Integrability, the qKZ equation, and connections

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The braid relation

Start with permutations:

- The transposition π_i swaps arguments i and $i+1$ of a function, e.g.

$$
\begin{tikzcd}
 & f(z_1, z_2, z_3) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
f(z_2, z_1, z_3) \arrow[d, "\pi_2"'] & & f(z_1, z_3, z_2) \arrow[d, "\pi_1"] \\
f(z_2, z_3, z_1) \arrow[dr, "\pi_1"'] & & f(z_3, z_1, z_2) \arrow[dl, "\pi_2"] \\
 & f(z_3, z_2, z_1) &
\end{tikzcd}
$$

- This is called the braid relation:

$$\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}$$

The Yang–Baxter relation

- The braid relation

$$\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}$$

- The Yang–Baxter relation generalises the braid relation

$$R_i(w) R_{i+1}(wz) R_i(z) = R_{i+1}(z) R_i(wz) R_{i+1}(w)$$

- $R_i(u)$ is called an R matrix, with complex parameter u.
- There are many known R matrices solving the Yang–Baxter relation. Each solution gives a different integrable system.

The qKZ equation

What can we do with an R matrix?

- We can look for a vector $|\Psi(z_1,\ldots,z_N)\rangle$ satisfying the qKZ equation

$$R_i(z_i/z_{i+1})|\Psi(z_1,\ldots,z_N)\rangle = \pi_i|\Psi(z_1,\ldots,z_N)\rangle$$

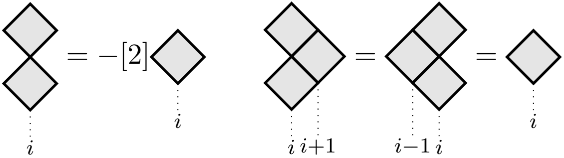
- The braid relation and Yang–Baxter relations ensure consistency, e.g. for $N=3$

$$\begin{gathered} R_1(z_2/z_3)R_2(z_1/z_3)R_1(z_1/z_2)|\Psi(z_1,z_2,z_3)\rangle \\ \| \\ \pi_1\pi_2\pi_1|\Psi(z_1,z_2,z_3)\rangle \\ \| \\ \pi_2\pi_1\pi_2|\Psi(z_1,z_2,z_3)\rangle \\ \| \\ R_2(z_1/z_2)R_1(z_1/z_3)R_2(z_2/z_3)|\Psi(z_1,z_2,z_3)\rangle \end{gathered}$$

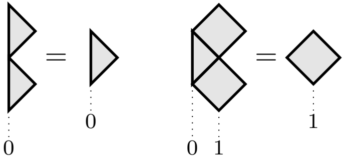
- But first we need an R matrix!

One boundary Temperley–Lieb algebra

- Generators $e_0, \ldots e_{N-1}$
- Bulk relations $e_i^2 = -[2]e_i$, $e_i e_{i\pm1} e_i = e_i$:



- Boundary relations $e_0^2 = e_0$, $e_1 e_0 e_1 = e_1$:

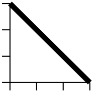
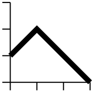
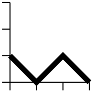


- With t-number

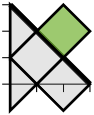
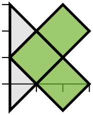
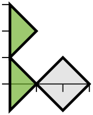
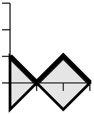
$$[u] = \frac{t^u - t^{-u}}{t - t^{-1}}.$$

Action on Ballot paths

- Ballot paths of length $N = 3$

$\Omega =$  $\alpha_1 =$  $\alpha_2 =$ 

- Example

$e_2|\Omega\rangle =$  $=$  $=$  $=$  $= |\alpha_2\rangle$

- Matrix form

$$e_2 = \begin{array}{c} \\ |\Omega\rangle \\ |\alpha_1\rangle \\ |\alpha_2\rangle \end{array} \begin{array}{c} \begin{array}{ccc} |\Omega\rangle & |\alpha_1\rangle & |\alpha_2\rangle \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & -[2] \end{pmatrix} \end{array}$$

