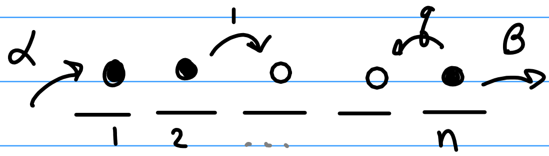
Goal: Explain how + why we can connect (multispecies) ASEP w/ open boundaries to combinatorics of ortho poly's / Macdonald poly's

Toy Example:  **Version 0**

(Later we add more species & more complicated bdy)

Matrix Ansatz (Derrida ...)

Suppose D, E matrices, $\langle w|$, $|V\rangle$ vectors s.t.

① $DE - qED = D + E$.

② $\alpha\langle w|E = \langle w|$

③ $\beta D|V\rangle = |V\rangle$

Then the steady state prob of e.g. state ● ● ○ ○ ● ○ is $\dfrac{\langle w|DDEEDE|V\rangle}{Z_6}$,

where $Z_n = \langle w|(D+E)^n|V\rangle$ is <u>partition</u> <u>function</u>.

<u>Idea:</u> Equivalent to showing that (up to scalar factor) steady state prob's satisfy recurrences.

eg: $\underset{\bullet\circ\bullet}{\langle w|DED|V\rangle} = \langle w|(qED+D+E)D|V\rangle$

$= q\underset{\circ\bullet\bullet}{\langle w|EDD|V\rangle} + \underset{\bullet\bullet}{\langle w|DD|V\rangle} + \underset{\circ\bullet}{\langle w|ED|V\rangle}$.

<u>Note:</u> In this example, we don't need to find matrices D, E to solve for prob's — for any word $X \in \{D,E\}^*$, can write it as sum of words of form $\langle w|E^\ell D^m|V\rangle$.

Then relations ②, ③ $\Rightarrow$

$\langle w|E^\ell D^m|V\rangle = \alpha^{-\ell}\beta^{-m}\langle w|V\rangle$.

$\boxed{\text{Orthog Poly}}$

However, one <u>can</u> find such D, E.
Define $\underline{d}$, $\underline{e}$ by $D = \frac{1}{1-q}(\mathbb{1}+\underline{d})$, $E = \frac{1}{1-q}(\mathbb{1}+\underline{e})$.
Then (1) $\Leftrightarrow$ $de - qed = (1-q)\mathbb{1}$

Various sol'ns found to Matrix Ansatz (DEHP), including some sol'ns where $\underline{d}+\underline{e}$ is tridiagonal.

Tridiagonal matrices: (under certain circumstances, e.g. real symm w/ pos. off-diag entries)

Rows $\rightsquigarrow$ 3-term recurrence rel'n for family of orthog poly's $\{P_n(x)\}$, ie.

$$\int P_\ell(x) P_m(x)\, dx = \begin{cases} 0 & \ell \neq m \\ \text{const} & \ell = m \end{cases}$$

<u>Sasamoto:</u> One sol'n to MA has property that $(\underline{d}+\underline{e})$ encodes Al-Salam-Chihara poly's.

<u>Calculation:</u> $\langle W| (\underline{d}+\underline{e})^n |V\rangle = \int x^n dx =: \underset{\text{moment}}{M_n}$

<u>Consequence:</u> Partition function $Z_n = \langle W| (D+E)^n |V\rangle$ of ASEP = alt sum of moments of orthog poly's.

<u>Rk:</u> Al-Salam Chihara = special case of Askey Wilson (when $c=d$)

AW polys = special case of multivariate Koornwinder poly's (when just 1 var)

Combinatorics

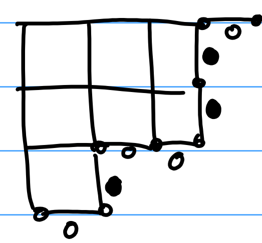
Ex: With $n=2$, our prob's are (up to normalization):

●●	○●	○○	●○
α^2	$\alpha\beta$	β^2	$\alpha\beta(\alpha+\beta+q)$

[Choose normalization s.t. we get poly's, & no common factor of all]

