Recent progress in formal theory

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RPP 2016
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Recent progress in ~~formal~~ theory

in planar N=4 SYM theory

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N=4 super Yang Mills theory

Maximally supersymmetric version of YM theory in 4d

$$\mathcal{L} = \frac{1}{4g^2}\mathrm{tr}\, F_{\mu\nu}F_{\mu\nu} + \frac{1}{2}\mathrm{tr}\, D_\mu \Phi_{AB} D_\mu \Phi_{AB} + \mathrm{tr}\, \bar{\Psi}_A \gamma_\mu D_\mu \Psi_A$$

$$+ \text{ Yukawa and quartic interactions}$$

For short, the theory of a massless spin 1 (extended) supermultiplet (everything else follows from it)

One thing to keep in mind : it is conformally invariant
Corollaries :
- no running of the coupling
- no confinement
- spectrum is made out of massless gluons + super-partners

Why studying it?
(or 3 good reasons to like it)

Theoretical laboratory : one can explore and identify mathematical and physical structures (at higher loops or strong coupling) more easily than in any other theory

AdS/CFT correspondence : it is one of these few theories for which we believe we know precisely what is the string theory dual (here it is IIB string theory in AdS5 * S5)

Integrability : it is believed to be "exactly solvable", in the 't Hooft planar limit at least, and referred to as the Ising model of 4d gauge theories

So let's try to solve it!

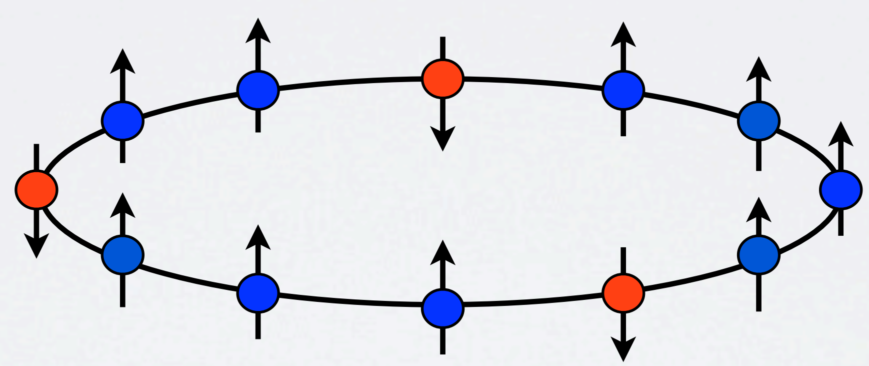
Progress report in integrability

Spectrum of scaling dimensions and spin chain

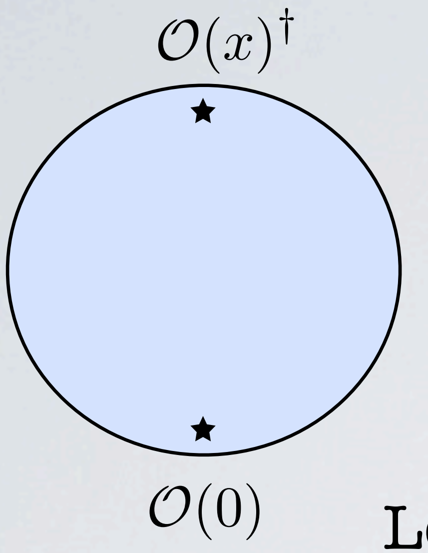
Gluon scattering amplitudes and Wilson loops

Structure constants and string splitting/joining

Scaling dimensions and spin chain



Spectral problem



Spectrum of scaling dimensions of local operators

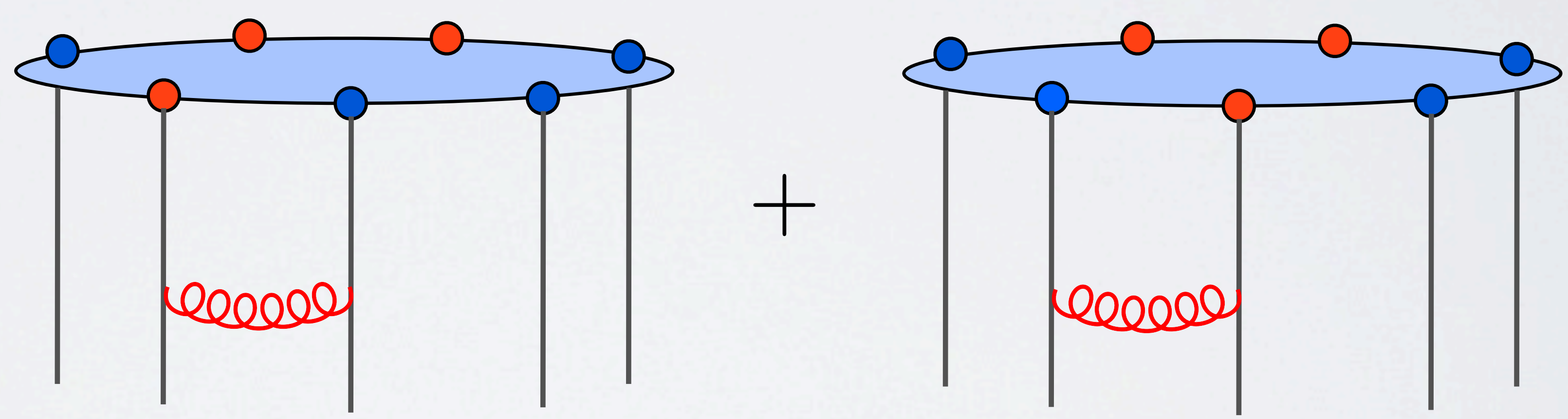
Local (single trace) operator

$$\langle \mathcal{O}(x)^\dagger \mathcal{O}(0) \rangle = \frac{1}{x^{2\Delta}}$$

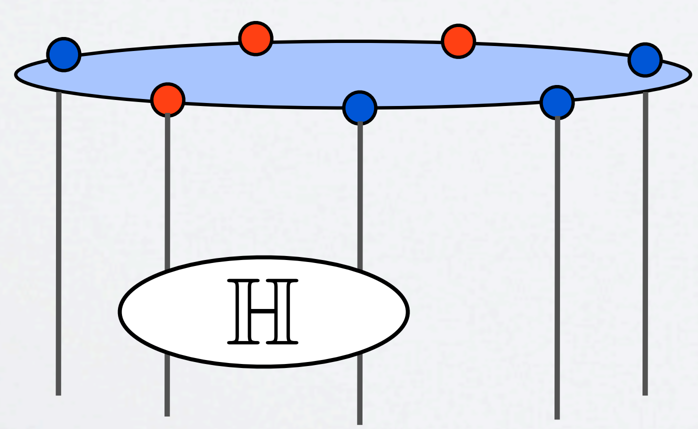
$$\mathcal{O} = \mathrm{tr}\, \Phi_1 \Phi_2 \ldots \Phi_L$$

Mixing problem

Radiative corrections induce mixing of operators



Equivalent to a spin chain problem



$$\Delta = \Delta_0 + 2g^2 H_{XXX}$$

Spectral problem

One-loop dilatation operator

[Minahan,Zarembo'02]
[Beisert,Staudacher'03]

$$H_{XXX} = \sum_{i=1}^{L} (I - P_{ii+1})$$

Werner Heisenberg

Hans Bethe

Heisenberg spin chain is integrable :

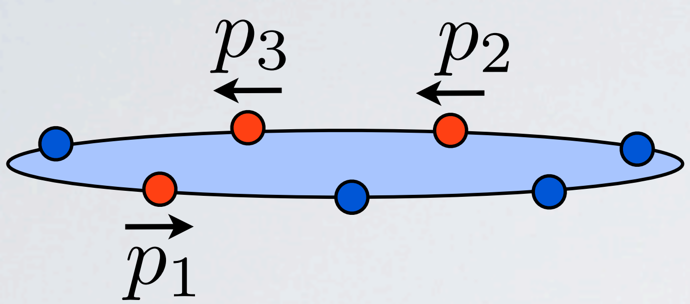
- As many commuting conserved charges as degrees of freedom (i.e., L for SU(2) spin chain)

- Fundamental excitations (magnons) about the ferro vacuum have a factorized S-matrix

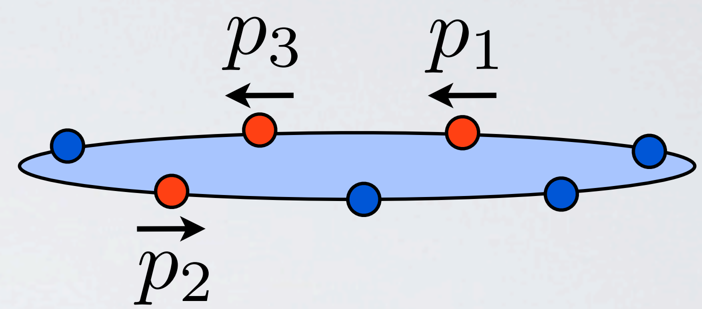
$$\mathbb{S}_{123} = \mathbb{S}_{23}\mathbb{S}_{13}\mathbb{S}_{12}$$

Spectral problem

Bethe wave function



$$+S(p_1, p_2)$$



$$+\ldots$$

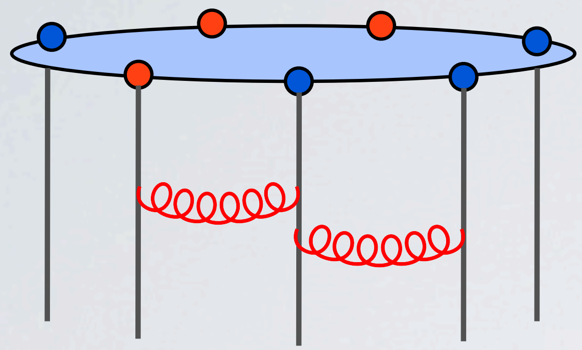
Periodicity conditions gives the Bethe ansatz equations
(i.e. quantization conditions for the magnon momenta)

$$e^{ip_i L} \prod_{j \neq i} S(p_i, p_j) = 1$$

And the spectrum of energies follows :

$$E = \sum_i E(p_i)$$

Higher loops?