The R-matrix

- Can check that this R-matrix satisfies the Yang-Baxter relation:
$$R_i(z) = \frac{t - t^{-1}z}{tz - t^{-1}}\mathbb{1} - \frac{z-1}{tz - t^{-1}}e_i$$
- With the boundary operator, define the K-matrix
$$K_0(z) = \frac{(1 - z^{-1}\zeta_1^{-1})(z - t\zeta_1)}{(z - \zeta_1)(t - z^{-1}\zeta_1^{-1})}\mathbb{1} - \frac{(1-t)(z - z^{-1})}{(z - \zeta_1)(t - z^{-1}\zeta_1^{-1})}e_0$$
which satisfies the reflection equation
$$K_0(z)R_1(wz)K_0(w)R_1(w/z) = R_1(w/z)K_0(w)R_1(wz)K_0(z)$$

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Mixed boundary qKZ equation

- Write a general vector in the Ballot path basis as

$$|\Psi(z_1,\ldots,z_N)\rangle = \sum_\alpha \psi_\alpha(z_1,\ldots,z_N)|\alpha\rangle$$

- The bulk part of the qKZ equation

$$R_i(z_i/z_{i+1})|\Psi(z_1,\ldots,z_N)\rangle = \pi_i|\Psi(z_1,\ldots,z_N)\rangle$$

- And in component form:

$$\sum_\alpha \psi_\alpha(z_1,\ldots,z_N)\Big(e_i|\alpha\rangle\Big) = \sum_\alpha \Big(T_i(-1)\psi_\alpha(z_1,\ldots,z_N)\Big)e_i|\alpha\rangle$$

where $T_i(u)$ is an operator acting on Laurent polynomials.

- The boundary equations

$$K_0(z_1^{-1})|\Psi(z_1,z_2,\ldots,z_N)\rangle = |\Psi(z_1^{-1},z_2,\ldots,z_N)\rangle,$$
$$|\Psi(z_1,\ldots,z_{N-1},z_N)\rangle = |\Psi(z_1,\ldots,z_{N-1},t^3 z_N^{-1})\rangle$$

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Solution of the qKZ equation

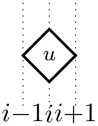
Theorem (de Gier, Pyatov 2010)

The solutions of the qKZ equation have a factorised form

$$\psi_\alpha(z_1,\ldots,z_N) = \prod_{i,j}^{\nearrow u_{i,j}} T_i(u_{i,j})\psi_\Omega(z_1,\ldots,z_N)$$

The product is constructed using a graphical representation of the Hecke generators

$T_0(u) =$ 

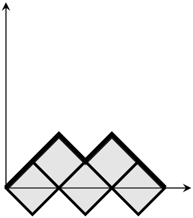
, $\quad T_i(u) =$ 

.

These are operators on Laurent polynomials, which also satisfy Yang–Baxter and reflection relations.

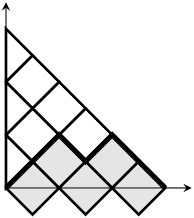
Factorised solutions

- Factorised solution for $\psi_\alpha(z_1, \ldots, z_N)$



Factorised solutions

- Factorised solution for $\psi_\alpha(z_1,\ldots,z_N)$
- Fill to maximal Ballot path $\Omega=(N,N-1,\ldots,0)$



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- Factorised solution for $\psi_\alpha(z_1, \ldots, z_N)$
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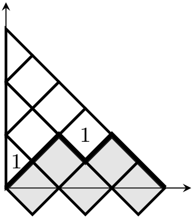

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- Factorised solution for $\psi_\alpha(z_1, \ldots, z_N)$
- Fill to maximal Ballot path $\Omega = (N, N-1, \ldots, 0)$
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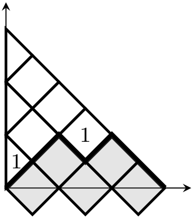
- Label remaining tiles by rule
  $u_{i,j} = \max\{u_{i+1,j-1}, u_{i-1,j-1}\} + 1$



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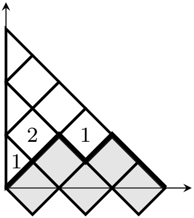

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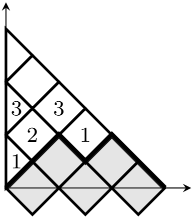