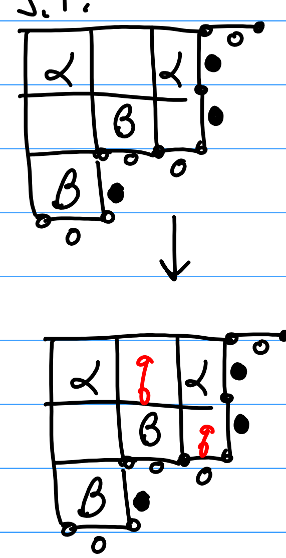
Monomial assoc to states of form $\circ^{\ell}\bullet^{m}$
More complicated when there are ●'s before ○'s.
Associate obvious Young diagram to states in $\{\bullet,\circ\}^*$, $\sigma \mapsto$ shape(σ)

○ ● ● ○ ○ ● ○ $\rightsquigarrow$



Goal: Identify prob's w/ fillings.

Rule: Fill boxes w/ α, β or ϕ s.t.
(1) boxes above α are ϕ
(2) boxes left of β are ϕ.
Now if box is forced to be empty by (1) or (2), leave blank.
Otherwise put q.
The weight of a tableaux is prod of boxes · monomial of border



Here: $\alpha^2\beta^2q^2\cdot(\beta\alpha^2\beta^2\alpha\beta)$

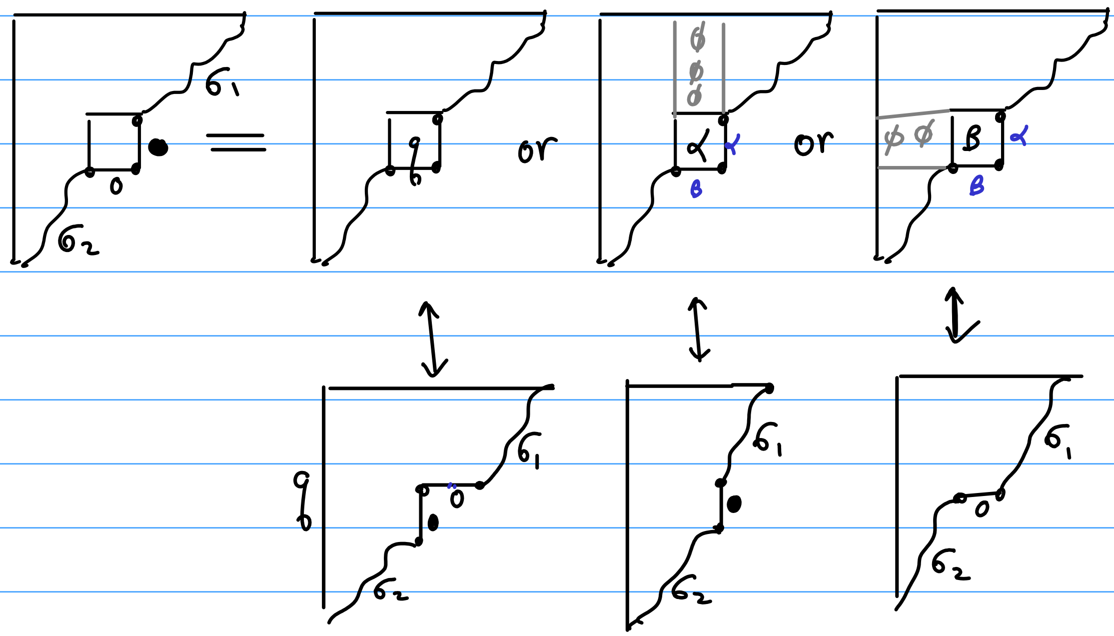
Def: Given word $\sigma \in \{\circ,\bullet\}^n$, let

$$\tilde{f}_\sigma(\alpha,\beta,q) = \sum_{\substack{\text{tableau} \\ \text{of type } \sigma}} wt(\sigma) \quad \text{and} \quad f_\sigma(\alpha,\beta,q) = (\alpha\beta)^{-n}\tilde{f}_\sigma(\alpha,\beta,q)$$