Increasing loop order = increasing range of the spin chain Hamitonian

Not much is known about the resulting long range spin chain
It is however believed to remain integrable

Hint :

We can still solve our problem by adding loop corrections to our previous ingredients : energy and S-matrix

Power of symmetry

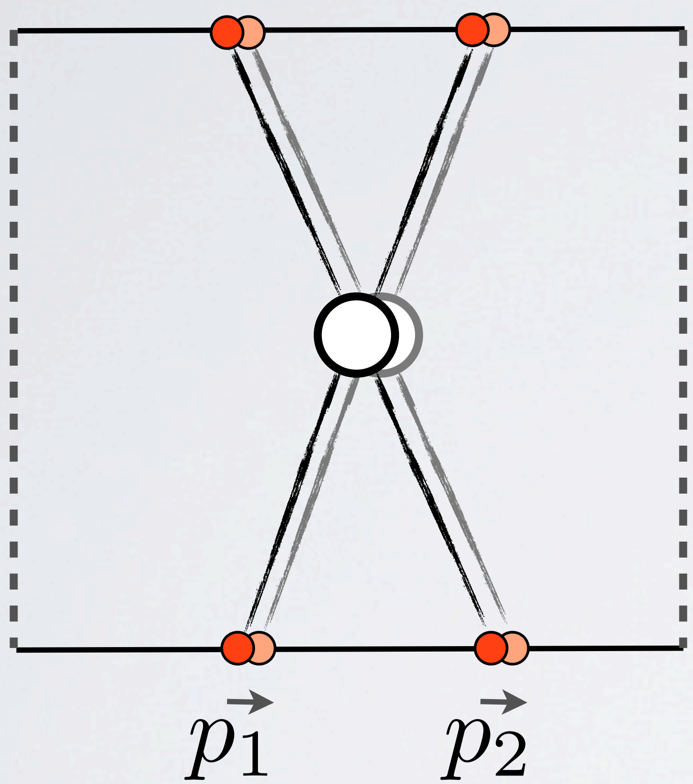
Way to go? Let the symmetry do the job

Residual symmetry group of BMN (ferro) vacuum :

[Beisert'05]

$$\underbrace{PSU(2|2)}_{\text{Left}} \times \underbrace{PSU(2|2)}_{\text{Right}} \ltimes \mathbb{R}^3$$

Central extensions : contain **energy** (and coupling constant)



Each magnon transforms in bi-fundamental irrep

$$\underbrace{\mathbf{2}|\mathbf{2}}_{\text{Left}} \otimes \underbrace{\mathbf{2}|\mathbf{2}}_{\text{Right}}$$

(Dimension = 16 = 8 bosons + 8 fermions)

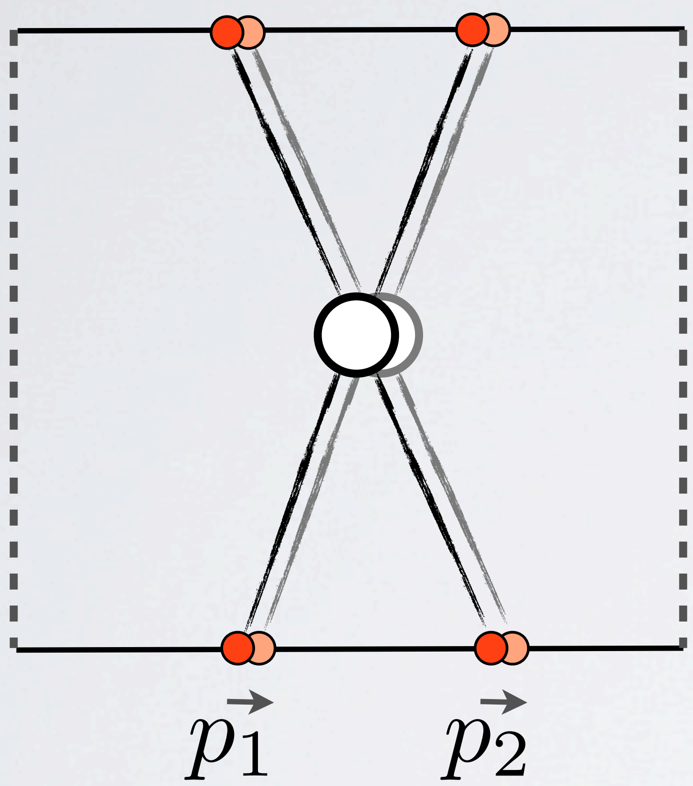
Dispersion relation

$$E = \sqrt{1 + 16\, g^2 \sin^2\left(\frac{p}{2}\right)}$$

Power of symmetry

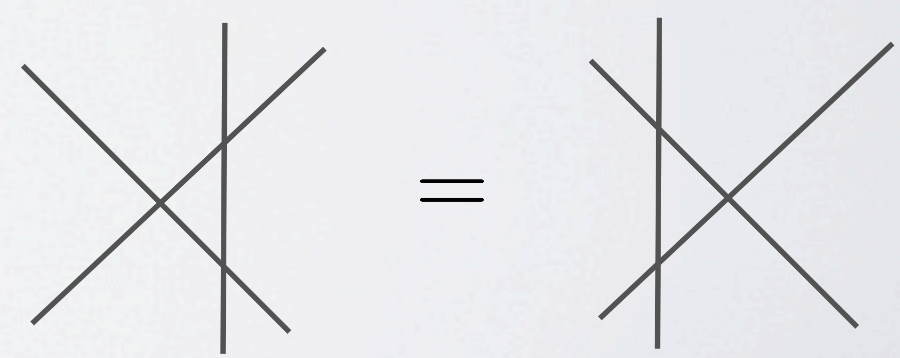
Way to go? Let the symmetry do the job
Residual symmetry group of BMN (ferro) vacuum :

[Beisert'05]



Symmetry fixes S-matrix
(up to overall scalar factor)

$$\mathbb{S}_{12} \sim S^0_{12}\,\mathcal{S}_{12} \times \dot{\mathcal{S}}_{12}$$



✓ Fulfills **Yang-Baxter** equation

✓ Scalar factor constrained by crossing symmetry [Janik'05]

Full solution?

Is it that simple?

2005

Symmetries
Beisert S-matrix

Beisert-Staudacher
Asymptotic Bethe Ansatz

BMN Vacuum
Spin Chain Picture

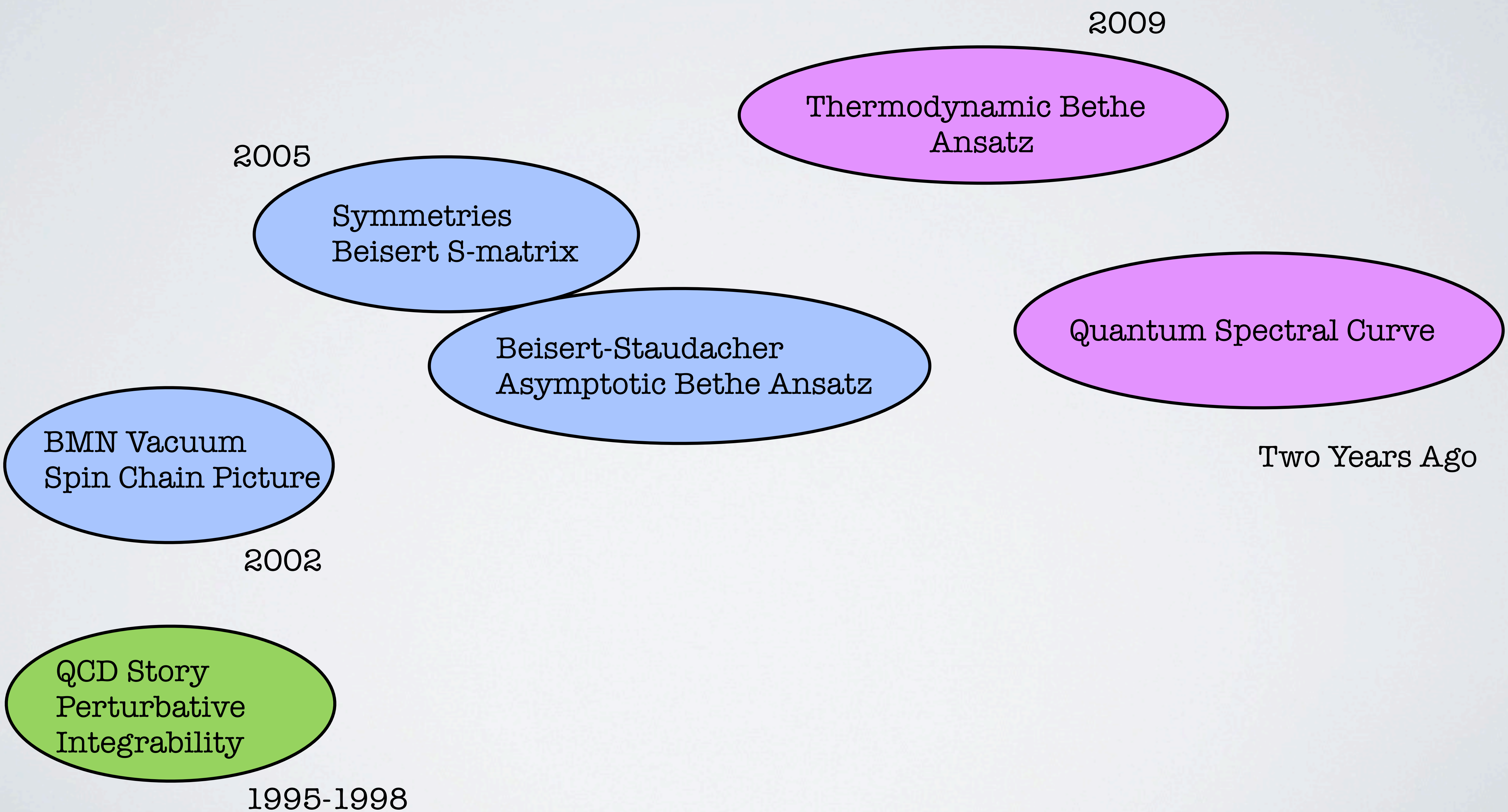
2002

QCD Story
Perturbative
Integrability

1995-1998

No, one must also account for finite size corrections (because spin chain has finite length)

