The ideas behind the construction of our covariant quark model *à la* Bakamjian-Thomas

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Introduction - Motivation

- Objective :
  the philosophy of the BT construction for heavy to heavy meson transitions

- Why : covariance of the transition amplitudes
- How : a lot of group theory. . . and a hint of dynamics

will avoid formulae as much as possible, only ideas

Major contribution of Jean-Claude in building this formalism

Sommaire

The Wigner Theorem

$\mathscr{H}$: Hilbert space of our model

	Inertial frame $\mathscr{R}$	Inertial frame $\mathscr{R}'$
	initial state $\lvert\psi\rangle$ final state $\lvert\phi\rangle$ (detector)	$\lvert\psi'\rangle$ $\lvert\phi'\rangle$
Probability	$\mathscr{P} = \lvert\langle\phi\,\vert\,\psi\rangle\rvert^2$	$\mathscr{P}' = \lvert\langle\phi'\,\vert\,\psi'\rangle\rvert^2$

Starting point

relativistic invariance $\Longleftrightarrow$ $\mathscr{P} = \mathscr{P}'$

How ?

Group theory : the Wigner theorem

The Wigner Theorem

Answer

$$\boxed{|\psi\rangle' = U(\Lambda, a)|\psi\rangle}$$

where

$U(\Lambda, a)$ is a unitary transformation
satisfies the usual group multiplication law :

$$U(\Lambda_2, a_2)U(\Lambda_1, a_1) = U(\Lambda_2\Lambda_1, a_2 + \Lambda_2 a_1)$$

Specifying $U(\Lambda, a)$ on $\mathscr{H}$ *defines* the relativistic quantum model

$\Longrightarrow$ New problem : find $U(\Lambda, a)$

Solution *à la* Bakamjian-Thomas

- Goal : implementing the Poincaré group transformations for a *finite* number of interacting particles (2 in our case)
- How : by redefining the generators of the Poincaré group such that the structure of the group is preserved
- In practice : a four step process

Solution *à la* Bakamjian-Thomas (II)

1. Change of variables : new set of internal variables

$$\begin{cases} \text{total momentum } \vec{P} = \vec{p}_1 + \vec{p}_2 \\ \text{internal momenta } \vec{k}_1 \text{ and } \vec{k}_2 \ (\vec{k}_1 + \vec{k}_2 = \vec{0}) \\ \text{internal spins } \vec{s}_1' \text{ and } \vec{s}_2' \end{cases} \qquad \begin{pmatrix} 1 \rightsquigarrow \text{heavy quark} \\ 2 \rightsquigarrow \text{light quark} \end{pmatrix}$$

2. Poincaré group generators

$$\begin{cases} \text{space translations} : \vec{P} \\ \text{time translation} : H = P^o = \sqrt{\vec{P}^2 + M^2} \\ \text{rotations} : \vec{J} = -i\vec{P} \times \dfrac{\partial}{\partial \vec{P}} + \vec{S} \\ \text{boosts} : \vec{K} = -\dfrac{i}{2}\left\{H, \dfrac{\partial}{\partial \vec{P}}\right\} - \dfrac{\vec{P} \times \vec{S}}{H + M} \end{cases}$$

Constraint : M (mass operator) function of the internal variables only and rotationally invariant

3. Expression of the wave function Ψ with these new variables

$$\Psi \xrightarrow[\text{transformation}]{\text{unitary}} \Psi^{\text{int}}$$

Solution *à la* Bakamjian-Thomas (III)

4. Introduction of the internal wave function at rest φ of the meson :

$$\Psi^{int}_{s_1,s_2}(\vec{P},\ \vec{k}_2) = (2\pi)^3\,\delta(\vec{p}_1+\vec{p}_2-\vec{P})\,\varphi_{s_1,s_2}(\vec{k}_2)$$

To sum it up

$$\Psi_{s_1,s_2}(\vec{p}_1,\vec{p}_2) \ = \ \sqrt{\frac{p_1^o+p_2^o}{M_o}}\,\frac{\sqrt{k_1^o\,k_2^o}}{\sqrt{p_1^o\,p_2^o}}\,\sum_{s_1',s_2'}\left(D(\boldsymbol{R}_1)_{s_1,s_1'}\,D(\boldsymbol{R}_2)_{s_2,s_2'}\right)\Psi^{int}_{s_1',s_2'}(\vec{P},\vec{k}_2)$$