Lemma: The f_σ's satisfy M.A. recurrences. In particular, if $\sigma_1, \sigma_2 \in \{\circ, \bullet\}^*$, then

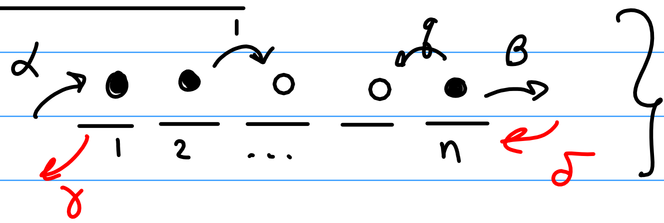
$$f_{\sigma_1 \bullet \circ \sigma_2} = q f_{\sigma_1 \circ \bullet \sigma_2} + f_{\sigma_1 \bullet \sigma_2} + f_{\sigma_1 \circ \sigma_2}$$

Pf Idea:



Cor: Steady state prob of state $\sigma \sim \sum_{\substack{\text{tableau} \\ \text{of type } \sigma}} wt(\sigma)$

More general model:



Version 1

$\exists$ MA as before, with

① $DE - qED = D + E$,

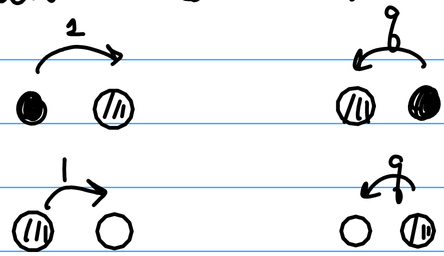
but now rel's ② & ③ become

(2') $\alpha \langle w | E - \gamma \langle w | D = \langle w |$

(3') $\beta D | V \rangle - \delta E | V \rangle = | V \rangle$

Can still use ① to write any word $\langle w|X|V\rangle$ as sum of words of form $\langle w|E^l D^m|V\rangle$ but now no way to use ②' or ③' to kill E's at left or D's at right.

Can also add another species of particle ⊘ with



<u>Don't</u> allow particles ⊘ to enter/exit at bdy, so number r of particles ⊘ is fixed.

<u>Version 2:</u> Bdy cond of Version 1 but w/ ⊘

<u>M.A. (Uchiyama)</u> Suppose $\exists$ matrices D, E, A vectors $\langle w|$, $|V\rangle$ s.t.

① $DE - qED = D + E$

①' $DA - qAD = A$

①'' $AE - qEA = A$

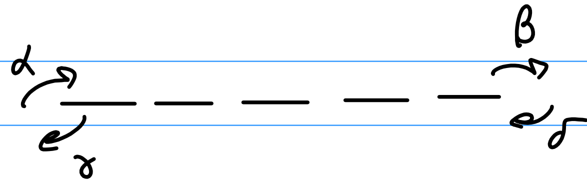
②' ③' as above.

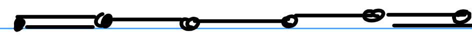
Then e.g. steady state prob of state ● ⊘ ⊘ ● ○ ○ $\sim \langle w|DAADEE|V\rangle$.

<u>Partition function</u> $Z_{n,r} = [y^r]\langle w|(D+E+yA)^n|V\rangle$

<u>Q:</u> Connection to orthog poly's & combinatorics?

<u>Rk:</u> Family of Macdonald polynomials assoc. to root systems.
Koornwinder poly's = Mac. poly's assoc to non-reduced root system of type BC.

Lattice 

reminiscent of type BC Dynkin diagram 

We'll see connection of Koornwinder w/ multispecies ASEP on this lattice.
[Result of C.W. & simultaneously Cantini]

Also, usual Macdonald poly's assoc to root system affine A. Connection of Macdonald to multispecies ASEP on this lattice
[Refer talk of Wheeler]

<u>Ortho poly's</u>: For Version 1, USW found sol'n to MA s.t. $(\underline{d}+\underline{e})$ is tridiagonal & encodes 3-term recurrence of AW poly's. $p_n(a,b,c,d|q)$

$\Longrightarrow$ $Z_n = \langle W|(D+E)^n|V\rangle$ closely related to moments $\mu_n(a,b,c,d|q) = \int x^n\,dx$.

