$$

and normalize the stationary "probability"

$$\psi_{w^*} = \chi_{\mathbf{m}}(\tau, \nu) := \prod_{\alpha<\beta} (\tau_\alpha + \nu_\beta)^{(\beta-\alpha-1)(m_\alpha+m_\beta-1)}$$

Positivity Conjecture [Lam & Williams, LC]

The components $\psi_w(\tau, \nu)$ are prime polynomials in τ, ν with positive integer coefficients

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Combinatorics: $\nu_\alpha = 0$ and multiline queues

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- A multiline queue (Ferrari et al.) of type $\mathbf{m}$ is a $\mathbb{Z} \times L$ array ($L = \sum m_i$), which has $\sum_{j \leq i} m_j$ particles on the i-th row.
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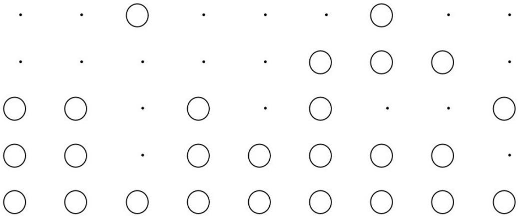

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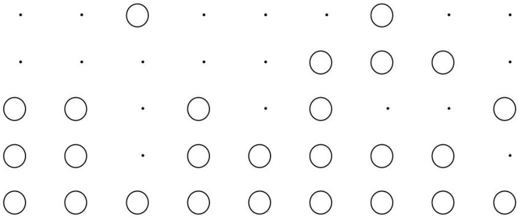

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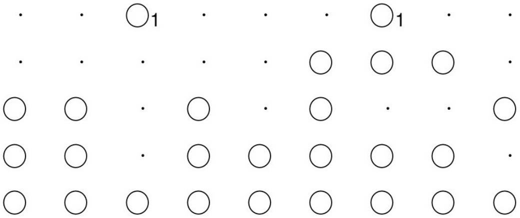

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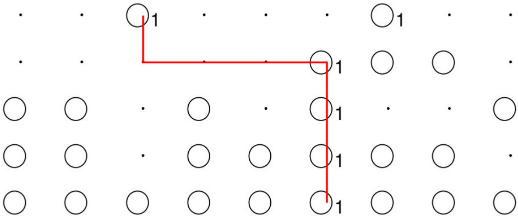

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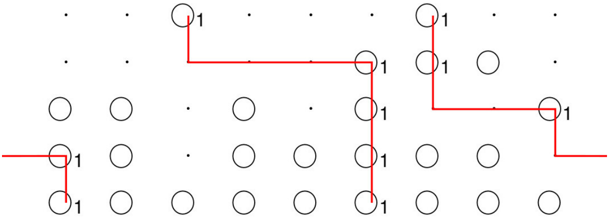

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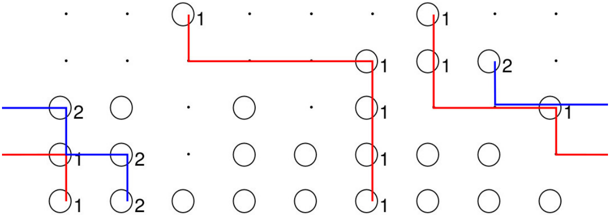

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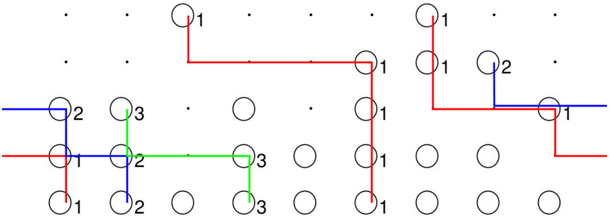

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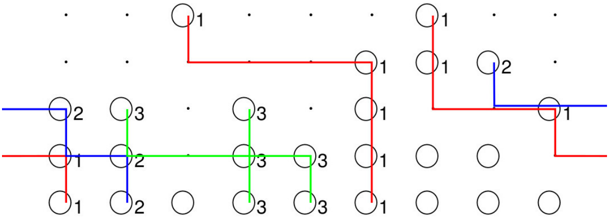