$D(R_i)$: *spin 1/2 Wigner rotations (combination of boosts)*

This is the "$\Psi' = U(\Lambda,\ a)\Psi$"-like type transformation we were looking for

Consequence : the transition amplitude

$$\langle\psi'|J|\psi\rangle = \int \frac{d\vec{p}_2}{(2\pi)^3}\sqrt{\frac{(p_1'^o+p_2^o)(p_1^o+p_2^o)}{M_o' M_o}}\,\frac{\sqrt{k_1'^o k_1^o}}{\sqrt{p_1'^o p_1^o}}\,\frac{\sqrt{k_2'^o k_2^o}}{\sqrt{p_2^o p_2^o}}$$
$$\times\sum_{s_1',s_2'}\sum_{s_1,s_2}\varphi'_{s_1',s_2'}(\vec{k}_2')^*[D(\mathbf{R}_1'^{-1})J(\vec{p}_1',\vec{p}_1)D(\mathbf{R}_1)]_{s_1',s_1}\,D(\mathbf{R}_2'^{-1}\mathbf{R}_2)_{s_2',s_2}\,\varphi_{s_1,s_2}(\vec{k}_2)$$

Comments :

- ψ and ψ' : heavy meson states
- wave functions are "covariantized"
- current J defined as acting on the heavy quark in a frame, BUT does not remain so in every frame...

$\Longrightarrow$ extra assumption required to get a *covariant* amplitude

The infinite mass limit

Heavy quark symmetries

- ⇝ flavour of the heavy quark irrelevant
- ⇝ independent conservation of the heavy quark spin $\vec{s}_1$ and of the "light component" $\vec{j} = \vec{\ell} + \vec{s}_2$

Consequences

1. Current J becomes covariant in this limit
2. Decomposition of the internal wave function :

$$\varphi_{s_1,s_2}(\vec{k}_2) = \underbrace{\frac{i}{\sqrt{2}}(\chi\sigma_2)_{s_1\,s_2}}_{\text{spin-orbit part}} \underbrace{\varphi(\|\vec{k}_2\|^2)}_{\text{radial part}}$$

(The spin-orbit part is calculated from angular momentum combination algebra)

Final result

$$\langle\psi'|J|\psi\rangle = \frac{1}{8}\,\frac{1}{\sqrt{v_o\,v'_o}}\int\frac{d\vec{p}_2}{(2\pi)^3}\,\frac{1}{p_2^o}\,\frac{\sqrt{(p_2.v')(p_2.v)}}{\sqrt{(p_2.v'+m_2)(p_2.v+m_2)}}$$
$$\times\,\mathrm{Tr}\left\{J\,(1+\not v)\,(m_2+\not p_2)\,\left(\boldsymbol{B}_{v'}\chi^\dagger\boldsymbol{B}_{v'}^{-1}\right)\,(1+\not v')\right\}$$
$$\times\,\phi_j(\|\overrightarrow{\boldsymbol{B}_{v'}^{-1}p_2}\|^2)^*\varphi(\|\overrightarrow{\boldsymbol{B}_{v}^{-1}p_2}\|^2)$$

Covariance is now obvious :

for the 4 D^{**} states

$$\begin{cases}
\boldsymbol{B}_{v'}\chi^\dagger_{(^3P_0)}\boldsymbol{B}_{v'}^{-1} &= \frac{1}{\sqrt{3}}\left[\not p_2 - (p_2\cdot v')\,\not v'\right]\gamma_5\\
\boldsymbol{B}_{v'}\chi^\dagger_{(^1P_1)}\boldsymbol{B}_{v'}^{-1} &= -(p_2\cdot\epsilon^*)\\
\boldsymbol{B}_{v'}\chi^\dagger_{(^3P_1)}\boldsymbol{B}_{v'}^{-1} &= -\frac{i}{\sqrt{2}}\,\epsilon^{\mu\nu\rho\sigma}v'_\mu\epsilon^*_\nu {p_2}_\rho\gamma_\sigma\gamma_5\\
\boldsymbol{B}_{v'}\chi^\dagger_{(^3P_2)}\boldsymbol{B}_{v'}^{-1} &= -\gamma^\mu {p_2}^\nu\epsilon^*_{\mu\nu}\gamma_5
\end{cases}$$