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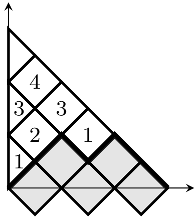

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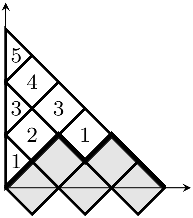

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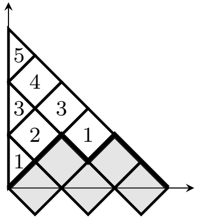


$$\psi_\alpha = T_0(1).T_1(2)T_0(3).T_3(1)T_2(3)T_1(4)T_0(5)\psi_\Omega$$

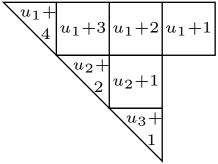
and

$$\psi_\Omega = \Delta_t^-(z_1, \ldots, z_N)\Delta_t^+(z_1, \ldots, z_N)$$

Alternate filling

Fill with consecutive integers along rows, e.g. for previous shape tilted by 45 °

$$\psi_{4,2,1}(u_1+1, u_2+1, u_3+1)$$

$=$ 

$$= \mathcal{T}_1(u_3+1)\mathcal{T}_2(u_2+1)\mathcal{T}_3(u_1+1)\psi_\Omega$$

where

$$\mathcal{T}_a(u+1) = T_{a-1}(u+1)\dots T_1(u+a-1)T_0(u+a)$$

gives a row of length a, numbered from $u+1$.

Sum rule

Theorem (de Gier, F)

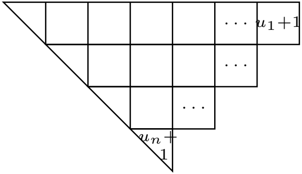
The staircase diagram has the expansion

$$\psi_{\bar{a}_1,\ldots,\bar{a}_n}(u_1+1,\ldots,u_n+1) = \sum_{\alpha} c_\alpha \psi_\alpha(z_1,\ldots,z_N),$$

where the coefficients c_α are non-zero and are monomials in

$$y_i = -\frac{[u_i]}{[u_i+1]}, \qquad \tilde{y}_i = -B_0(u_i+1).$$

$$\psi_{\bar{a}_1,\ldots,\bar{a}_n}(u_1+1,\ldots,u_n+1) =$$

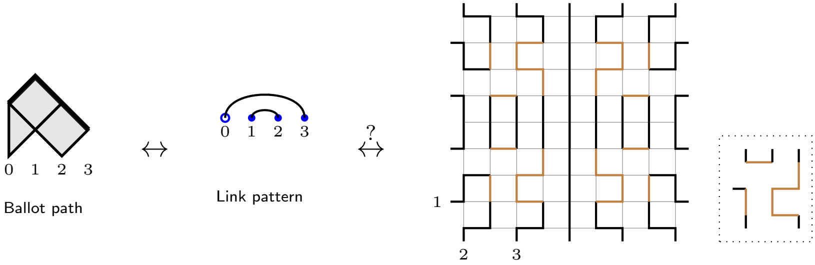


Specialisation of the sum rule

- At specialisation $u_i = 1$, $t = \mathrm{e}^{\pm 2\pi \mathrm{i}/3}$, all coefficients $c_\alpha = 1$.

$$\psi_{\bar{a}_1,\ldots,\bar{a}_n}(2,\ldots,2) = \sum_{\alpha} \psi_\alpha(z_1,\ldots,z_N),$$

  and at this point there is a closed form for the sum [Zinn-Justin 2007].
- Connections to Temperley–Lieb loop model and Razumov–Stroganov conjectures



Ballot path

Link pattern

Fully packed loops

Proof of the sum rule

- Recall the sum rule

$$\psi_{\bar{a}_1,\dots,\bar{a}_n}(u_1+1,\dots,u_n+1) = \sum_\alpha c_\alpha \psi_\alpha(z_1,\dots,z_N),$$

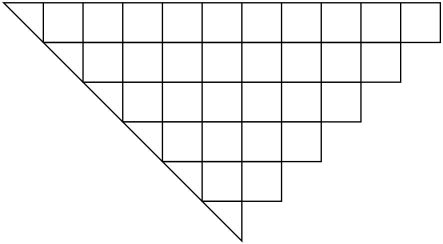
  where the coefficients c_α are non-zero and are monomials in y_i, $\tilde{y}_i$.
- Proof requires two steps:
  - Show how to expand the staircase diagram in terms of components ψ_α.
  - Show that each component ψ_α arises exactly once in the expansion.