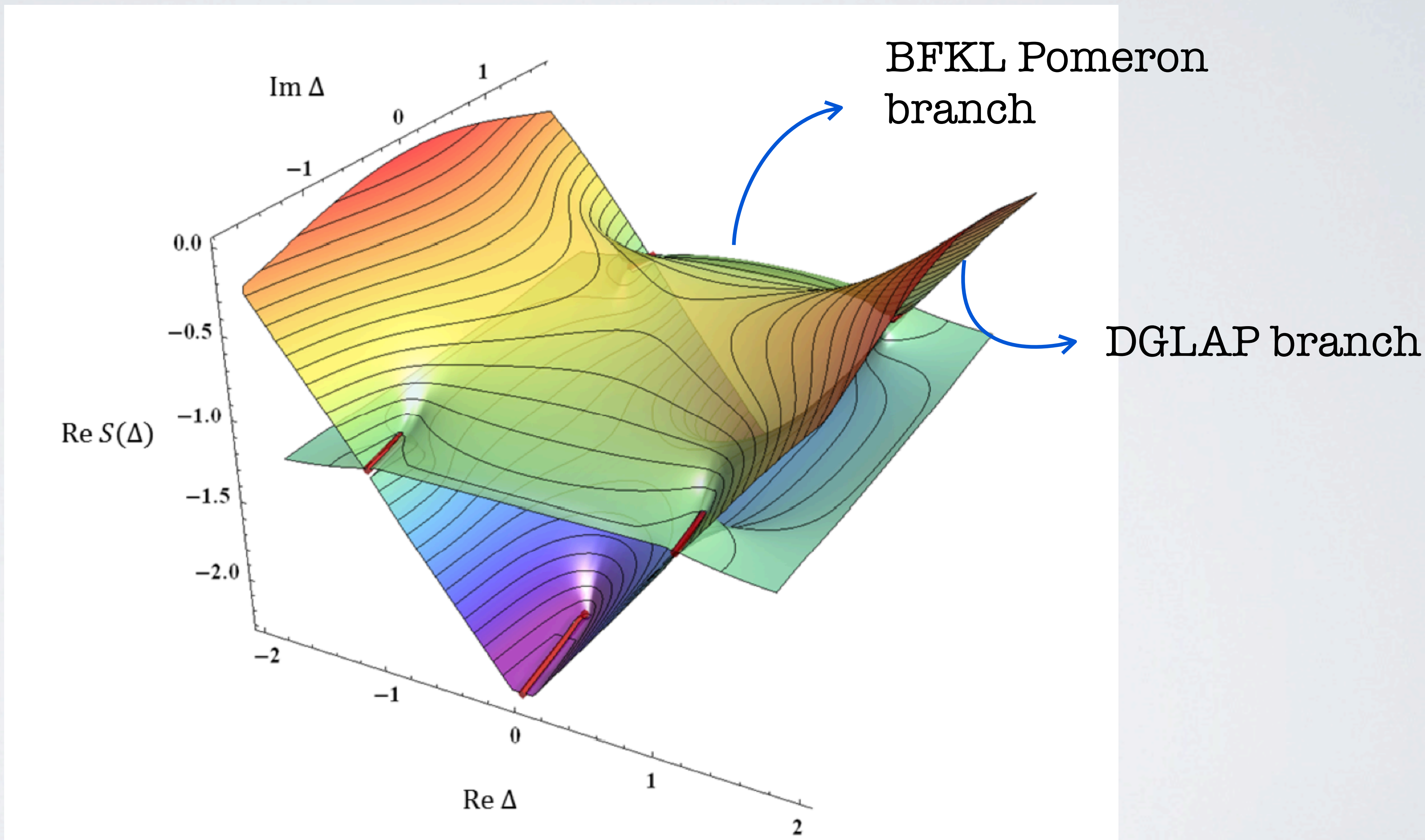
Full solution



Applications

Scaling dimension of twist two operator for complex spin

[Gromov,Levkovich-Maslyuk,Sizov'15]



Plot of real part of the spin S as a function of the scaling dimension Δ for 't Hooft coupling = 6.3

Applications

Scaling dimension of shortest unprotected operator
(so-called Konishi multiplet)

[Marboe,Volin'14]

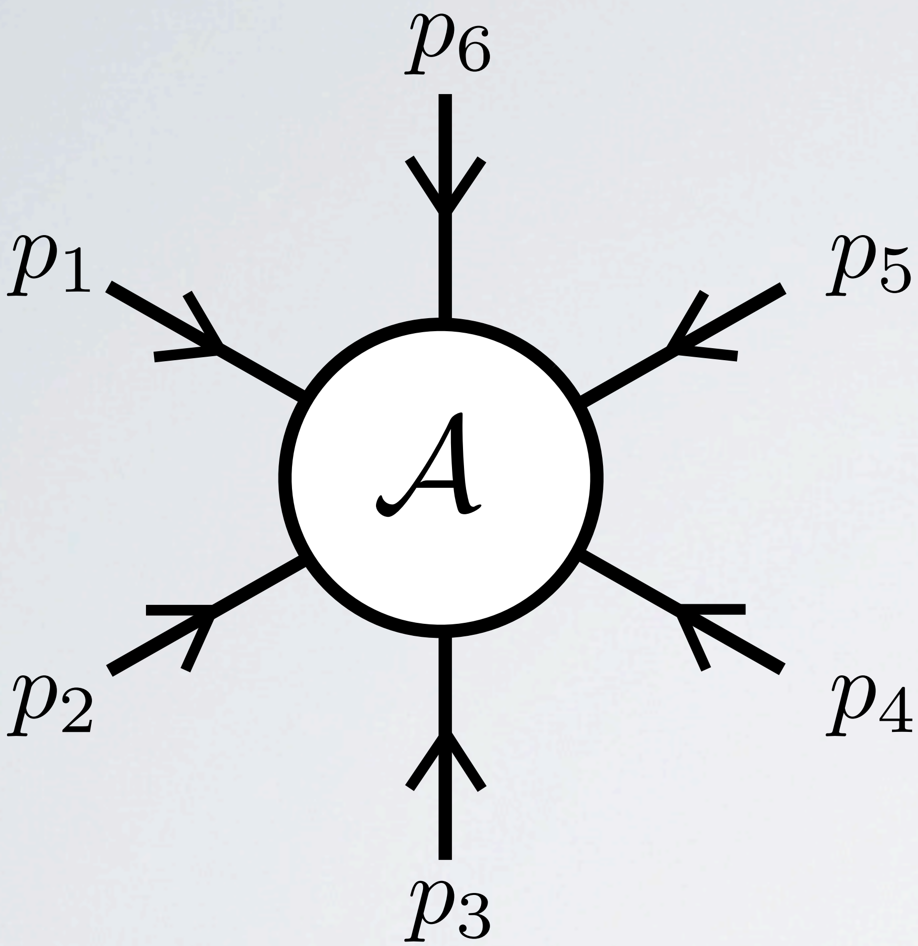
$$
\begin{aligned}
\Delta \;=\;& 4+12g^2-48g^4+336g^6+g^8\big(-2496+576\,\zeta_3-1440\,\zeta_5\big) \\
&+g^{10}\big(15168+6912\,\zeta_3-5184\,\zeta_3^2-8640\,\zeta_5+30240\,\zeta_7\big) \\
&+g^{12}\big(-7680-262656\,\zeta_3-20736\,\zeta_3^2+112320\,\zeta_5+155520\,\zeta_3\,\zeta_5+75600\,\zeta_7-489888\,\zeta_9\big) \\
&+g^{14}\big(-2135040+5230080\,\zeta_3-421632\,\zeta_3^2+124416\,\zeta_3^3-\;229248\,\zeta_5+411264\,\zeta_3\,\zeta_5 \\
&\quad -993600\,\zeta_5^2-1254960\,\zeta_7-1935360\,\zeta_3\,\zeta_7-835488\,\zeta_9+7318080\,\zeta_{11}\big) \\
&+g^{16}\Big(54408192-83496960\,\zeta_3+7934976\,\zeta_3^2+1990656\,\zeta_3^3-19678464\,\zeta_5-4354560\,\zeta_3\,\zeta_5 \\
&\quad -3255552\,\zeta_3^2\,\zeta_5+2384640\,\zeta_5^2+21868704\,\zeta_7-6229440\,\zeta_3\,\zeta_7+22256640\,\zeta_5\,\zeta_7 \\
&\quad +9327744\,\zeta_9+23224320\,\zeta_3\,\zeta_9+\frac{65929248}{5}\,\zeta_{11}-106007616\,\zeta_{13}-\frac{684288}{5}\,Z_{11}^{(2)}\Big) \\
&+g^{18}\Big(-1014549504+1140922368\,\zeta_3-51259392\,\zeta_3^2-20155392\,\zeta_3^3+575354880\,\zeta_5 \\
&\quad -14294016\,\zeta_3\,\zeta_5-26044416\,\zeta_3^2\,\zeta_5+55296000\,\zeta_5^2+15759360\,\zeta_3\,\zeta_5^2-223122816\,\zeta_7 \\
&\quad +34020864\,\zeta_3\,\zeta_7+22063104\,\zeta_3^2\,\zeta_7-92539584\,\zeta_5\,\zeta_7-113690304\,\zeta_7^2-247093632\,\zeta_9 \\
&\quad +119470464\,\zeta_3\,\zeta_9-245099520\,\zeta_5\,\zeta_9-\frac{186204096}{5}\,\zeta_{11}-278505216\,\zeta_3\,\zeta_{11}-253865664\,\zeta_{13} \\
&\quad +1517836320\,\zeta_{15}+\frac{15676416}{5}\,Z_{11}^{(2)}-1306368\,Z_{13}^{(2)}+1306368\,Z_{13}^{(3)}\Big)
\end{aligned}
$$