and J = combination of Dirac matrices

Conclusion and remarks

- Transition amplitudes written in a covariant way (group theoretical arguments *only*)
- NO dynamics used...yet...
- Known scaling behaviour of the infinite mass limit recovered, i.e. Isgur-Wise functions, and we even have expressions for them :

$$\tau_{1/2}^{(n)}(w) = \frac{1}{2\sqrt{3}} \int \frac{d\vec{p}_2}{(2\pi)^3} \frac{1}{p_2^o} \frac{\sqrt{(p_2 \cdot v')(p_2 \cdot v)}}{\sqrt{(p_2 \cdot v' + m_2)(p_2 \cdot v + m_2)}} \times \frac{(p_2 \cdot v)(p_2 \cdot v' + m_2) - (p_2 \cdot v')(p_2 \cdot v' + w\, m_2) + (1-w) m_2^2}{1-w} \times \phi_{\frac{1}{2}}^{(n)}(\|\overrightarrow{\boldsymbol{B}_{v'}^{-1} p_2}\|^2)^* \varphi^{(0)}(\|\overrightarrow{\boldsymbol{B}_{v}^{-1} p_2}\|^2)$$

- Sum rules (such as Bjorken) *spontaneously* satisfied
- Quantitative predictions require the knowledge of the radial part of the internal wave functions at rest (the orange thingies)

$\Longrightarrow$ choice of a mass operator M and resolution of a Schrödinger equation

Facteurs de Forme pour les processus $B \to D^{**}$ (avant)

Multiplet $j^P = 1/2^+$	
$J=0$	$\langle \frac{1}{2}\, 0^+ \vert\, V_\lambda \,\vert B \rangle = 0$ $\langle \frac{1}{2}\, 0^+ \vert\, A_\lambda \,\vert B \rangle = \frac{1}{2} \frac{1}{\sqrt{v_0\, v'_0}} \left[\tilde{u}_+ (v_\lambda + v'_\lambda) + \tilde{u}_- (v_\lambda - v'_\lambda) \right]$
$J=1$	$\langle \frac{1}{2}\, 1^+ \vert\, V_\lambda \,\vert B \rangle = \frac{1}{2} \frac{1}{\sqrt{v_0\, v'_0}} \left[\tilde{l}_{1/2}\, \epsilon^*_\lambda + \epsilon^*_\alpha\, v^\alpha \left[\tilde{c}_{1/2+} (v_\lambda + v'_\lambda) + \tilde{c}_{1/2-} (v_\lambda - v'_\lambda) \right] \right]$ $\langle \frac{1}{2}\, 1^+ \vert\, A_\lambda \,\vert B \rangle = \frac{1}{2} \frac{1}{\sqrt{v_0\, v'_0}}\, i\, \tilde{q}_{1/2}\, \epsilon_{\lambda\alpha\beta\gamma} (v^\beta + v'^\beta)(v^\gamma - v'^\gamma)$
Multiplet $j^P = 3/2^+$	
$J=1$	$\langle \frac{3}{2}\, 1^+ \vert\, V_\lambda \,\vert B \rangle = \frac{1}{2} \frac{1}{\sqrt{v_0\, v'_0}} \left[\tilde{l}_{3/2}\, \epsilon^*_\lambda + \epsilon^*_\alpha\, v^\alpha \left[\tilde{c}_{3/2+} (v_\lambda + v'_\lambda) + \tilde{c}_{3/2-} (v_\lambda - v'_\lambda) \right] \right]$ $\langle \frac{3}{2}\, 1^+ \vert\, A_\lambda \,\vert B \rangle = \frac{1}{2} \frac{1}{\sqrt{v_0\, v'_0}}\, i\, \tilde{q}_{3/2}\, \epsilon_{\lambda\alpha\beta\gamma} (v^\beta + v'^\beta)(v^\gamma - v'^\gamma)$
$J=2$	$\langle \frac{3}{2}\, 2^+ \vert\, V_\lambda \,\vert B \rangle = \frac{1}{2} \frac{1}{\sqrt{v_0\, v'_0}}\, i\, \tilde{h}_{3/2}\, \epsilon_{\lambda\alpha\beta\gamma}\, \epsilon^{\alpha\mu *}\, v_\mu\, (v^\beta + v'^\beta)(v^\gamma - v'^\gamma)$ $\langle \frac{3}{2}\, 2^+ \vert\, A_\lambda \,\vert B \rangle = \frac{1}{2} \frac{1}{\sqrt{v_0\, v'_0}} \left[\tilde{k}\, \epsilon^*_{\lambda\alpha} v^\alpha + \epsilon^*_{\alpha\beta} v^\alpha v^\beta \left[\tilde{b}_+ (v_\lambda + v'_\lambda) + \tilde{b}_- (v_\lambda - v'_\lambda) \right] \right]$