If we let $Z_n(\xi;\alpha,\beta,\gamma,\delta;q) = \langle W|(\xi D+E)^n|V\rangle$ be fugacity partition function, then

Keep

<u>Thm 1 (CSSW)</u>: $\mu_n(a,b,c,d|q) = \frac{(1-q)^n}{2^n i^n} Z_n(-1;\alpha,\beta,\gamma,\delta;q)$

where $\alpha = \frac{1-q}{1-ac+ai+ci}$, $\beta = \frac{1-q}{1-bd-bi-di}$, $\gamma = \alpha\cdot ac$, $\delta = \beta\cdot bd$

Multispecies ASEP + Koornwinder

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be partition. (Allow $\lambda_i = 0$)
Koornwinder poly's
$P_\lambda(z_1, \ldots, z_m;\ a, b, c, d; q, t)$ are multivariate Laurent poly's.

- $m = 1 \rightsquigarrow$ Askey Wilson poly's.
- Orthog w/ respect to Koornwinder density.
- Symm under $Sym(m)$ and $z_i \mapsto z_i^{-1}$.
- Honest poly's if use $x_i = \frac{z_i + z_i^{-1}}{2}$.

Analogue of $\mu_n = \int x^n dx$ when have m var's?

Maybe: Integrate homog. symm. poly in m var's.
More generally...

Def: Set $q = t$. For $\lambda = (\lambda_1, \ldots, \lambda_m)$, set

$$M_\lambda = I_K(s_\lambda(x_1, \ldots, x_m)).$$

Koornwinder moment

Integrate w/ resp to Koorn density.

Lemma:

$$M_\lambda = \frac{\det(\mu_{\lambda_i + m - i + m - j})_{i,j=1}^m}{\det(\mu_{2m-i-j})_{i,j=1}^m}$$

Koorn moment

Askey Wilson moment

Motivated by Thm 1, define

$$K_\lambda(\xi) = \frac{\det\left(Z_{\lambda_i+m-i+m-j}(\xi)\right)_{i,j=1}^{m}}{\det\left(Z_{2m-i-j}(\xi)\right)_{i,j=1}^{m}}$$

Prop (CW): $M_\lambda(a,b,c,d|q) = \left(\frac{1-q}{2i}\right)^{|\lambda|} K_\lambda(-1;\alpha,\beta,\gamma,\delta;q)$

Will primarily care about $K_\lambda(\xi)$. Call $K_\lambda(\xi)$ Koornwinder moments also.

Conj: $K_\lambda(\xi)$ is poly in $\alpha,\beta,\gamma,\delta,\xi,q$ w/ positive coefficients (up to factor).

Not obvious: $Z_n(\xi)$ is pos poly because $\exists$ formula (C.W.) as sum over tableaux. But ratio of 2 det's?

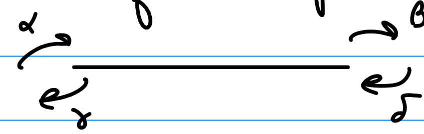
Thm (CW): When $q=\xi=1$,

$$K_\lambda = S^{|\lambda|} \prod_{z\in\lambda}(x+h(z)-1) \cdot \prod_{i=1}^{n-1}\prod_{j=i+1}^{n} \frac{(x+\lambda_i-\lambda_j+j-i-1)(\lambda_i-\lambda_j+j-i)}{(x+j-i-1)(j-i)}$$

where $S = \frac{(\alpha+\gamma)(\beta+\delta)}{\alpha\beta-\gamma\delta}$ and $x = \frac{\alpha+\beta+\gamma+\delta}{(\alpha+\gamma)(\beta+\delta)}$

($\Rightarrow$ Conj true when $q=\xi=1$)

<u>Thm:</u> Let $Z_{n,r}(\xi) = [y^r] \langle w | (D+E+yA)^n | V \rangle$ be the fugacity partition function of 2-species ⓪ ① ● ASEP on lattice



w/ exactly r particles ⑪.