Comments :

- Finite size corrections here starts at 4 loops
- Z.. stand for single valued multiple zeta values

Gluon scattering amplitudes and Wilson loops

Scattering amplitudes = Wilson loops

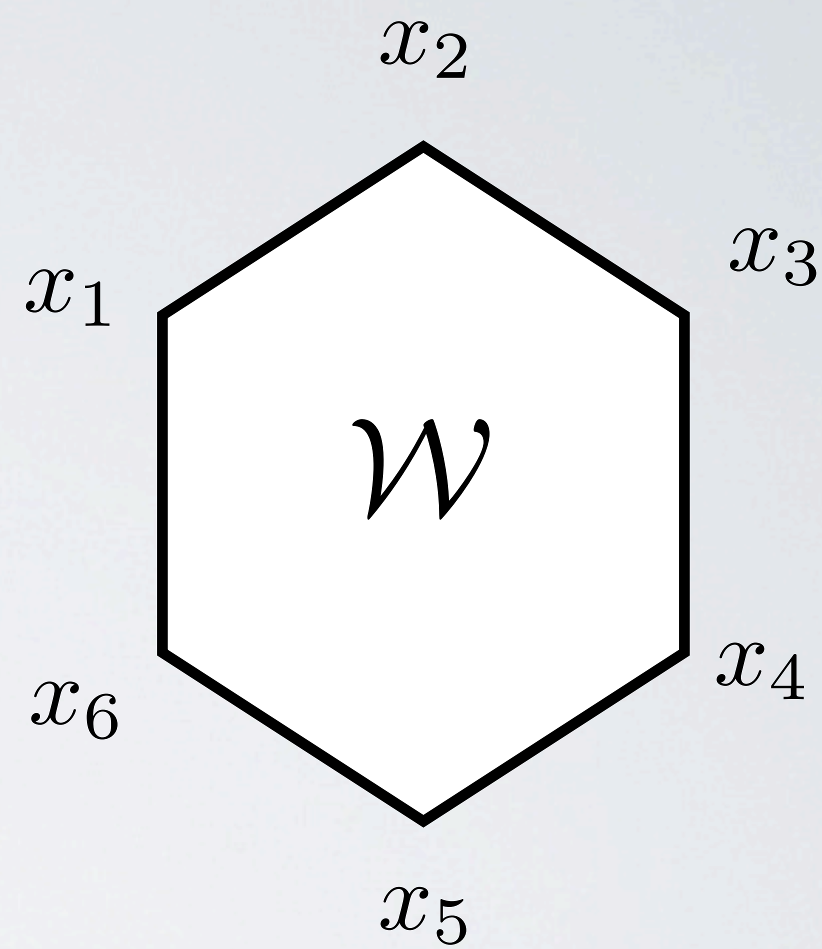


gluon scattering amplitude

$$=$$

$$p_i = x_{i+1} - x_i$$

[Alday,Maldacena'07]
[Drummond,Korchemsky,Sokatchev'07]
[Brandhuber,Heslop,Travaglini'07]
[Drummond,Henn,Korchemsky,Sokatchev'07]



light-like polygonal Wilson loop

In this theory they are the same
(in planar limit)

Combining symmetries

Super conformal + dual super conformal gives a Yangian symmetry (one of the hallmark of integrability)

[Drummond, Henn, Plefka'09]

Put severe constraints on the integrand of scattering amplitudes which can be constructed exactly

They lead to a purely geometrical reformulation of these integrands (Grassmannian, Amplituhedron)

[Arkani-Hamed, Bourjaily, Cachazo,Caron-Huot,Goncharov, Postnikov, Trnka'10'12]

Also put constraints on the full (integrated) scattering amplitudes

They lead to a bootstrap for constructing SA without any use of Feynman diagrams (proceeds from knowlege of space of functions + additional physical requirements)

[Dixon,Drummond, Henn'11]
[Dixon,Drummond, vonHippel, Pennington'13]
[Dixon,Drummond, Duhr, Pennington'13]
[Drummond,Papathanasiou, Spradlin'14]

Immediate consequences

Amplitudes are function of cross ratio only (up to divergent part) :

[Drummond,Henn,Korchemsky,Sokatchev'07]

$$\log W_n = \mathrm{BDS}_n + R_n(u_1, \ldots, u_{3n-15})$$

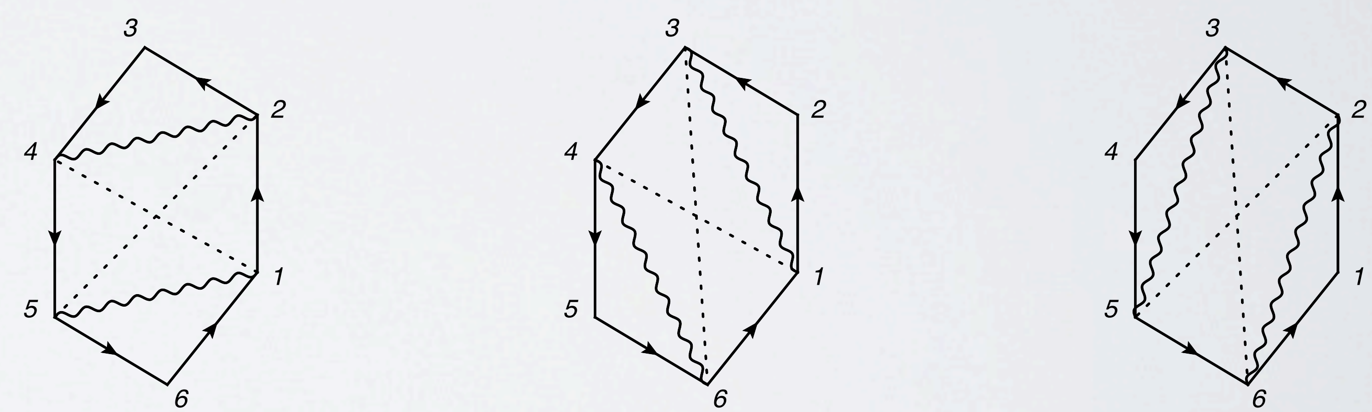
Bern-Dixon-Smirnov ansatz (contains all IR/UV divergences)

$R_n =$ remainder function $=$ function of 3n-15 cross ratios

In particular

$$R_4 = R_5 = 0$$

(simply because one cannot form cross ratios for 4- and 5-edge null WLs)



$u_2 = \frac{x_{15}^2 x_{24}^2}{x_{14}^2 x_{25}^2}$ $\quad$ $u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}$ $\quad$ $u_3 = \frac{x_{26}^2 x_{35}^2}{x_{25}^2 x_{36}^2}$

It means that 4- and 5-gluon amplitudes are known exactly and given by the BDS part only!

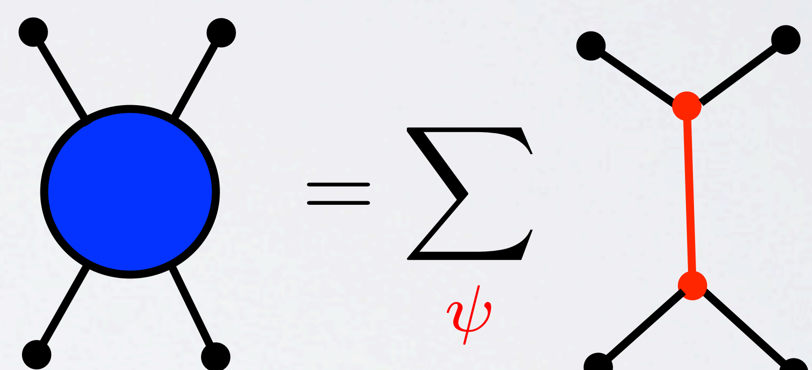
Immediate consequences

[Alday,Gaiotto,Maldacena,Sever,Vieira'09]

Yet another consequence :

WLs are some sort of non-local Green functions
and we can use the OPE for building big WLs out of smaller ones

OPE recap :