Expanding the staircase

- First expansion using result

$$\psi_{\bar{a}_1,\ldots,\bar{a}_n}(u_1+1,\ldots,u_n+1) \\ = \prod_{i=n,n-1,\ldots,1} \left(\mathcal{T}_{\bar{a}_i}(1) + y_i \mathcal{T}_{\bar{a}_i-1}(1) + \tilde{y}_i\right) \psi_\Omega.$$

- Example term from expansion with $N=12$



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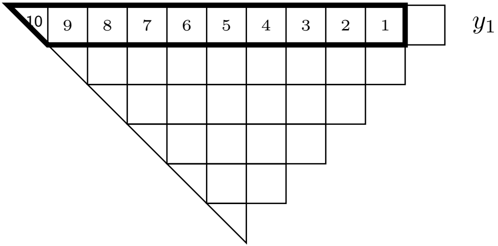

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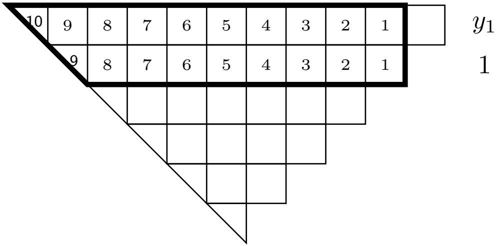

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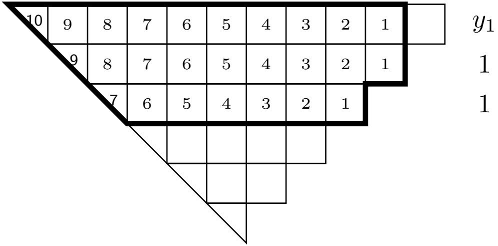

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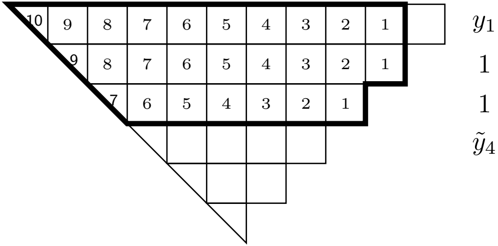

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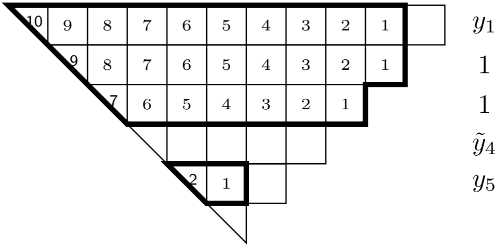

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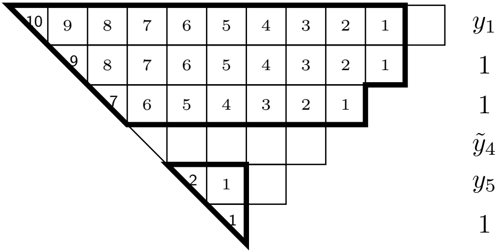

Expanding the staircase

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$$\begin{aligned}&\psi_{\bar{a}_1,\ldots,\bar{a}_n}(u_1+1,\ldots,u_n+1)\\&=\prod_{i=n,n-1,\ldots,1}\left(\mathcal{T}_{\bar{a}_i}(1)+y_i\mathcal{T}_{\bar{a}_i-1}(1)+\tilde{y}_i\right)\psi_\Omega.\end{aligned}$$

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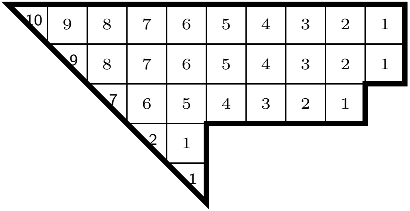
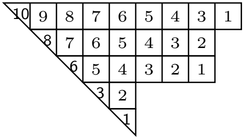


- Coefficient $y_1\tilde{y}_4y_5$, but a second expansion is required for terms like this one.

Working backwards - an algorithm

- Work backwards from ψ_α to term from staircase expansion.

