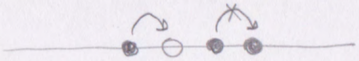
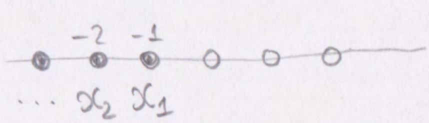
Foreword and disclaimer These are the handwritten notes used by Alexei Borodin to give his mini-course *Integrable probability: between determinantal and nondeterminantal models* at the school *Quantum integrable systems, conformal field theories and stochastic processes* held in Cargèse from 12 to 23 September 2016. We the organizers of the school had not requested in advance the lecturers to provide notes for their courses, but Alexei kindly agreed his to be posted on the school website[1] together with the present disclaimer that they might contain plenty of errors, missed references, etc.

Abstract A recently discovered link between determinantal (free fermion) and nondeterminantal integrable probabilistic models allows to reduce some asymptotic questions about the ASEP with step initial condition to those for TASEP. More generally, a relation exists between the higher spin stochastic six vertex model and the Schur (and more generally, Macdonald) processes. The goal of the lectures is to discuss this connection. Based on arXiv:1608.01553, arXiv:1608.01557, arXiv:1608.01564.

[1] `https://indico.in2p3.fr/event/12461/`

TASEP 

Step (packed) IC 

Laguerre/Wishart ensemble on $\mathbb{R}_+$

$$\text{j.p.d.} = \text{const} \cdot \prod_{i<j} (y_i - y_j)^2 \prod_{i=1}^{N} w(y_i)\, dy_i$$

with $w(y) = y^{\beta-1} e^{-y}$, $y_1 \geq y_2 \geq \dots \geq y_N > 0$.

notation: Laguerre (N, β).

<u>Theorem</u> (Johansson '98, limit of RSK)

$$\boxed{\text{Prob}\{x_N(t) \geq x\} = \text{Prob}\{y_1 < t\}}$$

<u>Corollary</u> (Johansson '98)

$$\boxed{\lim_{t\to\infty} \text{Prob}\left\{\frac{x_m(t) - c_1 \cdot t}{c_2 t^{1/3}} \leq r\right\} = F_{GUE}(r)} = \det(\mathbb{1} - \mathbb{A})_{L^2(r,+\infty)}$$

with $\frac{m}{t} = s \in (0,1)$ $\quad c_1 = c_1(s) = 1 - 2\sqrt{s}$

$\quad c_2 = c_2(s) = s^{-1/6}(1-\sqrt{s})^{2/3}$

$$\mathbb{A}(x,y) = \int_0^{+\infty} A(x+z)A(y+z)\, dz$$

$\mathbb{A}$ is the spectral projection for $\left(\frac{d^2}{dx^2} - x\right)$ corr. to $\mathbb{R}_+$;

$$A''(x+z) - x\, A(x) = z \cdot A(x)$$

Determinantal structure of the Laguerre ensemble

$$\rho_n(y_1, \dots, y_n) = \det\left[K_N(y_i, y_j)\right]_{i,j=1}^{n},$$

$$K_N(x,y) = \sum_{n=0}^{N-1} P_n(x) P_n(y) \sqrt{w(x) w(y)},$$

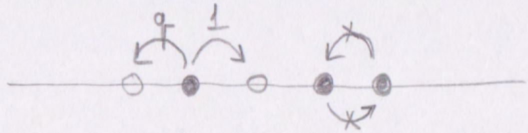
where $\{P_n\}$ are <u>orthonormal</u> Laguerre poly's.

K_N is the projection onto $\text{Span}(1, y, \dots, y^{N-1})\sqrt{w(y)}$.

$$\text{Prob}\{y_1 < t\} = \det(1 - K_N)_{L^2(t,+\infty)}$$

Convergence $K_N \to \mathbb{A}$ finishes the proof.

ASEP



$0<q<1$

Theorem (Tracy-Widom '08) The asymptotic behaviour of x_m is exactly the same as for TASEP (above) with $x_m(t)$ replaced by $x_m\left(\frac{t}{1-q}\right)$.

Define the discrete Laguerre ensemble as the determinantal pt process on $\mathbb{Z}_+$ with corr. kernel

$$K_r(x,y)=\int_r^{+\infty} P_x(a)P_y(a)w(a)\,da$$

This is the matrix of projection $L^2(\mathbb{R}_+)\to L^2((r,+\infty))$ in the basis $\{P_n(a)\sqrt{w(a)}\}_{n\geq 0}$.

Theorem (B-Olshanski '16)

$h(x)=\#$ of particles $\geq x$

$x\geq 0$

$\zeta\in\mathbb{C}\setminus q^{-\mathbb{Z}_{\geq 0}}$

$$\boxed{\mathbb{E}_{ASEP}\prod_{i\geq 0}\frac{1}{1+\zeta q^{h(x)+i}}=\mathbb{E}_{Y\in \mathrm{DLaguerre}(\overset{r}{\overset{\shortparallel}{(1-q)t}},\overset{\beta}{\overset{\shortparallel}{x+1}})}\prod_{y\in Y}\frac{1}{1+\zeta q^{y}}}$$

Let us check what happens as $q\to 0$.