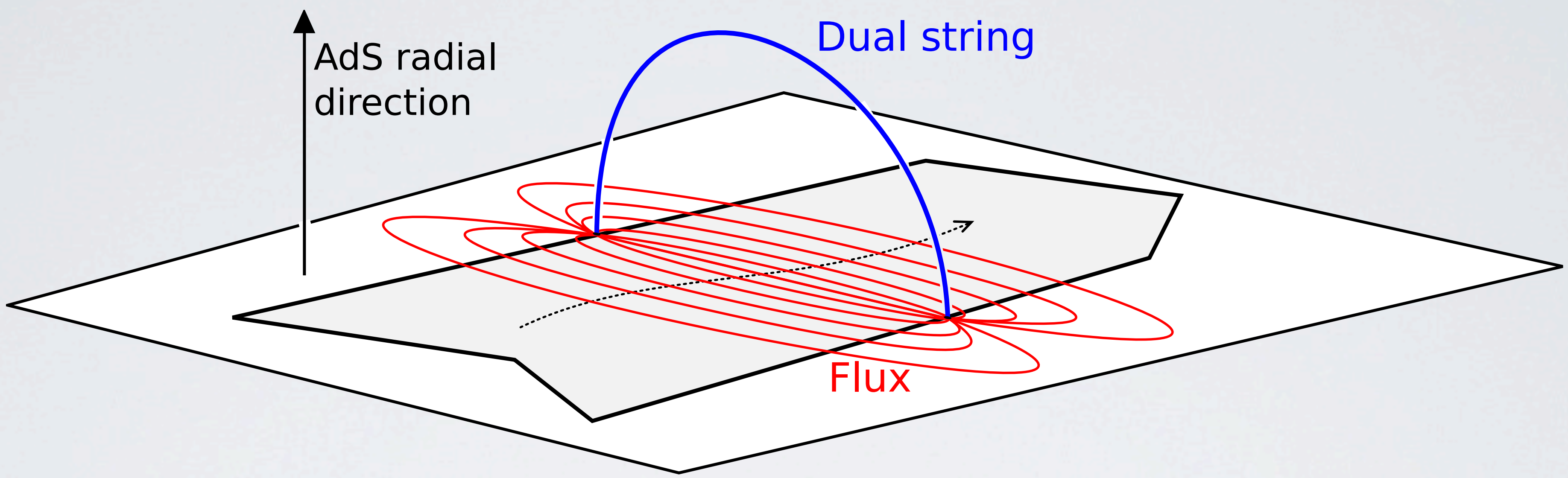


$$\mathcal{W} = \sum_{\text{states } \psi} C_{\text{bot}}(\psi) \times e^{-E(\psi)\tau + ip(\psi)\sigma + im(\psi)\phi} \times C_{\text{top}}(\psi)$$

Wilson loops at finite coupling

Hamiltonian picture
for OPE

[Alday,Gaiotto,Maldacena,Sever,Vieira'09]



1+1d background : flux tube sourced by two parallel null lines

bottom&top cap excite the flux tube out of its ground state

→ *Sum over all flux-tube eigenstates*

$$\mathcal{W} = \sum_{\text{states } \psi} C_{\text{bot}}(\psi) \times e^{-E(\psi)\tau + ip(\psi)\sigma + im(\psi)\phi} \times C_{\text{top}}(\psi)$$

Can we combine this OPE picture with integrability and obtain a finite coupling representation of scattering amplitudes in this theory?

Pentagon way : main ideas

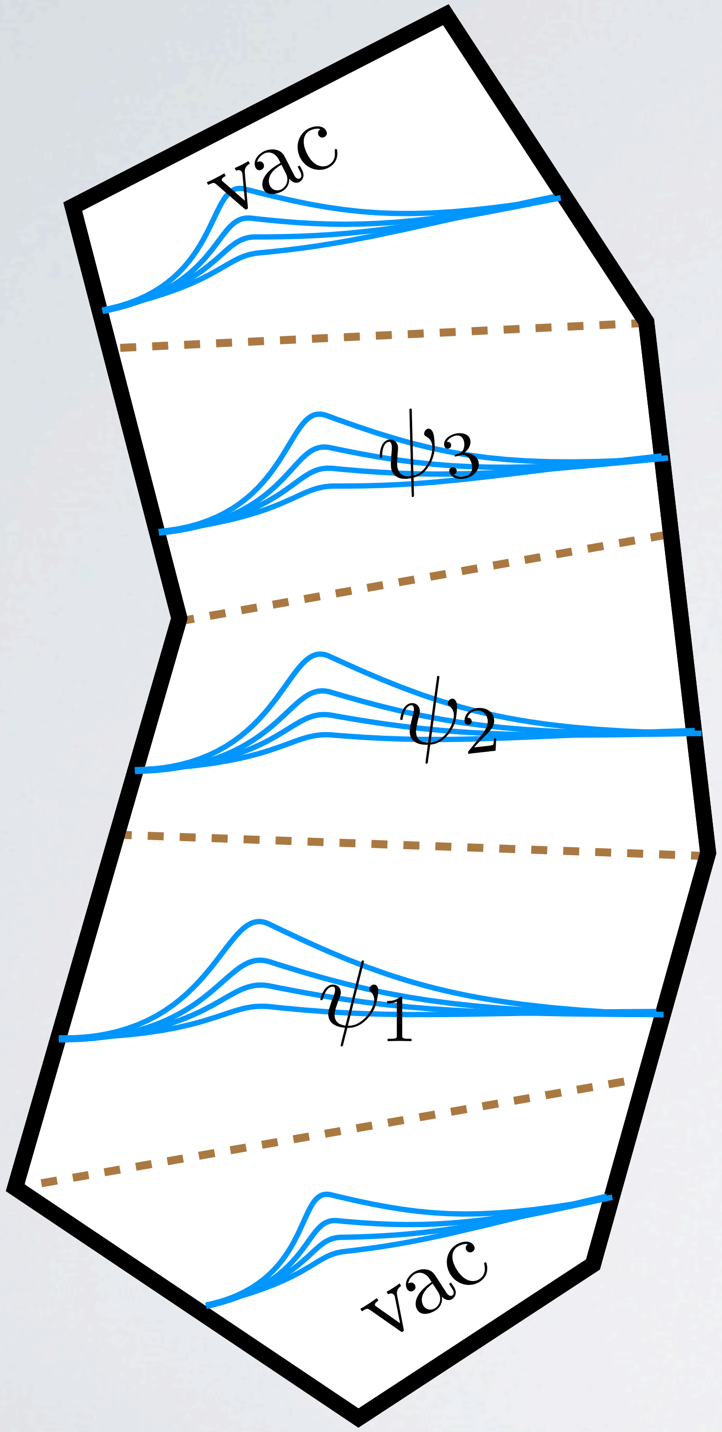
[BB,Sever,Vieira'13]

Remember : use small objects
to build bigger ones

Here smallest objects : squares and pentagons
(no cross ratios = fixed by conformal symmetry)

Analogy with OPE data for local operators :
Square = 2pt function = spectral data
Pentagon = 3pt function = coupling

Implementation

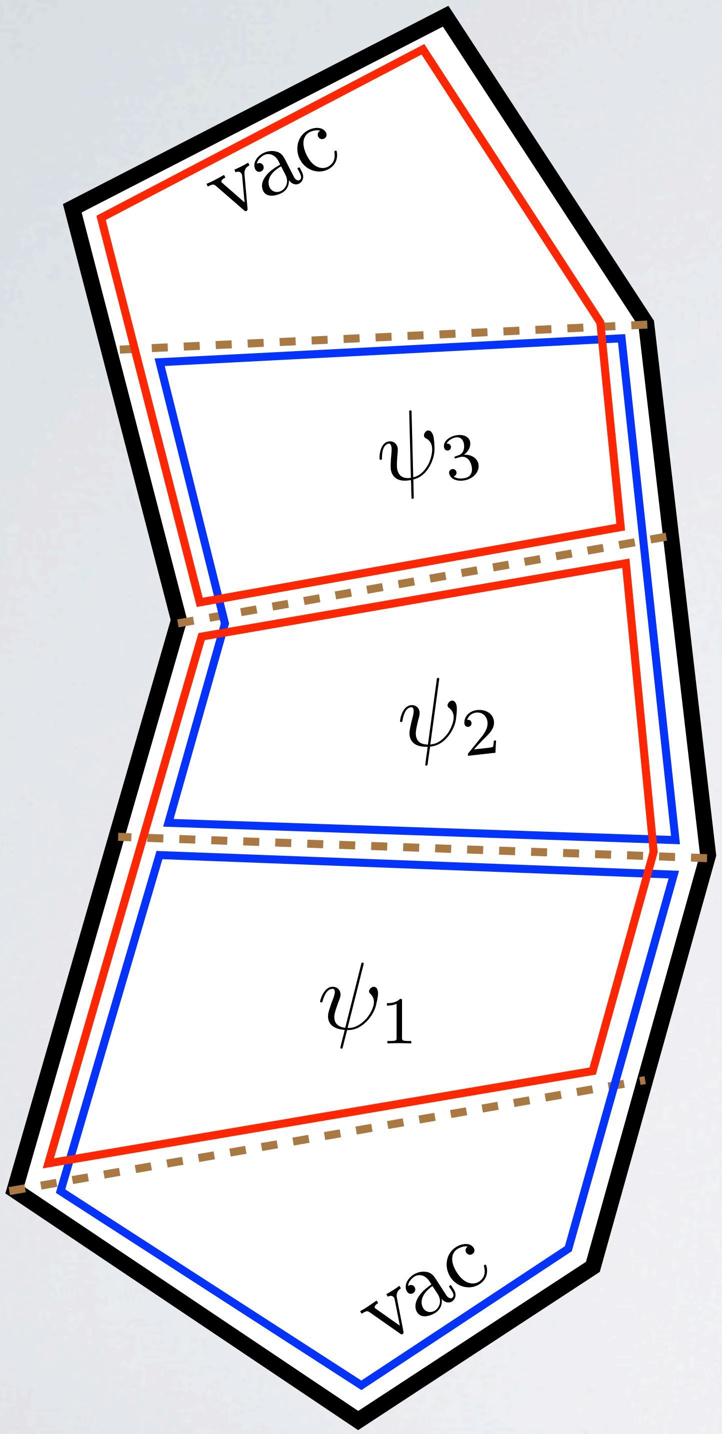


Step 1:
Pick a polygon and
divide it into squares

and think about each square
as hosting the flux tube
in a particular state

Step 2:
Decompose the flux tube state
over a basis of eigenstates
(w.r.t symmetries of the square)