Then
$$\underset{\text{2-spec ASEP}}{Z_{n,r}(\xi)} = (1-q)^r \underbrace{K_{(n-r,\overbrace{0,0,\dots,0}^{r\ 0\text{'s}})}(\xi)}_{\text{Koornwinder moment}}$$

$$= \frac{\det\left(Z_?(\xi)\right)_{i,j=1}^{r+1}}{\det\left(Z_{2m-i-j}(\xi)\right)_{i,j=1}^{r+1}}$$

(specialize at $\xi = -1$ gives $I_K\left(h_{n-r}(x_1, x_2, \dots, x_{r+1})\right)$)

<u>Intermediate step:</u> Let $\langle w | = (1,0,0\dots)$, $|V\rangle = \begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}$ D, E be MA sol'n of USW connected to AW poly's. Let $|V^r\rangle = \begin{pmatrix}0\\\vdots\\1\\0\\\vdots\end{pmatrix} \leftarrow r$.

Then

$$Z_{n,r} \sim \langle w | (D+E)^n | V^r \rangle \sim K_{(n-r,0,\dots,0)}(\xi)$$

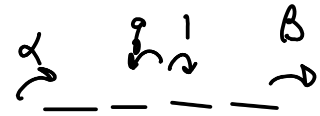
↑ algebra ↑ EV + lattice path comb.

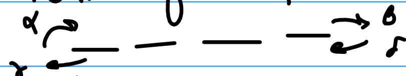
<u>Rk:</u> Cantini uses affine Hecke alg of type C to show $Z_{n,r} \sim P_{(\underbrace{1,1,\dots,1}_{n-r},\underbrace{0\dots0}_{r})}(x_i = 1, q = t)$.

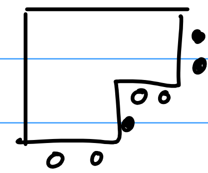
<u>Q:</u> ASEP interpret for general moments $K_\lambda(\xi)$?

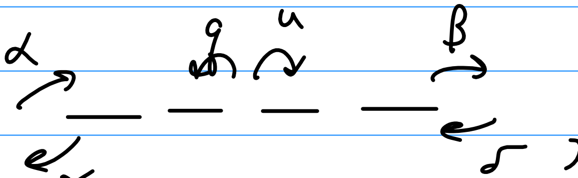
<u>CMW:</u> Combinatorial formula for stationary dist of 2-species ASEP. Hence get comb for for the above special Koornwinder moments/poly's.

No details here: just some motivation for results in Olya's talk

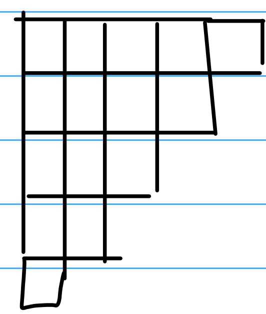
Combinatorics: Saw comb form for ASEP on 

If we try to extend to 

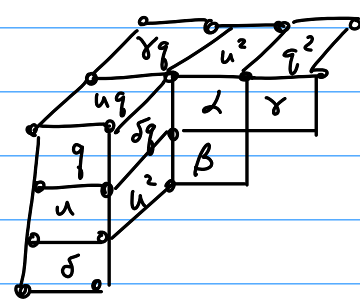
, and naively put γ's and δ's into shape , it doesn't work, because prob's are poly's w/ degree > # boxes ↗

If you replace 1 by u 

, find that Z_n is poly in $\alpha \beta \gamma \delta\ q\ u$ which is homog. of degree $\binom{n}{2}$.

Motivates using staircase shape: 

For Version 2, since we have extra species of particle ⓵ in between ○ and ●, use shapes w/ additional slope /

Rhombic staircase tableaux: 

(For details, see Olya's talk —
Thm (CMW): Comb formula for stationary dist of 2-species ASEP in terms of rhombic staircase tableaux.