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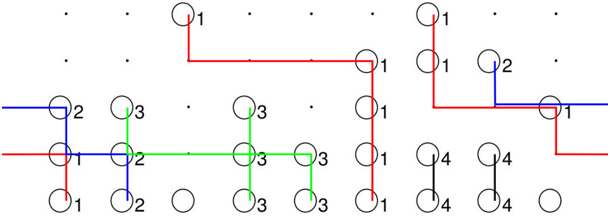

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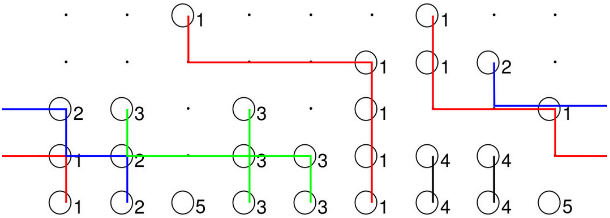

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Theorem [Arita Mallick]

$$\psi_w \propto \sum_{q|BP(q)=w} \prod_{\alpha<\beta} \left(\frac{\tau_\beta}{\tau_\alpha} \right)^{z_{\alpha,\beta}(q)}$$

where $z_{\alpha,\beta}(q)$ is the number of vacancies on row j that are covered by a i Bully Path.

Open question

Generalize such a construction to the case $\nu_\alpha \neq 0$?

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Double Schubert polynomials [Lascoux-Schützenberger]

Let $\mathbf{t} = t_1, t_2, \ldots$ and $\mathbf{v} = v_1, v_2 \ldots$ two infinite sets of commuting variables

Definition: double Schubert polynomials

For the longest permutation $\sigma_0 \in S_n$

$$\mathfrak{S}_{\sigma_0}(\mathbf{t}, \mathbf{v}) := \prod_{i+j \leq n} (t_i - v_j)$$

for generic $\sigma \in S_n$

$$\mathfrak{S}_{\sigma}(\mathbf{t}, \mathbf{v}) = \partial_{\sigma^{-1}\sigma_0} \mathfrak{S}_{\sigma_0}(\mathbf{t}, \mathbf{v})$$

where $\partial_\sigma = \partial_{s_{i_1}} \partial_{s_{i_2}} \ldots \partial_{s_{i_\ell}}$, ($s_{i_1} \cdot s_{i_2} \cdots s_{i_\ell}$ is a reduced decomposition of σ) and

$$\partial_{s_{i_1}} = \frac{1 - s_i^{\mathbf{t}}}{t_i - t_{i+1}}, \qquad s_i^{\mathbf{t}} : t_i \leftrightarrow t_{i+1}.$$

Conjecture

- The functions $\psi_w(\tau,\nu)$ can be expressed as polynomials of double Schubert polynomials with the variables $\mathbf{t},\mathbf{v}$ choosen as

$$\mathbf{t} = \overbrace{\tau_1,\dots,\tau_1}^{m_1},\overbrace{\tau_2,\dots,\tau_2}^{m_2},\dots,\overbrace{\tau_{r-1},\dots,\tau_{r-1}}^{m_{r-1}}$$

$$\mathbf{v} = \underbrace{-\nu_r,\dots,-\nu_r}_{m_r},\underbrace{-\nu_{r-1},\dots,-\nu_{r-1}}_{m_{r-1}},\dots,\underbrace{-\nu_2,\dots,-\nu_2}_{m_2}$$

  with *positive integer coefficients*.
- The double Schubert polynomials appearing in the expression of $\psi_w(\tau,\nu)$ correspond to permutations in $\sigma \in S_{L(w)}$ such that

$$L - m_r < i < j \longrightarrow \sigma_i < \sigma_j$$

$$L - m_1 < i < j \longrightarrow \sigma_i^{-1} < \sigma_j^{-1}.$$

Multispecies ASEP: Integrability

Suppose that we have a matrix $\check{R}(x, y)$ depending on two formal commuting variables, such that

$$\check{R}(x, x) = \mathbf{1}, \qquad \frac{d}{dx}\check{R}(x, y)|_{x=y=0} \propto \sum_{1 \leq \alpha \neq \beta, N} p_{\alpha,\beta} M_{\alpha,\beta}$$

and a vector

$$\psi(\mathbf{z}) \in V_{\mathbf{m}} \otimes \mathbb{C}[\mathbf{z}], \qquad \mathbf{z} = \{z_1, \ldots, z_L\}$$

that satisfies the following

Exchange equations

$$\check{R}_i(z_i, z_{i+1})\psi(\mathbf{z}) = s_i \circ \psi(\mathbf{z})$$

where s_i acts on the polynomial part $\mathbb{C}[\mathbf{z}]$ by the exchange $z_i \leftrightarrow z_{i+1}$.