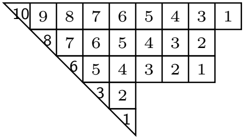
$$\psi_\alpha(z_1,\ldots,z_N) =$$



Working backwards - an algorithm

- Work backwards from ψ_α to term from staircase expansion.

$$\psi_\alpha(z_1, \ldots, z_N) =$$



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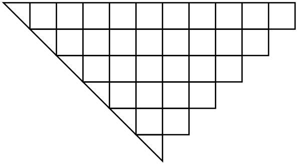

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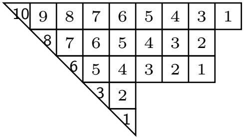
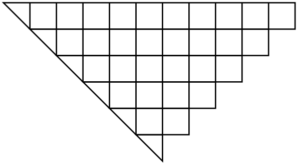


- Draw empty maximal staircase
- Add rows to staircase, bottom up, in lowest place each fits



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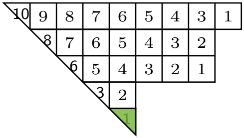

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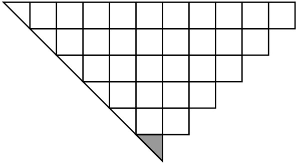

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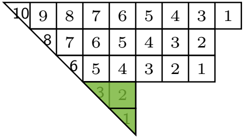

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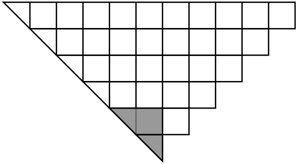

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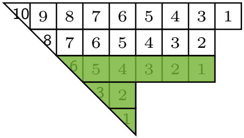

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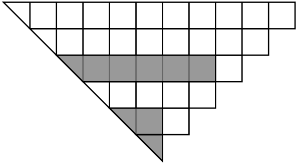

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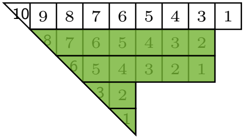

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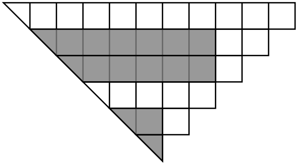

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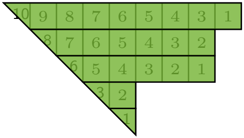
$\psi_\alpha(z_1, \ldots, z_N) =$ 

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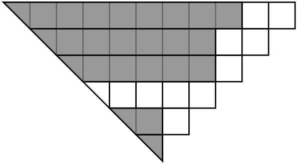

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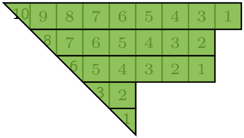

- Draw empty maximal staircase
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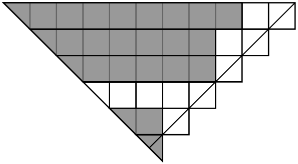

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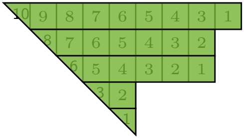

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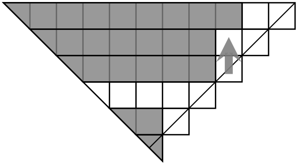

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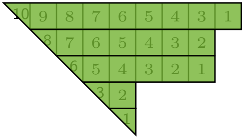

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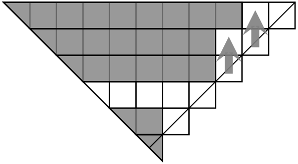

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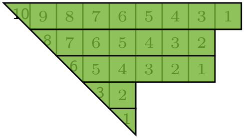
$$\psi_\alpha(z_1, \ldots, z_N) =$$ 

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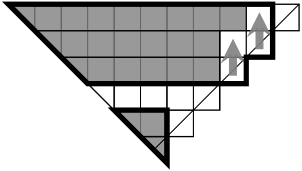

- Coefficient $c_\alpha = y_1 \tilde{y}_4 y_5$.

Conclusion and future work

- We have found a factorised form for a sum rule for the mixed boundary qKZ equation.
- We would like to find a way to evaluate the general sum, or at certain specialisations. This may shed light on Razumov–Stroganov conjectures.
- There are connections to special functions (Macdonald and Koornwinder polynomials) and special bases of the Hecke algebra, which need to be clarified.