Implementation



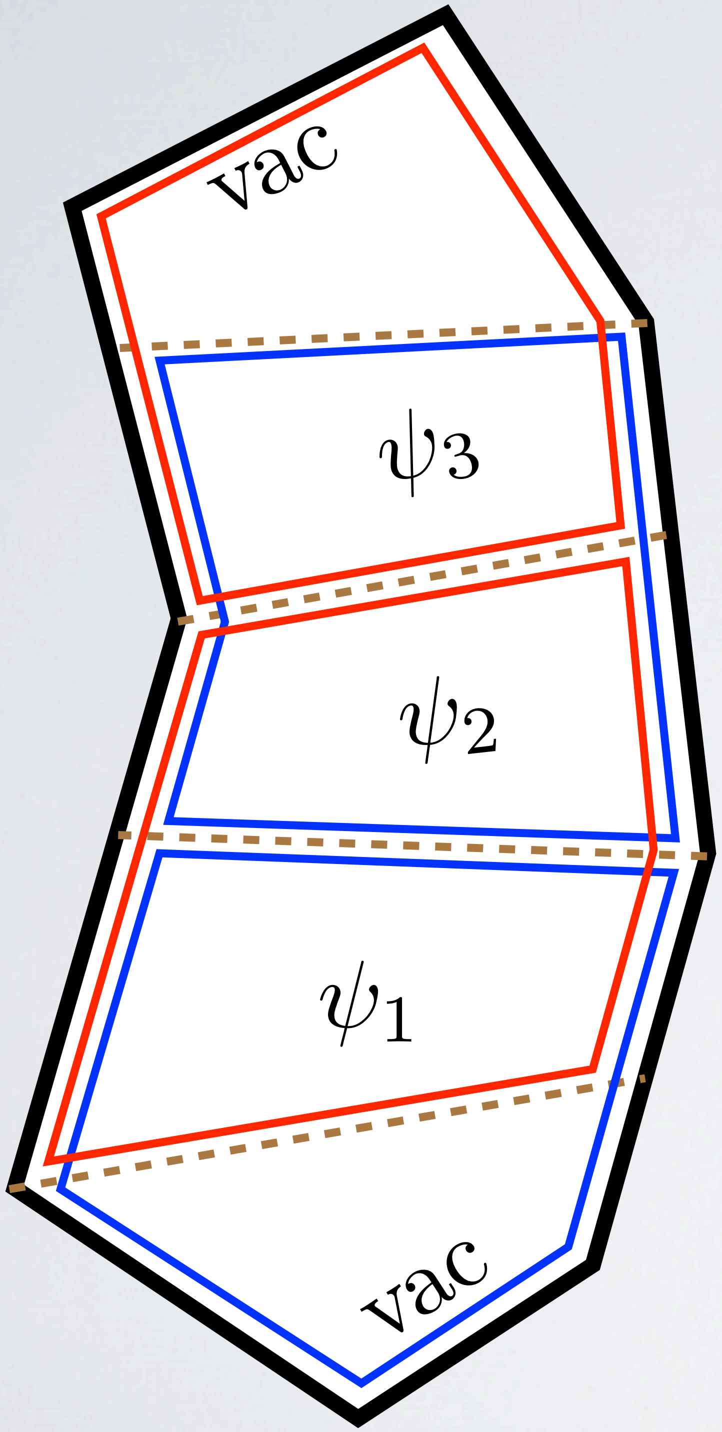
energy momentum angular momentum

$$= \sum_{\psi_i} \left[\prod_i e^{-E_i\tau_i + i p_i \sigma_i + i m_i \phi_i} \right] \times$$

$$P(0|\psi_1)P(\psi_1|\psi_2)P(\psi_2|\psi_3)P(\psi_3|0)$$

Pentagon transition :
measures the amplitude for
a transition from one state
to another

Pentagon way



$$= \sum_{\psi_i} \left[\prod_i e^{-E_i \tau_i + i p_i \sigma_i + i m_i \phi_i} \right] \times$$

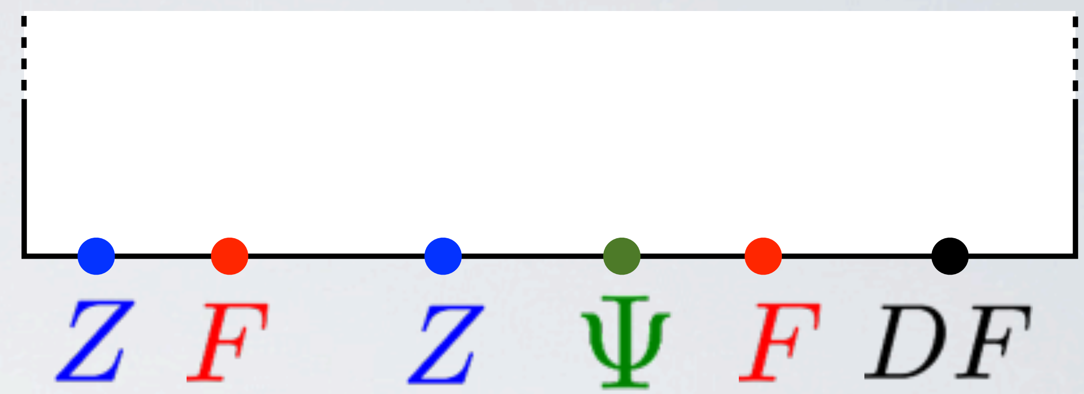
$$P(0|\psi_1) P(\psi_1|\psi_2) P(\psi_2|\psi_3) P(\psi_3|0)$$

To compute amplitudes we need :

- The spectrum of flux-tube states ψ
- All the pentagon transitions $P(\psi_1|\psi_2)$

The flux-tube excitations

$\psi = N$ particles state



Field insertions along a light-ray:
create/annihilate state on the flux tube

Not so much different from spin chain...
... in fact it is the same problem as before but expanded around a different vacuum

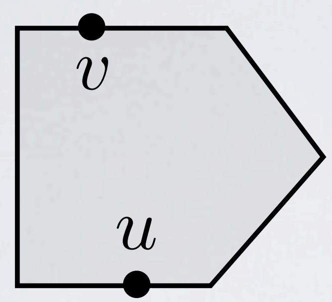
Discretized version of light-ray:
bath of covariant derivatives

$$\mathcal{O} = \mathrm{tr}\,(Z\,DDDD\ldots DDDD\,\overset{\xrightarrow{p_1}}{F}\,DDDD\ldots DDDD\,\overset{\xrightarrow{p_2}}{F}\,DDDD\ldots DDDD\,Z)$$

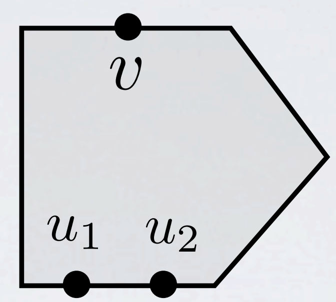
Flux tube states $\longleftrightarrow$ Large spin operators

The pentagon transitions

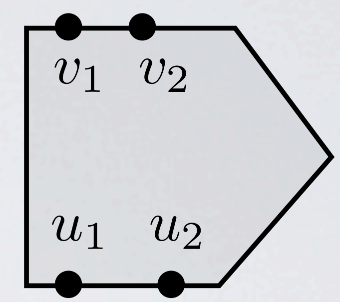
Field insertions on pentagon WL :



$P(u|v)$



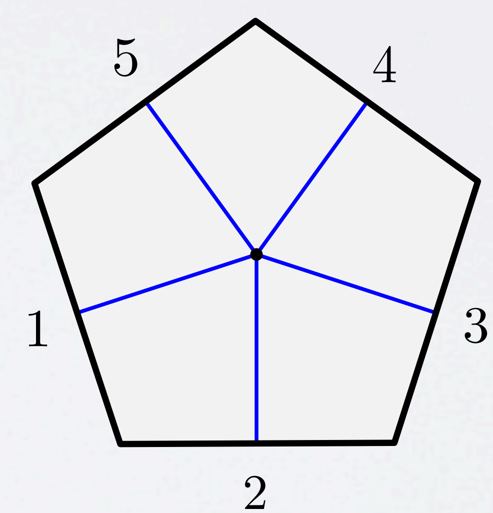
$P(u_1, u_2|v)$



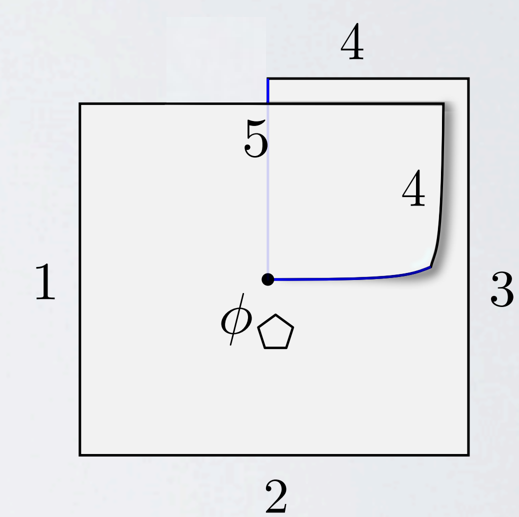
$P(u_1, u_2|v_1, v_2)$

. . .