Multispecies ASEP: Integrability

Lemma

The specialization $\psi(\mathbf{0})$ is proportional to the M-ASEP stationary probability

$$\mathcal{M}\psi(\mathbf{0}) = 0$$

Proof.

Differentiating the exchange equations we get

$$\frac{d}{dz_i}\check{R}(z_i, z_{i+1})|_{z_i=z_{i+1}=0}\psi(\mathbf{0}) = \partial_{i+1}\psi(\mathbf{0}) - \partial_i\psi(\mathbf{0})$$

These are terms of a telescopic sum □

Multispecies ASEP: Integrability

- Consistency of the exchange equations is ensured by the unitarity relation

$$\check{R}_i(x,y)\check{R}_i(y,x) = \mathbf{1}$$

and the braid Yang-Baxter equation

$$\check{R}_i(y,z)\check{R}_{i+1}(x,z)\check{R}_i(x,y) = \check{R}_{i+1}(x,y)\check{R}_i(x,z)\check{R}_{i+1}(y,z)$$

- We search the $\check{R}-$matrix of the "baxterized" form

$$\check{R}(x,y) = \mathbf{1} + \sum_{1\leq\alpha\neq\beta\leq N} g_{\alpha,\beta}(x,y)M_{\alpha,\beta}$$

- Suppose that $\forall\alpha\neq\beta$, $g_{\alpha,\beta}\neq 0$ then the only solution (up to permutation of the species) corresponds to

$$p_{\alpha,\beta} = \begin{cases} p & \text{for} \quad \alpha<\beta \\ q & \text{for} \quad \alpha>\beta \end{cases}$$

multispecies ASEP introduced by Rittenberg et al.

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Multispecies TASEP: baxterized form of R-matrix

Proposition

If for some $\alpha \neq \beta$, $g_{\alpha,\beta} = 0$ then, up to species relabelling, the most general baxterized $R-$matrix is of the form

$$\check{R}(x,y) = \mathbf{1} + \sum_{1 \leq \alpha < \beta \leq N} g_{\alpha,\beta}(x,y) M_{\alpha,\beta}$$

with

$$g_{\alpha,\beta}(x,y) = \frac{(y-x)(\tau_\alpha + \nu_\beta)}{(\tau_\alpha y - 1)(\nu_\beta x + 1)} \rightarrow p_{\alpha<\beta} = \tau_\alpha + \nu_\beta$$

Lemma

The exchange equations corresponding the the $\check{R}$ matrix of the Multispecies TASEP admit a polynomial solution, unique up to multiplication of a completely symmetric polynomial in the $\mathbf{z}$.

Multispecies TASEP: baxterized form of R-matrix

Proposition

If for some $\alpha \neq \beta$, $g_{\alpha,\beta} = 0$ then, up to species relabelling, the most general baxterized $R-$matrix is of the form

$$\check{R}(x,y) = \mathbf{1} + \sum_{1\leq\alpha<\beta\leq N} g_{\alpha,\beta}(x,y) M_{\alpha,\beta}$$

with

$$g_{\alpha,\beta}(x,y) = \frac{(y-x)(\tau_\alpha + \nu_\beta)}{(\tau_\alpha y - 1)(\nu_\beta x + 1)} \to p_{\alpha<\beta} = \tau_\alpha + \nu_\beta$$

Lemma

The exchange equations corresponding the the $\check{R}$ matrix of the Multispecies TASEP admit a polynomial solution, unique up to multiplication of a completely symmetric polynomial in the $\mathbf{z}$.