$$\lim_{q\to 0}\prod_{i\geq 0}\frac{1}{1+q^{-N+\frac{1}{2}}q^{h(x)+i}}=\mathbb{1}_{h(x)\geq N}.$$

$$\mathbb{E}(\cdot)\to \mathrm{Prob}\{x_N^{ASEP}\geq x\}.$$

$$\lim_{q\to 0}\prod_{y\in Y}\frac{1}{1+q^{-N+\frac{1}{2}}q^{y}}=\mathbb{1}_{\min Y\geq N}$$

$$\mathbb{E}(\cdot)\to \mathrm{Prob}\{\text{no particles in }\{0,1,\dots,N-1\}\}=$$

$$=\det\left(1-K_r^{DL}\right)_{L^2(\{0,\dots,N-1\})}=\begin{pmatrix}\det(1-AB)=\\ =\det(1-BA)\end{pmatrix}$$

$$=\det\left(1-K_N^{L}\right)_{L^2(r,+\infty)}=\mathrm{Prob}\{y_1<t\}$$

In fact, from the asymptotic point of view, $q \in (0,1)$ has little difference from $q=0$.
Let us say that a sequence $\{\xi_n\}$ of r.v.'s spreads if $\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \mathrm{Prob}\{x < \xi_n \le x+1\} = 0$.

Let us also say that $\{\xi_n\}$ and $\{\eta_n\}$ are asymptotically equivalent if
(a) $\{\xi_n\}$ spreads iff $\{\eta_n\}$ spreads; (b) if they spread, $\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left|\mathrm{Prob}\{\xi_n \le x\} - \mathrm{Prob}\{\eta_n \le x\}\right| = 0$

Corollary $h(x)$ for the ASEP and the left-most particle $\min Y$ of $\mathrm{DLaguerre}((1-q)t, x+1)$ are asymptotically equivalent.

This and the convergence of either TASEP or any of the two Laguerre ensembles to the Airy limit proves Tracy-Widom's theorem.

Let us go further. The above application of the $\mathbb{E}\Pi = \mathbb{E}\Pi$ was based on the fact that either product converges to 0 or 1 with high probability. We can arrange for these products and their factors to have nontrivial limit $\in (0,1)$.

Theorem (B-Olshanski '16) Assume $r, \beta \to +\infty$ so that $1+\varepsilon < r/\beta < \varepsilon^{-1}$ for some $\varepsilon > 0$.
Then
$$\frac{\sigma - \mathrm{DLaguerre}(r,\beta)}{\tau} \longrightarrow \text{Airy det. pt. process}$$
with $\sigma = \frac{(r-\beta)^2}{4r}$, $\tau = \frac{(r^2-\beta^2)^{2/3}}{2^{4/3} r}$. Note that $\tau \sim \sigma^{1/3}$.

$$\mathrm{ASEP}(\text{time } t, \text{position } x) \rightsquigarrow \mathrm{DLaguerre}((1-q)t, x+1) \approx \sigma - \tau\, \mathrm{Airy} \approx$$

$q = (1-\varepsilon)$, $t = \varepsilon^{-4}\hat{t}$, $x = \varepsilon^{-3}\hat{x}$

$$\approx \varepsilon^{-3}\hat{\sigma} - \varepsilon^{-1}\hat{\tau} \cdot \mathrm{Airy}$$
$$\hat{\sigma} = \frac{(\hat{t}-\hat{x})^2}{4\hat{t}}, \quad \hat{\tau} = \frac{(\hat{t}^2 - \hat{x}^2)^{2/3}}{2^{4/3}\hat{t}}$$

$$\prod_{i\ge 0}\frac{1}{1+\zeta q^{h+i}} = \prod_{i\ge 0}\frac{1}{1+\hat\zeta e^{\ln q(-\varepsilon^{-3}\hat\sigma+h+i)}} =$$

$$= \prod_{i\ge 0}\frac{1}{1+\varepsilon\hat\zeta e^{\hat h} q^{i}} \longrightarrow e^{-\hat\zeta e^{\hat h}}$$

where we replaced $h=\varepsilon^{-3}\hat\sigma-\varepsilon^{-1}\ln\varepsilon-\varepsilon^{-1}\hat h$

The last convergence is due to the fact that

$$\prod_{i\ge 0}\frac{1}{1+(1-q)zq^{i}} = \sum_{k\ge 0}\frac{(-1)^k(1-q)^k}{(q;q)_k}z^k \to e^{-z}$$

(this side goes first)

$$\prod_{y\in Y}\frac{1}{1+\zeta q^{y}} = \prod_{a\in Airy}\frac{1}{1+\zeta e^{\ln q(\varepsilon^{-3}\hat\sigma-\varepsilon^{-1}\hat\tau\cdot a)}} \longrightarrow$$

$$\to \prod_{a\in Airy}\frac{1}{1+\hat\zeta e^{\hat\tau a}}, \qquad \hat\zeta=\lim_{\varepsilon\to 0}\zeta\cdot e^{\ln q\cdot\varepsilon^{-3}\hat\sigma}$$

(choose $\zeta=\zeta(\varepsilon)$ so that this holds)

Conclusion As $\varepsilon\to 0$, we obtain

with $\hat h=\varepsilon^{-2}\hat\sigma-\ln\varepsilon-\varepsilon h$.

$$\boxed{\mathbb{E}\, e^{-\hat\zeta e^{\hat h}} = \mathbb{E}\prod_{a\in Airy}\frac{1}{1+\hat\zeta e^{\hat\tau a}}}$$

This is a result of Amir-Corwin-Quastel and Sasamoto-Spohn (2010).
In fact, $e^{\hat h-T/24}$ has the same distribution as the stochastic heat equation $Z_T=\frac{1}{2}Z_{xx}-Z\cdot\dot W$ with the $\delta(x)$ initial condition and $T=2\hat\tau^3$. $\mathcal{H}=-\ln Z$ solves the KPZ equation.
This essentially goes back to Bertini-Giacomin (1997). A direct moment comparison is in B-Gorin'16.

may be skipped

In the same spirit one can look at the front edge of the ASEP.