Pentagon as form factors



$=$



Geometrical picture :

excess angle$= \frac{\pi}{2}$

Hamiltonian picture:

twist operator $\phi_{\pentagon}$

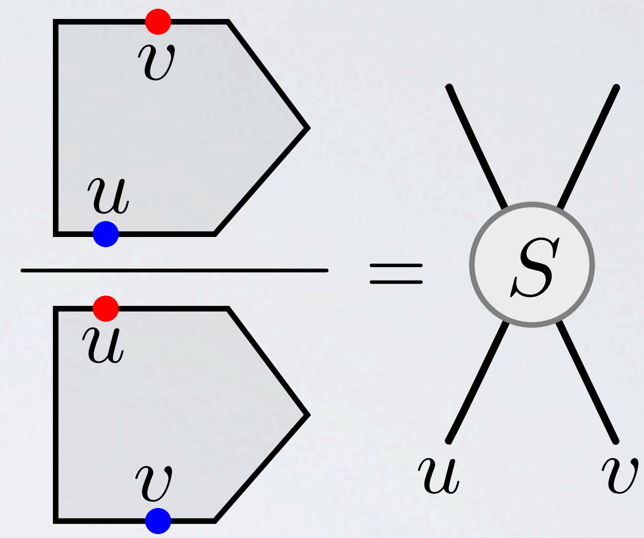
E.g.,

$$P(u_1, u_2|v_1) = \langle v_1| \; \phi_{\pentagon} \, |u_1, u_2\rangle$$

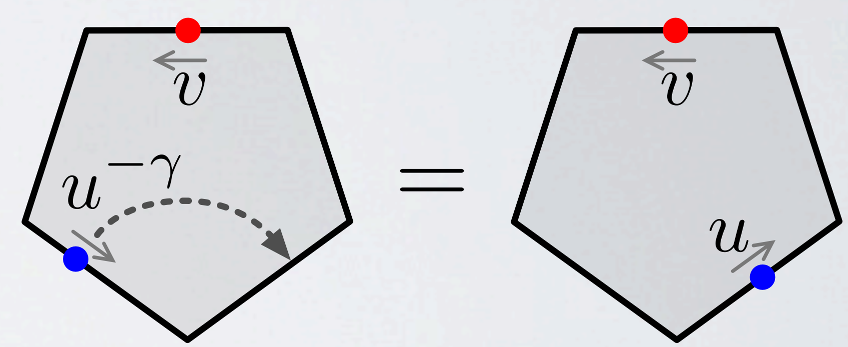
The pentagon transitions

We can use integrable bootstrap
for finding them :

- Fundamental axiom : $\dfrac{P(u|v)}{P(v|u)} =$



- Mirror axiom : $P(u^{-\gamma}|v) = P(v|u)$



This is enough to find the transitions
in terms of S-matrix :

$$P(u|v)^2 = \frac{S(u,v)}{(u-v)(u-v+i)S(u^{\gamma},v)}$$

All pentagon transitions

$$P_{A|B}(u|v)^2 = \mathcal{F}_{A|B}(u|v) \times \frac{S_{AB}(u,v)}{S_{AB}(u^\gamma,v)}$$

ϕ : scalar

ψ : fermion

F : gluon

[BB,Sever,Vieira'13'14]
[BB,Caetano,Cordova,Sever,Vieira'15]
[Belitsky'14'15]

$$\mathcal{F}_{\phi F}(u|v) = 1\,,$$

$$\mathcal{F}_{\phi\psi}(u|v) = -\frac{1}{(u-v+\frac{i}{2})}\,,$$

$$\mathcal{F}_{\phi\phi}(u|v) = \frac{1}{(u-v)(u-v+i)}\,,$$

$$\mathcal{F}_{FF}(u|v) = \frac{(x^+y^+-g^2)(x^+y^--g^2)(x^-y^+-g^2)(x^-y^--g^2)}{g^2x^+x^-y^+y^-(u-v)(u-v+i)}\,,$$

$$\mathcal{F}_{F\psi}(u|v) = -\frac{(x^+y-g^2)(x^-y-g^2)}{g\sqrt{x^+x^-}y(u-v+\frac{i}{2})}\,,$$

$$\mathcal{F}_{F\bar{\psi}}(u|v) = -\frac{g\sqrt{x^+x^-}y(u-v+\frac{i}{2})}{(x^+y-g^2)(x^-y-g^2)}\,,$$

$$\mathcal{F}_{F\bar{F}}(u|v) = \frac{g^2x^+x^-y^+y^-(u-v)(u-v+i)}{(x^+y^+-g^2)(x^+y^--g^2)(x^-y^+-g^2)(x^-y^--g^2)}\,,$$

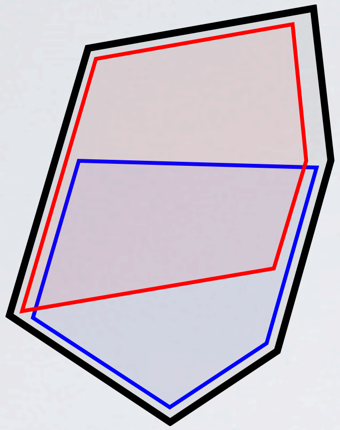
$$\mathcal{F}_{\psi\psi}(u|v) = -\frac{(xy-g^2)}{\sqrt{gxy}(u-v)(u-v+i)}\,,$$

$$\mathcal{F}_{\psi\bar{\psi}}(u|v) = -\frac{\sqrt{gxy}}{(xy-g^2)}\,,$$

Full 6-gluon amplitude

[BB, Sever, Vieira'15]

OPE series :

$\mathcal{W}_{\rm hex} =$  $$= \sum_n \frac{1}{S_n} \int \frac{du_1 \ldots du_n}{(2\pi)^n} \Pi(\{u_i\})$$

Flux tube integrand :

That's it!
(everything here is known at any coupling)

$$\Pi(\{u_i\}) = \Pi_{\rm dyn} \times \Pi_{\rm mat}$$

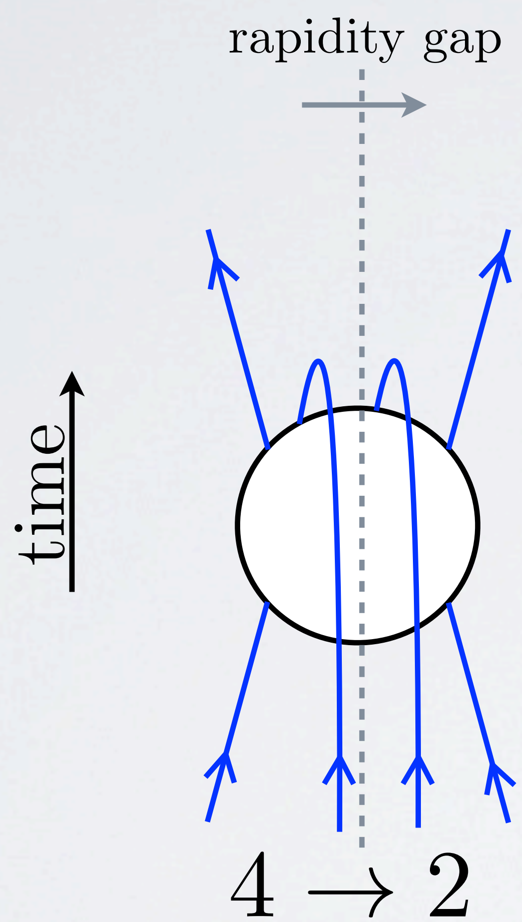
$$\Pi_{\rm dyn} = \prod_i \mu(u_i)\, e^{-E(u_i)\tau + ip(u_i)\sigma + im_i\phi} \prod_{i<j} \frac{1}{|P(u_i|u_j)|^2}$$

Some applications...

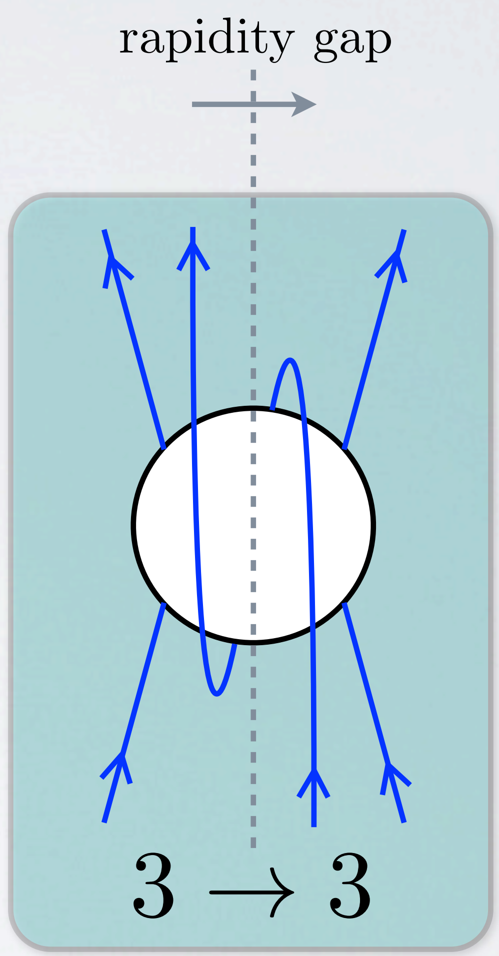
Application : multi Regge kinematics

- High energy scattering

- Becomes interesting in the so-called Mandelstam regions



[Bartels,Lipatov,Sabio Vera'15]

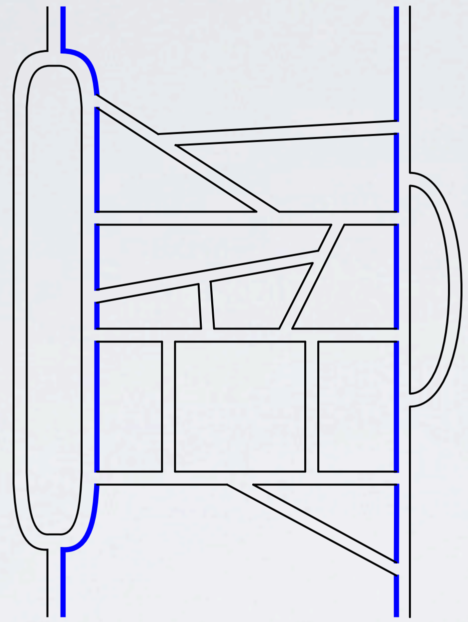


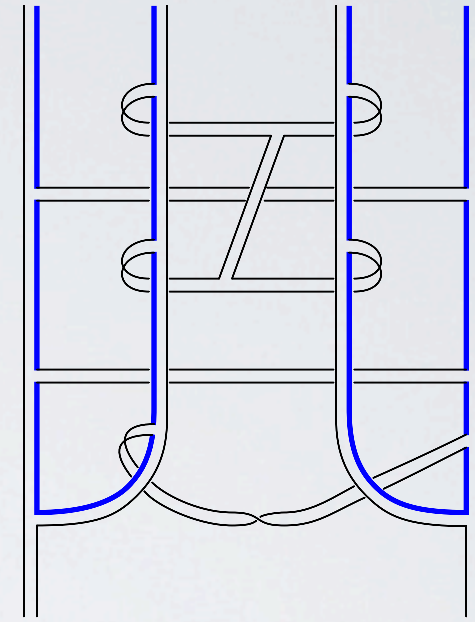
[Bartels,Lipatov,Prygarin'10]