Exchange equations in components

Once expanded in components, the exchange equations read as follows

$$\psi_{\dots,w_i=w_{i+1},\dots}(\mathbf{z}) = s_i \circ \psi_{\dots,w_i=w_{i+1},\dots}(\mathbf{z})$$

$$\psi_{\dots,w_i>w_{i+1},\dots}(\mathbf{z}) = \hat{\pi}_i(w_i, w_{i+1})\psi_{\dots,w_{i+1},w_i,\dots}(\mathbf{z})$$

and

$$\hat{\pi}_i(\alpha,\beta) = \frac{(\tau_\alpha z_{i+1} - 1)(\nu_\beta z_i + 1)}{\tau_\alpha + \nu_\beta} \frac{1 - s_i}{z_i - z_{i+1}}$$

This system of equation is cyclic: if $\psi_w(\mathbf{z})$ is known for a given configuration w, one can obtain $\psi_{w'}(\mathbf{z})$ for any other w' by acting with the $\hat{\pi}$ operators.

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Affine 0-Hecke algebra with spectral parameters

The operators $\hat{\pi}_i(\alpha,\beta)$ satisfy a spectral parameter deformation (not baxterization!) of the 0-Hecke algebra (recovered for t_α and ν_α independent of α)

$$\hat{\pi}_i^2(\alpha,\beta) = -\hat{\pi}_i(\alpha,\beta)$$

$$\hat{\pi}_i(\beta,\gamma)\hat{\pi}_{i+1}(\alpha,\gamma)\hat{\pi}_i(\alpha,\beta) = \hat{\pi}_{i+1}(\alpha,\beta)\hat{\pi}_i(\alpha,\gamma)\hat{\pi}_{i+1}(\beta,\gamma)$$

$$[\hat{\pi}_i(\alpha,\beta),\hat{\pi}_j(\gamma,\delta)] = 0 \quad |i-j| > 2$$

Simple consequences of the exchange equations

• If the configuration w has a sub-sequence
$w_\ell \leq w_{\ell+1} \leq \cdots \leq w_{k-1} \leq w_k$ then

$$\psi_w(\mathbf{z}) = \prod_{i=\ell}^{k} \left(\prod_{\substack{\alpha \in w_{\ell,k} \\ \alpha < w_i}} (\tau_\alpha z_i - 1) \prod_{\substack{\alpha \in w_{\ell,k} \\ \beta > w_i}} (\nu_\beta z_i + 1) \right) \tilde{\psi}_w(\mathbf{z})$$

where $\tilde{\psi}_w(\mathbf{z})$ is symmetric in the variable $\{z_\ell, \dots, z_k\}$
• In particular if $w = w^*$ has minimum number of descents
$w_\ell \leq w_\ell \leq \cdots \leq w_{\ell-2} \leq w_{\ell-1}$ then $\tilde{\psi}_{w^*}(\mathbf{z})$ is symmetric in the whole set of variables $\mathbf{z}$ and by cyclicity is a common factor of all the $\psi_w(\mathbf{z})$.

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Simple consequences of the exchange equations

- Normalization choice

$$\psi_{w^*}(\mathbf{z}) = \chi_{\mathbf{m}}(\tau,\nu) \prod_{i=1}^{L} \left(\prod_{\alpha < w_i^*} (1 - \tau_\alpha z_i) \prod_{\beta > w_i^*} (1 + \nu_\beta z_i) \right)$$

- The solution of the exchange equation of minimal degree in the sector **m** has degree

$$\deg_{z_i} \psi^{(\mathbf{m})}(\mathbf{z}) = r - 1$$

Theorem

With the normalization given above, the components ψ_w are polynomials in all their variables $(\mathbf{z}, \tau, \nu)$ with no common factors.

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Recursions

Proposition

By specializing $z_L = \tau_1^{-1}$ or $z_L = -\nu_r^{-1}$ we have the following recursion

$$\psi_{w1}(\mathbf{z})|_{z_L=\tau_1^{-1}} = K^-(\mathbf{z} \setminus z_L)\psi_w(\mathbf{z} \setminus z_L)$$

$$\psi_{wr}(\mathbf{z})|_{z_L=-\nu_r^{-1}} = K^+(\mathbf{z} \setminus z_L)\psi_w(\mathbf{z} \setminus z_L)$$

where the factors $K^\pm(\mathbf{z} \setminus z_L)$ can be easily computed by inspection of $\psi_{w^*}(\mathbf{z})$.