Theorem (B-Olshanski '16) If $\beta\to\infty$ and $\tilde r=\tilde r(\beta)$ varies so that $\frac{\beta-\tilde r}{2\sqrt\beta}\to -r$ then DLaguerre$(\tilde r,\beta)\longrightarrow$ DHermite(r).

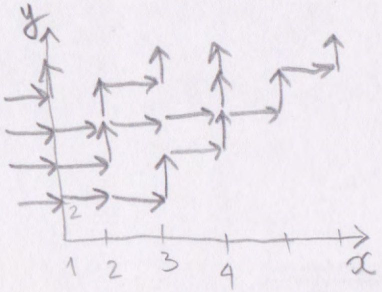
Corollary As $t\to\infty$, for the ASEP at time t, $h\left((1-q)t-2\sqrt{(1-q)t}\cdot r\right)\longrightarrow\xi_r$ where the r.v. ξ_r is characterized by

$$\boxed{\mathbb{E}\prod_{i\ge 0}\frac{1}{1+\zeta q^{\xi_r+i}} = \mathbb{E}_{\mathrm{DHermite}(r)}\prod_{y\in Y}\frac{1}{1+\zeta q^{y}}}$$

Tracy-Widom (2008) had a different expression for ξ_r, establishing equivalence may be interesting.

A more general algebraic framework

The stochastic 6-vertex model (Gwa-Spohn '92)



$\mathrm{Prob}(\;) = \mathrm{Prob}(\;) = 1.$

$\mathrm{Prob}(\;) = \delta_1 \quad \mathrm{Prob}(\;) = 1 - \delta_1$

$\mathrm{Prob}(\;) = \delta_2 \quad \mathrm{Prob}(\;) = 1 - \delta_2$

Change $(\delta_1, \delta_2) \to (q, \xi, u)$ by

$$\delta_1 = q\delta_2, \quad \delta_1 = \frac{1 - q^{1/2}\xi u}{1 - q^{-1/2}\xi u}.$$

Inhomogeneous case: $\xi \to \xi_x$; $u \to u_y$

(x,y) are the coordinates in the quadrant $\mathbb{Z}^2_{\geq 1}$

The Schur measures (Okounkov '99)

Cauchy identity

$$\prod_{i,j=1}^{N} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_\lambda(x_1, x_2, \dots)\, s_\lambda(y_1, y_2, \dots)$$

$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq 0)$, $s_\lambda(z_1, z_2, \dots) = \dfrac{\det\left[z_i^{N+\lambda_j - j}\right]_{i,j=1}^{N}}{\det\left[z_i^{N-j}\right]_{i,j=1}^{N}}$

$$\mathrm{Prob}\{\lambda\} = \prod_{i,j}(1 - x_i y_j) \cdot s_\lambda(x) s_\lambda(y).$$

Theorem (Okounkov) The random point configuration $\{N+\lambda_1 - 1, N + \lambda_2 - 2, \dots, \lambda_N\}$ forms a det. pt. process

Theorem (B'16)

$$\boxed{\mathbb{E}_{6V} \prod_{i \geq 0} \frac{1}{1 + \zeta q^{h(M,N)+i}} = \mathbb{E}_{\mathrm{Schur}} \prod_{y \in Y_\lambda} \frac{1}{1 + \zeta q^{y}}}$$

where $h(M,N)$ is the number of paths weakly to the right of $(M,N) \in \mathbb{Z}^2_{\geq 1}$, and

$Y_\lambda = \mathbb{Z}_{\geq 0} \setminus \{N+\lambda_1 - 1, N + \lambda_2 - 2, \dots, \lambda_N\}$; $(x_1, \dots, x_N) = (u_1^{-1}, \dots, u_N^{-1})$, $(y_1, \dots, y_{M-1}) = \{q^{-1/2}\xi_i^{-1}\}_{i=1}^{M-1}$.

In the homogeneous case, the pt. conf. $\{N+\lambda_1 - 1, \dots, \lambda_N\}$ forms an N-pt Meixner orth. poly. ensemble, and taking an appropriate limit one reaches the ASEP $\Leftrightarrow$ DLaguerre correspondence.

Taylor coefficients of the product identity. The main identity is

$$\mathbb{E}_{6V}\prod_{i\ge 0}\frac{1}{1+\zeta q^{h+i}}=\mathbb{E}_{Schur}\prod_{y\in\mathbb{Z}_{\ge0}\setminus Y_\lambda}\frac{1}{1+\zeta q^{y}},$$

or, equivalently,

$$\mathbb{E}_{6V}\frac{(-\zeta;q)_\infty}{(-q^h\zeta;q)_\infty}=\mathbb{E}_{Schur}\prod_{y\in Y_\lambda}(1+\zeta q^y).$$

$$\sum\frac{(a;q)_n}{(q;q)_n}z^n=\frac{(az;q)_\infty}{(z;q)_\infty}$$

For the LHS we use the q-binomial theorem:

$$\mathbb{E}_{6V}\frac{(-\zeta;q)_\infty}{(-q^h\zeta;q)_\infty}=\sum_{n\ge0}\mathbb{E}_{6V}\frac{(q^{-h};q)_n\cdot q^{h\cdot n}(-1)^n}{(q;q)_n}\cdot\zeta^n.$$