Normalisation : $\langle M(\vec{v}\,') \vert M(\vec{v}) \rangle \;=\; (2\pi)^3\, \delta^3(\vec{p} - \vec{p}\,')$

Facteurs de Forme pour les processus $B \to D^{**}$ (après)

Multiplet $j^P = 1/2^+$	
$J = 0$	$\langle \frac{1}{2}\, 0^+ \vert\, V_\lambda \,\vert B\rangle = 0$ $\langle \frac{1}{2}\, 0^+ \vert\, A_\lambda \,\vert B\rangle = -\frac{1}{\sqrt{v_0\, v'_0}}(v_\lambda - v'_\lambda)\, \tau_{1/2}(w)$
$J = 1$	$\langle \frac{1}{2}\, 1^+ \vert\, V_\lambda \,\vert B\rangle = \frac{1}{\sqrt{v_0\, v'_0}}\Big[(v\cdot\epsilon^*)v'_\lambda \;+\; (1 - v\cdot v')\epsilon^*_\lambda\Big]\, \tau_{1/2}(w)$ $\langle \frac{1}{2}\, 1^+ \vert\, A_\lambda \,\vert B\rangle = \frac{i}{\sqrt{v_0\, v'_0}}\, \epsilon_{\lambda\sigma\rho\tau}\, v^\sigma\, \epsilon^{*\rho}\, v'^\tau\, \tau_{1/2}(w)$
Multiplet $j^P = 3/2^+$	
$J = 1$	$\langle \frac{3}{2}\, 1^+ \vert\, V_\lambda \,\vert B\rangle = \frac{1}{2\sqrt{2}}\,\frac{1}{\sqrt{v_o\, v'_o}}\Big[(v\cdot\epsilon^*)\Big\{v'_\lambda(v\cdot v' - 2) - 3\, v_\lambda\Big\} \;+\; \epsilon^*_\lambda(1 - v\cdot v')(1 + v\cdot v')\Big]\, \tau_{3/2}(w)$ $\langle \frac{3}{2}\, 1^+ \vert\, A_\lambda \,\vert B\rangle = \frac{i}{2\sqrt{2}}\,\frac{1}{\sqrt{v_o\, v'_o}}\,(1 + v\cdot v')\, \epsilon_{\lambda\sigma\rho\tau}\, v^\sigma\, \epsilon^{*\rho}\, v'^\tau\, \tau_{3/2}(w)$
$J = 2$	$\langle \frac{3}{2}\, 2^+ \vert\, V_\lambda \,\vert B\rangle = -i\frac{\sqrt{3}}{2}\,\frac{1}{\sqrt{v_o\, v'_o}}\, \epsilon_{\lambda\alpha\mu\beta}\, \epsilon^{*\mu\nu}\, v_\nu\, v^\alpha\, v'^\beta\, \tau_{3/2}(w)$ $\langle \frac{3}{2}\, 2^+ \vert\, A_\lambda \,\vert B\rangle = \frac{\sqrt{3}}{2}\,\frac{1}{\sqrt{v_o\, v'_o}}\Big[(1 + v\cdot v')\, \epsilon^*_{\lambda\nu}\, v^\nu \;-\; v^\mu\, v^\nu\, \epsilon^*_{\mu\nu}\, v'_\lambda\Big]\, \tau_{3/2}(w)$

Normalisation : $\langle M(\vec{v}\,')\vert M(\vec{v})\rangle \;=\; (2\pi)^3\, \delta^3(\vec{p} - \vec{p}\,')$

On a posé : $w = v\cdot v'$

The magic. . .

The dynamics actually used in the model does not need to be relativistic ; it is the way of writing the matrix elements which is covariant ! ! !

MERCI POUR VOTRE ATTENTION