Simplest non trivial component

Let $w^{(\alpha)}$ be a configuration such that for $i \leq j \leq L - m_\alpha$

$$w_i \neq \alpha \qquad \text{and} \qquad w_i \leq w_j$$

For example

$$w^{(3)} = 1\ 1\ 2\ 4\ 4\ 4\ 5\ 6\ 6\ 3\ 3\ 3$$

Then

$$\psi^{(\mathbf{m})}_{w^{(\alpha)}}(\mathbf{z}) = (\text{Trivial Factors}) \times \phi^{(\mathbf{m})}_{\alpha}(z_1, \ldots, z_{L-m_\alpha})$$

where $\phi^{(\mathbf{m})}_{\alpha}(z_1, \ldots, z_{L-m_\alpha})$ is a symmetric polynomial in $z_1, \ldots, z_{L-m_\alpha}$ of degree 1 in each variable separately.

- Thanks to the recursion relations they can be computed explicitly
- These polynomials turn out to be the building blocks of more general components

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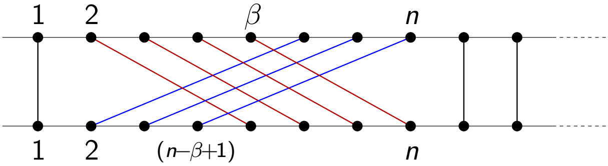
Simplest non trivial component

For any $n > 0$ and $1 \leq \beta \leq n$ define the following polynomials

$$\Phi^n_\beta(\mathbf{z};\mathbf{t};\mathbf{v}) := \Delta(\mathbf{t},\mathbf{v}) \oint_{\mathbf{t}} \frac{dw}{2\pi i} \frac{\prod_{i=1}^{n-1}(1-wz_i)}{\prod_{1\leq\rho\leq\beta}(w-t_\rho)\prod_{1\leq\sigma\leq n-\beta+1}(w-v_\sigma)}$$

For $\mathbf{z}=0$ these specialize to the double Schubert Polynomials

$$\Phi^n_\beta(\mathbf{0};\mathbf{t};\mathbf{v}) = \mathfrak{S}_{1,\beta+1,\beta+2,\ldots n,2,3,\ldots,\beta}(\mathbf{t};\mathbf{v})$$



Proposition

$$\phi_{\alpha}^{(\mathbf{m})}(z_1, \dots, z_{L-m_\alpha}) = \Phi_{\beta}^{L-m_\alpha}(\mathbf{z}; \mathbf{t}; \mathbf{v})$$

with $\beta = 1 + \sum_{\gamma<\alpha} m_\gamma$, and

$$\mathbf{t} = \{\dots, \overbrace{\tau_\gamma, \dots, \tau_\gamma}^{m_\gamma}, \dots, \overbrace{\tau_{\alpha-1}, \dots, \tau_{\alpha-1}}^{m_{\alpha-1}}, \tau_\alpha\}$$

$$\mathbf{v} = \{-\nu_\alpha, \underbrace{-\nu_{\alpha+1}, \dots, -\nu_{\alpha+1}}_{m_{\alpha+1}}, \dots, \underbrace{-\nu_\gamma, \dots, -\nu_\gamma}_{m_\gamma}, \dots\}$$

Factorization of components with least ascending

We have seen that to each "ascent" in a configuration w one has a bunch of trivial factors, therefore the intuition is that the more ascents w has the "simpler" is its component ψ_w.

Actually the configurations $\tilde{w}$ which have minimal number of ascent are also computable

Exm

$$\tilde{w} = 6\ 6\ 5\ 4\ 4\ 4\ 3\ 3\ 3\ 2\ 1\ 1$$

Theorem

Calling $\mathbf{z}_\alpha = \{z_i | w_i = \alpha\}$

$$\psi_{\tilde{w}} = \prod_\alpha \phi_\alpha^{(\mathbf{m})}(\mathbf{z} \setminus \mathbf{z}_\alpha)$$

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Factorization of components with least ascending: corollaries

Consider the case $m_\alpha = 1$ for $1 \leq \alpha \leq L$ and specialize $\mathbf{z} = 0$