For the RHS we have

$$\mathbb{E}_{Schur}\prod_{y\in Y_\lambda}(1+\zeta q^y)=\sum_{n\ge0}\mathbb{E}_{Schur}\,e_n(q^{N+\lambda_1-1},\dots,q^{\lambda_N})\cdot\zeta^n.$$

$$e_n(x_1,x_2,\dots)=\sum_{i_1<\dots<i_n}x_{i_1}x_{i_2}\cdots x_{i_n}.$$

We thus want to see that

$$\mathbb{E}_{6V}\frac{(-1)^n(q^h-1)\cdots(q^h-q^{n-1})}{(1-q)\cdots(1-q^n)}=\mathbb{E}_{schur}\,e_n(q^{N+\lambda_1-1},\dots,q^{\lambda_N}).$$

In fact both sides can be computed, and they are equal to

$$\frac{1}{(2\pi i)^n n!}\oint_{\text{around}\{x_m\}}\cdots\oint\det\left[\frac{1}{qz_a-z_b}\right]_{a,b=1}^n\prod_{i=1}^n\left(\prod_{m=1}^N\frac{qz_i-x_m}{z_i-x_m}\cdot\prod_{j=1}^{M-1}\frac{1-y_jz_i}{1-qy_jz_i}\right)dz_i=$$

$$=\frac{1}{(2\pi i)^n n!}\oint_{\text{around}\{u_m^{-1}\}}\cdots\oint\det\left[\frac{1}{qz_a-z_b}\right]_{a,b=1}^n\prod_{i=1}^n\left(\prod_{m=1}^N\frac{1-qz_iu_m}{1-z_iu_m}\cdot\prod_{j=1}^{M-1}\frac{q^{1/2}\xi_j-z_i}{q^{-1/2}\xi_j-z_i}\right)dz_i$$

We shall check this for $n=1$, and our methods will be extendable to the case of general $n\ge1$.

<u>The Cauchy identity</u> Take two N-tuples of reals $\{x_i, y_i\}_{i=1}^N$ and assume $|x_i y_j| < 1$ for any $i, j = 1, \ldots, N$. The Cauchy determinant identity gives

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \frac{1}{\prod_{i<j} (x_i - x_j)(y_i - y_j)} \cdot \det \left[\frac{1}{1 - x_i y_j} \right]_{i,j=1}^N .$$

Using $\sum_{k \geq 0} z^k = (1-z)^{-1}$, we can write

$$\det \left[\frac{1}{1 - x_i y_j} \right] = \sum_{k_1 > \ldots > k_N \geq 0} \; \sum_{\sigma, \tau \in S(N)} \operatorname{sgn}(\sigma\tau) \left(x_{\sigma(1)} y_{\tau(1)} \right)^{k_1} \cdots \left(x_{\sigma(n)} y_{\tau(n)} \right)^{k_N} =$$

$$= \sum_{k_1 > \ldots > k_N \geq 0} \det \left[x_i^{k_j} \right]_{i,j=1}^N \cdot \det \left[y_i^{k_j} \right]_{i,j=1}^N$$

Denoting $S_\lambda(z_1, \ldots, z_N) = \det \left[z_i^{N + \lambda_j - j} \right]$, we reach the Cauchy identity

$$\prod_{i,j=1}^N \frac{1}{1 - x_i y_j} = \sum_{\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq 0)} S_\lambda(x) S_\lambda(y).$$

The measure on partitions λ with weights $\mathrm{Prob}\{\lambda\} = \prod (1 - x_i y_j) S_\lambda(x) S_\lambda(y)$ is called the Schur measure (Okounkov, 1999).

<u>Determinantal structure</u> To any partition λ assign an infinite point configuration $\mathcal{L}(\lambda) = \{\lambda_i - i\}_{i \geq 1} \subset \mathbb{Z}$. The resulting pt. process is determinantal:

<u>Th.</u> (Okounkov '99) $\mathrm{Prob}\{\{x_1, \ldots, x_n\} \subset \mathcal{L}(\lambda)\} = \det \left[K(x_i, x_j) \right]_{i,j=1}^n$

with

$$K(x,y) = \frac{1}{(2\pi i)^2} \oint \oint \frac{H(x; v)}{H(y; v^{-1})} \frac{H(y; w^{-1})}{H(x; w)} \frac{1}{v - w} \frac{dv\, dw}{v^{i+1} w^{-j}},$$

with contours $|w| = R_1 < R_2 = |v|$, and $H(x; v) = \prod \frac{1}{(1 - x_i v)}$ and similarly for the other H's. Equivalently, denoting $\mathcal{L}(\lambda) = \{l_1 > l_2 > \ldots\}$,

$$\mathbb{E} \left[\sum_{\substack{i_1, \ldots, i_n \\ \text{pairwise distinct}}} F(l_{i_1}, \ldots, l_{i_n}) \right] = \sum_{m_1, \ldots, m_n \in \mathbb{Z}} F(m_1, \ldots, m_n) \cdot \det \left[K(m_i, m_j) \right]_{i,j=1}^n .$$

The averages $\mathbb{E}_{\mathrm{Schur}}\, e_n(q^{N+\lambda_1-1},\dots,q^{\lambda_N})$ are exactly of the same type,

$$\mathbb{E}_{\mathrm{Schur}}\, e_n(q^{N+\lambda_1-1},\dots,q^{\lambda_N})=\frac{1}{n!}\sum_{m_1,\dots,m_n\in\mathbb{Z}_{\geq 0}} q^{m_1+\dots+m_n}\cdot \det\left[K(m_i,m_j)\right]_{i,j=1}^{n}$$

But rather than using the formula for the correlation kernel above, we'll use a more direct approach.

Difference operators

Consider q-difference operators (here $(T_{q,x}g)(x)=g(qx)$)

$$(\mathcal{D}_n f)(x_1,\dots,x_N)=\frac{1}{\prod_{i<j}(x_i-x_j)}\sum_{i_1<\dots<i_n} T_{q,x_{i_1}}\cdots T_{q,x_{i_n}}\prod_{i<j}(x_i-x_j)\, f(x_1,\dots,x_N).$$

It is essentially obvious that for $\lambda=(\lambda_1\geq\dots\geq\lambda_N\geq 0)$,

$$\mathcal{D}_n S_\lambda = e_n(q^{N+\lambda_1-1},\dots,q^{\lambda_N})\cdot S_\lambda .$$

Indeed, this follows from the fact that for any $\sigma\in S_N$,

$$\sum_{i_1<\dots<i_n} T_{q,x_{i_1}}\cdots T_{q,x_{i_n}}\, x_{\sigma(1)}^{N+\lambda_1-1}\cdots x_{\sigma(N)}^{\lambda_N} = e_n(q^{N+\lambda_1-1},\dots,q^{\lambda_N})\cdot x_{\sigma(1)}^{N+\lambda_1-1}\cdots x_{\sigma(N)}^{\lambda_N}.$$

The the expectation of e_n can be computed as

$$\mathbb{E}_{\mathrm{Schur}}\, e_n(q^{N+\lambda_1-1},\dots,q^{\lambda_N})=\frac{\sum e_n(\dots)\, S_\lambda(x)S_\lambda(y)}{\sum S_\lambda(x)S_\lambda(y)}=\frac{\mathcal{D}_n^{(x)}\prod_{i,j}(1-x_iy_j)^{-1}}{\prod_{i,j}(1-x_iy_j)^{-1}}.$$

Let us now set $n=1$. Then we can use the following (easy)

Lemma For a (locally) holomorphic function $f:\mathbb{C}\to\mathbb{C}$

$$\frac{\mathcal{D}_1\, f(x_1)\cdots f(x_N)}{f(x_1)\cdots f(x_N)}=\sum_{r=1}^{N}\prod_{j\neq N}\frac{qx_r-x_j}{x_r-x_j}\,\frac{f(qx_r)}{f(x_r)}=\frac{1}{2\pi i}\oint_{\text{around }\{x_i\}}\prod_{i=1}^{N}\frac{qw-x_i}{w-x_i}\,\frac{1}{qw-w}\,\frac{f(qw)}{f(w)}\,dw$$

Applying to $f(x)=\prod_j(1-xy_j)^{-1}$ yields

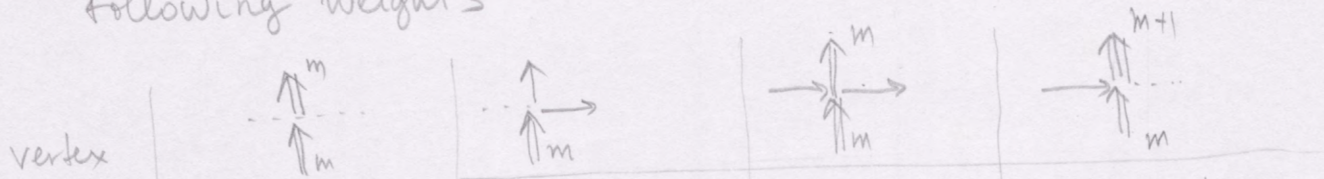
$$\mathbb{E}_{\mathrm{Schur}}\, e_1(q^{N+\lambda_1-1},\dots,q^{\lambda_N})=\frac{1}{2\pi i}\oint_{\text{around }\{x_i\}}\frac{1}{qw-w}\prod_{i=1}^{N}\frac{qw-x_i}{w-x_i}\cdot\prod_{j\geq 1}\frac{1-wy_j}{1-qwy_j}\,dw,$$

as desired. The argument for $n>1$ is very similar, just more w's are added.

Note that the above argument can actually be used for computing the correlation functions: one needs to apply the composition of several $\mathcal{D}_1$'s with different q's to the partition function, yielding $\mathbb{E}\prod_i\left(\sum_j q_i^{\lambda_j+N-j}\right)$, and then extract the coefficient of $\prod q_i^{x_i}$. This returns the determinantal formula above, as was shown by Amol Aggarwal in arXiv: 1401.6979.

We now switch to the vertex models.

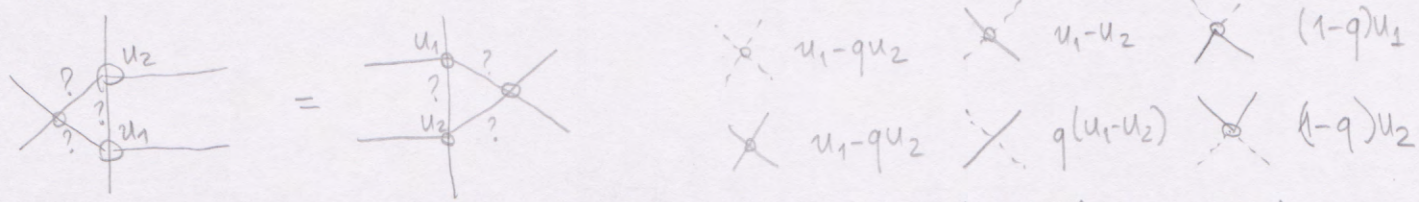
<u>Vertex models</u> Rather than considering the 6 vertex model, it is more natural to allow more general vertices with arbitrarily many vertical arrows (the arrow conservation still holds). Consider the following weights