Corollary

The formula for the least ascending component implies and generalizes a formula conjectured by Lam and Williams which expresses $\psi_{\tilde{w}}$ as a product of double-Schubert Polynomials of τ, ν

$$\psi_{L,L-1,\dots,1} = \mathfrak{S}_{1,2,3\dots,L}\mathfrak{S}_{1,3,4\dots,L,2}\mathfrak{S}_{1,4,5,\dots,L,2,3}\mathfrak{S}_{1,L,2,3\dots,L-1}$$

Actually we get something more: suppose that

$$w = w^L\ j\ w^R$$

with $w^L_i > j > w^R_h$ then

$$\psi_w \propto \mathfrak{S}_{1,j+1,j+2,\dots,L,2,\dots,j,L+1,L+2,\dots}$$

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Factorization of components with least ascending: corollaries

Suppose that w splits as $w^{(k)}w^{(k-1)}\dots w^{(2)}w^{(1)}$,

$$w_i^{(r)} < w_j^{(s)} \qquad \text{for} \quad r < s$$

Exm:

$$w = 6\ 5\ 6 | 4\ 3\ 4\ 4\ 3\ 3 | 2\ 1\ 1$$

Then

$$\psi_w = \bar{\psi}_{w^{(k)}} \bar{\psi}_{w^{(k-1)}} \cdots \bar{\psi}_{w^{(2)}} \bar{\psi}_{w^{(1)}}$$

This means that under conditioning of splitting $w = w^{(k)}w^{(k-1)}\dots w^{(2)}w^{(1)}$ the $w^{(j)}$ are independent.

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Normalization

In order to compute actual probabilities we need *the partition function*

$$\mathcal{Z}^{(\mathbf{m})}(\mathbf{z}) = \sum_{w|\mathbf{m}(w)=\mathbf{m}} \psi_w(\mathbf{z})$$

Thanks to the exchange relations this polynomial turns out to be symmetric in $\mathbf{z}$.

For simplicity let me present the general formula for the case $m_\alpha = 1$

$$\mathcal{Z}^{(\mathbf{m})}(\mathbf{z}) = \mathrm{Sym}_{\mathbf{z}} \left[\frac{\psi_{1\cdots(L-1)L}(\mathbf{z}_{\sigma_0})\psi_{L(L-1)\cdots 1}(\mathbf{z})}{\prod_{1\leq\alpha<\beta\leq L}(\tau_\alpha - \nu_\beta)^{\beta-\alpha}\Delta(\mathbf{z})} \right]$$

Where σ_0 is the longest permutation in $\mathcal{S}_L$, $\sigma_0 = (L, L-1, \ldots, 2, 1)$.

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Factorization of the sum rule

If for some γ we have

$$\begin{array}{lll} \nu_\alpha = \nu & \text{for} & 1 \leq \alpha \leq \gamma \\ \tau_\alpha = \tau & \text{for} & \gamma \leq \alpha \leq r \end{array}$$

Then the partition function factorizes

$$\mathcal{Z}^{(\mathbf{m})}(\mathbf{z}) = \prod_{\alpha=1}^{\gamma-1} \phi_\alpha^{(\mathbf{m}_\alpha^{\uparrow})}(\mathbf{z}) \prod_{\alpha=\gamma+1}^{r} \phi_\alpha^{(\mathbf{m}_\alpha^{\downarrow})}(\mathbf{z})$$

where

$$\mathbf{m}_\alpha^{\downarrow} = \Pi^{\alpha-1}_{\alpha,\alpha+1,\ldots}\mathbf{m}, \qquad \mathbf{m}_\alpha^{\uparrow} = \Pi^{\alpha+1}_{\ldots,\alpha-1,\alpha}\mathbf{m}$$

Some open questions

- Correlation functions, currents, etc.
- Do the components $\psi_w(\mathbf{z})$ have a combinatorial expression?
- What is the "right" context for the 0-Hecke algebra with spectral parameters?
  The operators $\hat{\pi}(\alpha, \beta)$ can be used for example to define a family of deformed Grothendieck "polynomials" which depend on the parameters τ, ν. Do they have any geometric meaning?
- Deal with others Weyl groups.

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