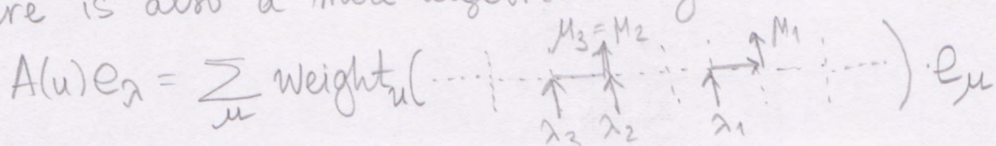
w	$\dfrac{1-sq^m u}{1-su}$	$\dfrac{(1-s^2q^{m-1})u}{1-su}$	$\dfrac{u-sq^m}{1-su}$	$\dfrac{1-q^{m+1}}{1-su}$

The normalization is such that $w(\cdots\!\!\!\cdot\!\!\cdots)=1$. The parameter s needs to be set to $q^{-1/2}$, then double edges do not appear as long as there are none on the top boundary, b/c $w(\to\!\Uparrow\cdots)=0$.

This specific choice of weights is explained by the fact that they satisfy the Yang-Baxter equation:



There is also a more algebraic way to state this relation. Define



$B(u)e_\lambda=\sum_\mu \mathrm{weight}_u(\to(\text{same})\cdots)\,e_\mu$

$C(u)e_\lambda=$ — " — $(\cdots(\text{same})\to)\,e_\mu$

$D(u)e_\lambda=$ — " — $(\to(\text{same})\to)\,e_\mu$

so far on a finite piece of the lattice.

Form the monodromy matrix $T(u)=\begin{bmatrix} A(u) & B(u) \\ C(u) & D(u)\end{bmatrix}$. Then YB reads

$$T(u_1)\otimes T(u_2) = Y\,\big(T(u_2)\otimes T(u_1)\big)\,Y^{-1},$$

where
$$Y=\frac{1}{(u_1-qu_2)(u_2-qu_1)}\begin{bmatrix} u_2-qu_1 & & & \\ & u_2-u_1 & (1-q)u_2 & \\ & (1-q)u_1 & q(u_2-u_1) & \\ & & & u_2-qu_1\end{bmatrix}$$

and $T(u_1)\otimes T(u_2)=\begin{bmatrix} A(u_1)T(u_2) & B(u_1)T(u_2) \\ C(u_1)T(u_2) & D(u_1)T(u_2)\end{bmatrix}$, $T(u_2)\otimes T(u_1)=\begin{bmatrix} T(u_2)A(u_1) & T(u_2)B(u_1) \\ T(u_2)C(u_1) & T(u_2)D(u_1)\end{bmatrix}$

In particular, A's commute between themselves, same for B's, C's, D's, and we also have more complicated relations like (matrix el't (2,4))

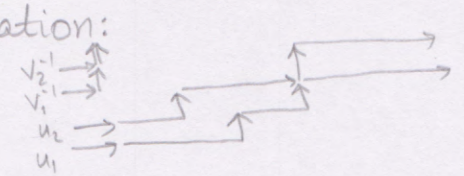
$$B(u_1)D(u_2)=\frac{u_1-u_2}{qu_1-u_2}\,D(u_2)B(u_1)+\frac{(1-q)u_2}{u_2-qu_1}\,B(u_2)D(u_1).$$

As we increase the length of the strip, the D-operator needs to be normalized:
$$\overline{D}(u)=\lim_{L=\#\text{ of vertices}\to\infty}\frac{D^{(L)}(u)}{\big(w_u(\rightarrow\!\circ\!\rightarrow)\big)^L}.$$

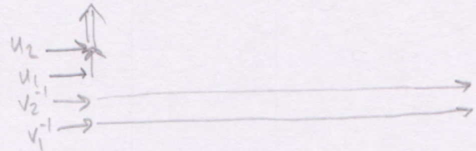
The above commutation relation then simplifies in the limit:

if $\left|\dfrac{w_{u_1}(\rightarrow\rightarrow)}{w_{u_2}(\rightarrow\rightarrow)}\right|<1$ then $B(u_1)\overline{D}(u_2)=\dfrac{u_1-u_2}{qu_1-u_2}\,\overline{D}(u_2)B(u_1)$.

We can now iterate this commutation relation:



$$\big(\overline{D}(v_N^{-1})\cdots\overline{D}(v_1^{-1})B(u_M)\cdots B(u_1)e_\varnothing,\,e_{0^M}\big)=$$



$$=\prod_{i,j}\frac{1-qu_iv_j}{1-u_iv_j}\big(B(u_M)\cdots B(u_1)\overline{D}(v_N^{-1})\cdots\overline{D}(v_1^{-1})e_\varnothing,e_M\big)=$$

$$=\prod_{i,j}\frac{1-qu_iv_j}{1-u_iv_j}\cdot\prod_i\frac{1-q^i}{1-su_i}.$$

The Cauchy identity In order to rewrite the above relation more explicitly, we need the following symmetrization formula:

$$B(u_M)\cdots B(u_1)e_\varnothing = \sum_{\lambda=(\lambda_1\geq\ldots\geq\lambda_M)} F_\lambda(u_1,\ldots,u_M)\, e_\lambda$$

with
$$F_\lambda(u_1,\ldots,u_M) = \frac{(1-q)^M}{\prod_i(1-su_i)}\cdot\sum_{\sigma\in S_M}\sigma\left(\prod_{i<j}\frac{u_i-qu_j}{u_i-u_j}\cdot\prod_i\left(\frac{u_i-s}{1-su_i}\right)^{\lambda_i}\right)$$

One could use the definition of B's to check this formula by induction on M, or use the algebraic Bethe ansatz formalism to derive it. The above identity then reads

$$\sum_\lambda F_\lambda(u_1,\ldots,u_M)\cdot\left(\bar{\mathcal{D}}(v_N^{-1})\cdots\bar{\mathcal{D}}(v_1^{-1}); e_\lambda, e_{0^M}\right) = \prod_i\frac{1-q^i}{1-su_i}\prod_{i,j}\frac{1-qu_iv_j}{1-u_iv_j}$$

At $q=s=0$ this turns into the familiar Cauchy identity for the Schur functions, and F_λ turns into s_λ. The case $s=0$ but $q\neq0$ corresponds to the Hall-Littlewood polynomials.

The 2nd term in the summation can also be evaluated via a similar symmetrization formula, and one way to obtain it is via the following orthogonality relation.

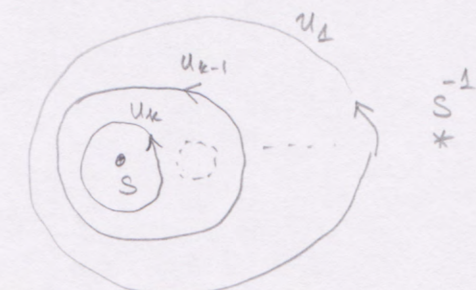
Proposition [B-Corwin-Petrov-Sasamoto '14, Povolotsky '13] For any $k\geq1$, $\lambda=(\lambda_1\geq\ldots\geq\lambda_k)\in\mathbb{Z}^k$ and similar μ,

$$\frac{c(\lambda)}{(2\pi i)^k}\oint\cdots\oint\prod_{\alpha<\beta}\frac{u_\alpha-u_\beta}{u_\alpha-qu_\beta}\cdot F_\lambda(u_1,\ldots,u_k)\cdot\prod_i\frac{1}{u_i-s}\left(\frac{1-su_i}{u_i-s}\right)^{\mu_i}du_i = \mathbb{1}_{\lambda=\mu}$$

where $c(\lambda)=\prod\frac{(s^2;q)_{\ell_i}}{(q;q)_{\ell_i}}$ if $\lambda=0^{\ell_0}1^{\ell_1}\ldots$, and the integration contours are as shown. Equivalently,

$$\frac{c(\lambda)}{(1-q)^k k!}\oint\cdots\oint\prod_{\alpha\neq\beta}\frac{u_\alpha-u_\beta}{u_\alpha-qu_\beta}F_\lambda(u)F_\mu(u^{-1})\,du = \mathbb{1}_{\lambda=\mu}$$

where all the contours are the same γ that includes $q\gamma$, encircles s and leaves s^{-1} outside.



The contours are q-nested.

Thus, one can think of $\{F_\lambda\}$ as of a Fourier basis.

<u>The specialization ρ.</u> We actually need a simplified version of the above Cauchy identity. It can be obtained by taking

$$(v_1,\dots,v_N)=(\varepsilon, q\varepsilon,\dots,q^{N-1}\varepsilon)\ \xrightarrow[\substack{\text{analytic continuation}\\ \text{in } q^N \text{ to } q^N=(\varepsilon s)^{-1}}]{\varepsilon\to 0}\ \text{"specialization } \rho\text{"}.$$

Then the RHS of the Cauchy identity becomes (the u-term doesn't change)

$$\prod_{i,j}\frac{1-qu_iv_j}{1-u_iv_j}\longrightarrow\prod_i\frac{1-q^Nu_i\varepsilon}{1-u_i\varepsilon}\longrightarrow\prod_i\frac{1-u_i/s}{1-u_i\varepsilon}\longrightarrow\prod_i\left(1-\frac{u_i}{s}\right),$$

and using the orthogonality we obtain the identity

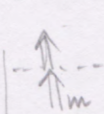
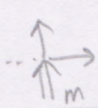
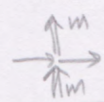
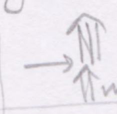
$$\sum_{\lambda=(\lambda_1\geq\dots\geq\lambda_M\geq 1)} F_\lambda(u_1,\dots,u_M)\,(-s)^{\lambda_1+\dots+\lambda_M}\cdot c(\lambda)=\prod_i\frac{s(s-u_i)}{1-su_i}.$$

This implies that if we modify the weights that produced F's via

$$w\left(\begin{smallmatrix} & i_2 & \\ j_1 & & j_2 \\ & i_1 & \end{smallmatrix}\right)\rightsquigarrow(-s)^{j_2}\cdot\frac{(s^2;q)_{i_2}}{(q;q)_{i_2}}\Big/\frac{(s^2;q)_{i_1}}{(q;q)_{i_1}}\cdot w\left(\begin{smallmatrix} & i_2 & \\ j_1 & & j_2 \\ & i_1 & \end{smallmatrix}\right)=:L$$

and also remove the trivial 0^{th} column from the definition then so defined F_λ^{stoch} will have the property that $\sum_{\lambda_1\geq\dots\geq\lambda_M\geq 1}F_\lambda^{\text{stoch}}(u)\equiv 1$.

Furthermore, this is obvious since the weights

vertex				
L	$\dfrac{1-sq^mu}{1-su}$	$\dfrac{-su+sq^mu}{1-su}$	$\dfrac{-su+s^2q^m}{1-su}$	$\dfrac{1-s^2q^m}{1-su}$

are now stochastic: $\sum_{i_2,j_2}L_u\left(\begin{smallmatrix} & i_2 & \\ j_1 & & j_2 \\ & i_1 & \end{smallmatrix}\right)\equiv 1$. This gives rise to the <u>stochastic six vertex model</u>.

<u>A slightly more general Cauchy identity</u> The idea is to add one (or a few) extra normal variable to the specialization ρ. This will slightly deform the summands, which we will interpret as multiplying the weight of λ by an observable, and will also add $\prod_i\frac{1-qu_iw}{1-u_iw}$ to the partition function, where w is the extra variable.

The extra factor (the observable) for a single extra w reads

$$q^M + \sum_{i=1}^{M} \frac{q^{i-1}}{(-s)^{\lambda_i}} \frac{1-q}{1-s^{-1}w} \left(\frac{w-s}{1-sw}\right)^{\lambda_i} =: \mathcal{O}_\lambda(w)$$

This looks a little complicated, but w is generic, and we can integrate over it. For example, we can use

$$\frac{1}{2\pi i} \oint_{\text{around } s^{-1}} \frac{1}{w-s} \cdot \left(\frac{1-sw}{w-s}\right)^{\theta-1} \mathcal{O}_\lambda(w)\, dw = \sum_{i=1}^{M} \frac{s q^{i-1}}{(-s)^{\lambda_i}} \cdot \frac{1-q}{1-s^2} \mathbb{1}_{\lambda_i=\theta}$$

Define the 1^{st} q-correlation function by

$$q\text{-corr}(\theta) = \mathbb{E} \sum_{i:\, \lambda_i=\theta} q^i . \qquad \left(\begin{array}{l}\text{More generally,}\\ q\text{-corr}(\theta_1,\dots,\theta_k)\end{array} = \mathbb{E} \sum_{\substack{i_1\dots i_k \\ \lambda_{i_1}=\theta_1,\dots,\lambda_{i_k}=\theta_k}} q^{i_1+\dots+i_k}\right)$$

The above arguments prove the following formula:

$$q\text{-corr}(\theta) = \frac{-q(-s)^{\theta-1}(1-s^2)}{1-q} \frac{1}{2\pi i} \oint_{\text{around } s^{-1}, u^{-1}} \frac{1}{w-s} \left(\frac{1-sw}{w-s}\right)^{\theta-1} \cdot \prod_{j=1}^{M} \frac{1-qu_jw}{1-u_jw}\, dw$$

for the convergence of Cauchy

More generally, similar arguments prove

$$q\text{-corr}(\theta_1,\dots,\theta_k) = (-1)^k q^{\frac{k(k+1)}{2}} (-s)^{\sum(\theta_i-1)} \cdot c(\theta) \oint\dots\oint_{\substack{q^{-1}\text{ nested around } s^{-1}, u^{-1}\\ w_1 \text{ inside}}} \prod_{\alpha<\beta} \frac{w_\alpha-w_\beta}{w_\alpha-qw_\beta} \prod_{i=1}^{k} \frac{1}{w_i-s}\left(\frac{1-sw_i}{w_i-s}\right)^{\theta_i-1}$$

$$* \prod_{j=1}^{M} \frac{1-qu_jw_i}{1-u_jw_i}\, dw_i .$$

Finally, what we actually need is

$$q^{\text{height}(x)} - 1 = \frac{1-q}{-q} \sum_{\theta \geq x} q\text{-corr}(\theta).$$

Since $\displaystyle\sum_{\theta\geq x} \left(\frac{1-sw}{w-s}\right)^{\theta-1} (-s)^{\theta-1} = \left(\frac{1-sw}{w-s}\right)^{x-1} (-s)^{x-1} \frac{1}{1+s(1-sw)/w-s} = \left(\frac{s(sw-1)}{w-s}\right)^{x-1} \frac{w-s}{(1-s^2)w}$

we obtain

$$\mathbb{E}\left(q^{h(x)}-1\right) = \frac{1}{2\pi i} \oint_{\text{around } u^{-1}} \left(\frac{1-sw}{1-s^{-1}w}\right)^{x-1} \cdot \prod_{j=1}^{M} \frac{1-qu_jw}{1-u_jw} \cdot \frac{dw}{w}$$

Similarly, one proves

$$\mathbb{E}\prod_{i=1}^{\ell}\left(q^{h(x)}-q^{i-1}\right)=q^{\frac{\ell(\ell-1)}{2}}\oint_{\text{around } u^{-1}}\cdots\oint\prod_{\alpha<\beta}\frac{w_\alpha-w_\beta}{w_\alpha-qw_\beta}\cdot\prod_{i=1}^{\ell}\left(\frac{1-sw_i}{1-s^{-1}w_i}\right)^{x_i-1}\prod_{j=1}^{M}\frac{1-qu_jw_i}{1-u_jw_i}\frac{dw_i}{w_i}.$$

The needed summation under the integral to get this from the q-correlation functions is again a version of the Cauchy identity. Note that the nesting of integrals around u^{-1} does not actually matter as the corresponding residues vanish.

To bring the above formula to the form on page 6, we need a simple identity

$$\frac{(-1)^\ell}{(2\pi i)^\ell}\oint\cdots\oint\prod_{a<b}\frac{w_a-w_b}{w_a-qw_b}\prod_i\frac{f(w_i)\,dw_i}{w_i}=$$

$$=\frac{q^{-\ell(\ell-1)/2}(q;q)_\ell}{(2\pi i)^\ell\cdot\ell!}\oint\cdots\oint\det\left[\frac{1}{qw_i-w_j}\right]_{i,j=1}^{\ell}\prod_i f(w_i)\,dw_i$$

It is proved by symmetrization of the integration variables, and thus all the integration contours are the same here (f is arbitrary).