- Energy dependence in this regime is governed by so-called Regge cuts, with BFKL eigenvalue $\omega(\nu)$

Regge cut = color dipole

Feynman diagram :



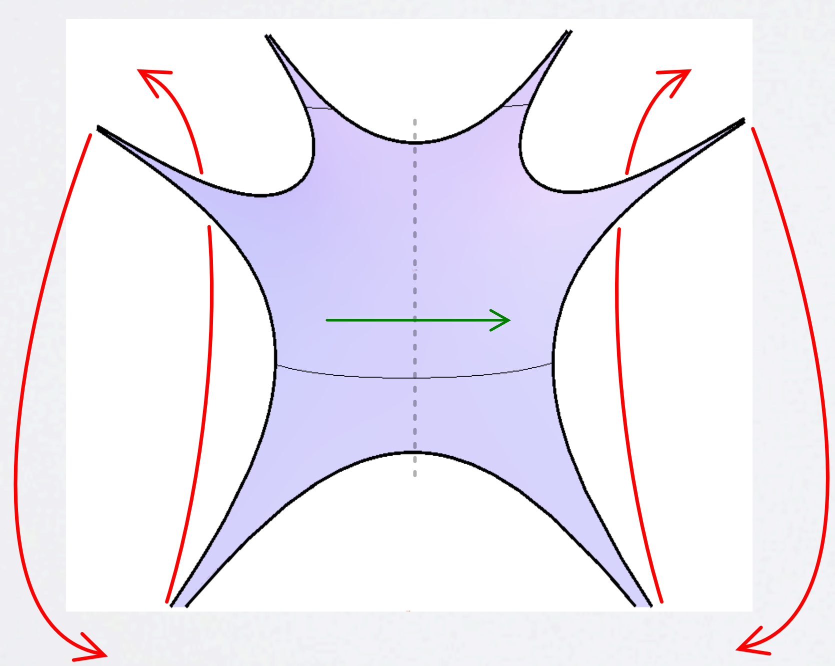


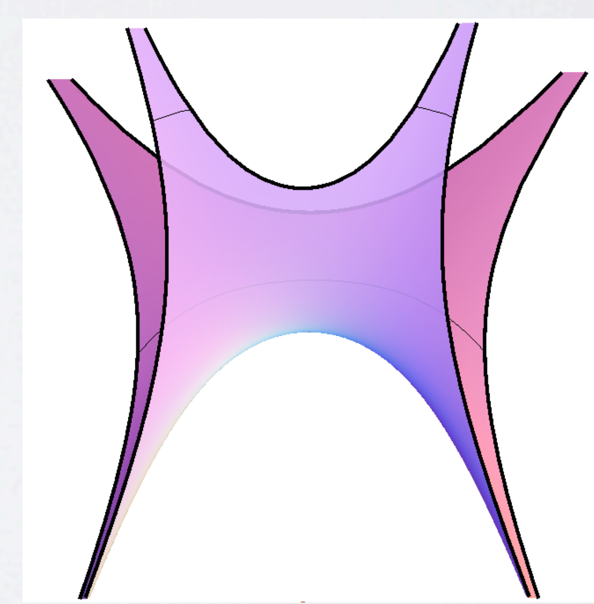
Similar to dipole considered by Stéphane Munier, except that here dipoles are in the adjoint of the gauge group

Regge pole
(euclidean sheet)

Regge cut or dipole
(minkowksian sheet)

String picture :





$\Rightarrow \quad \omega(\nu)$

This is what BFKL is about

OPE versus BFKL

OPE

$$\mathcal{W}_{\rm hex} = 1 + \sum_{m\neq 0} (-1)^m e^{im\phi} \int_{-\infty}^{+\infty} \frac{dp}{2\pi}\, \hat{\mu}_m(p)\, e^{ip\sigma - \tau E_m(p)} + \ldots$$

leading term dominate at large τ

BFKL

$$\mathcal{W}^{\circlearrowleft}_{\rm hex} e^{-i\pi\delta'} = \sum_{m=-\infty}^{\infty} (-1)^m e^{im\phi} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi}\, \hat{\mu}_{\rm BFKL}(\nu, m)\, e^{i(\sigma-\tau)\nu + (\sigma+\tau)\omega(\nu,m)} + \ldots$$

leading term dominate at large $\tau + \sigma$

- Both are valid at any coupling

- They look pretty much the same but diagonalize different symmetry generators...

$$\nu \overset{?}{=} \frac{1}{2}(p - iE) \qquad \omega \overset{?}{=} -\frac{1}{2}(E - ip)$$

... but the two expansions operate in different kinematical domains or sheets

OPE versus BFKL

OPE

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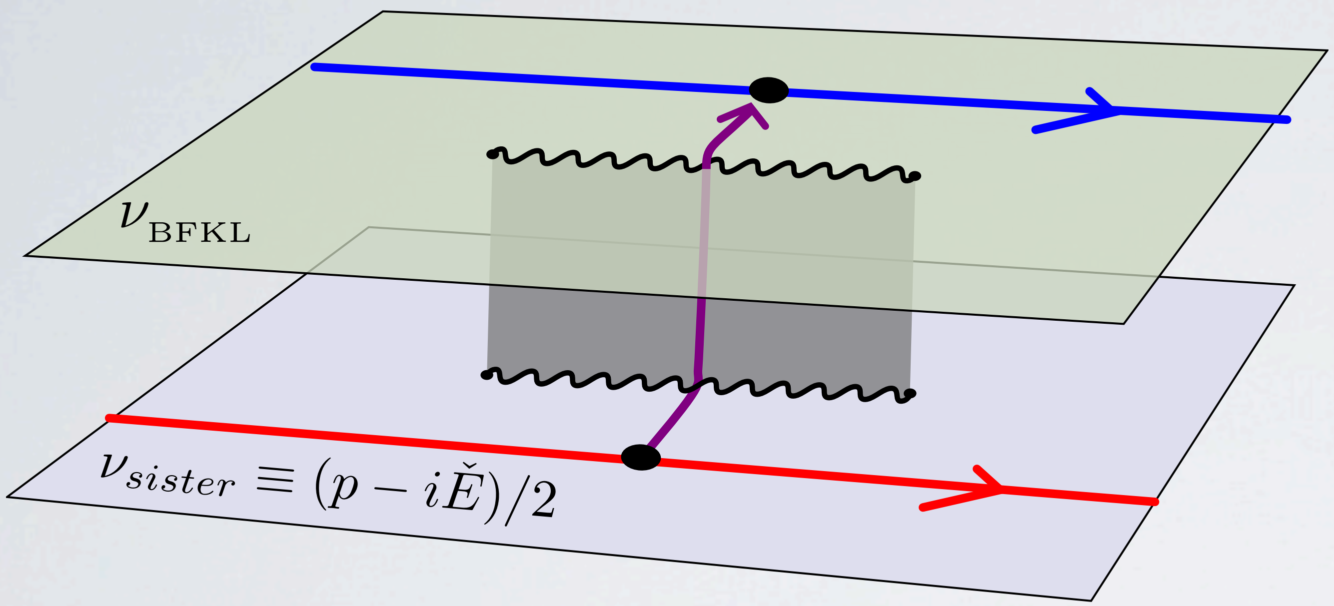
leading term dominate at large $\tau + \sigma$

Akin to DGLAP versus BFKL resummation

Leading term in one expansion $\Leftrightarrow$ Resummation of **infinitely many** terms in the other

From collinear to Regge kinematics

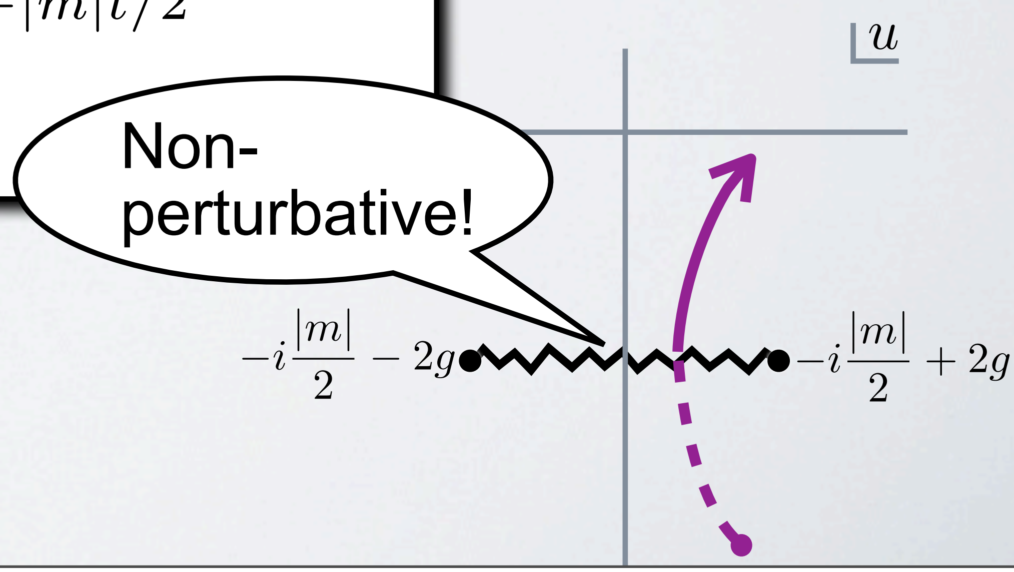
[BB,Caron-Huot,Sever'14]



At finite coupling it is easy to navigate between the two pictures

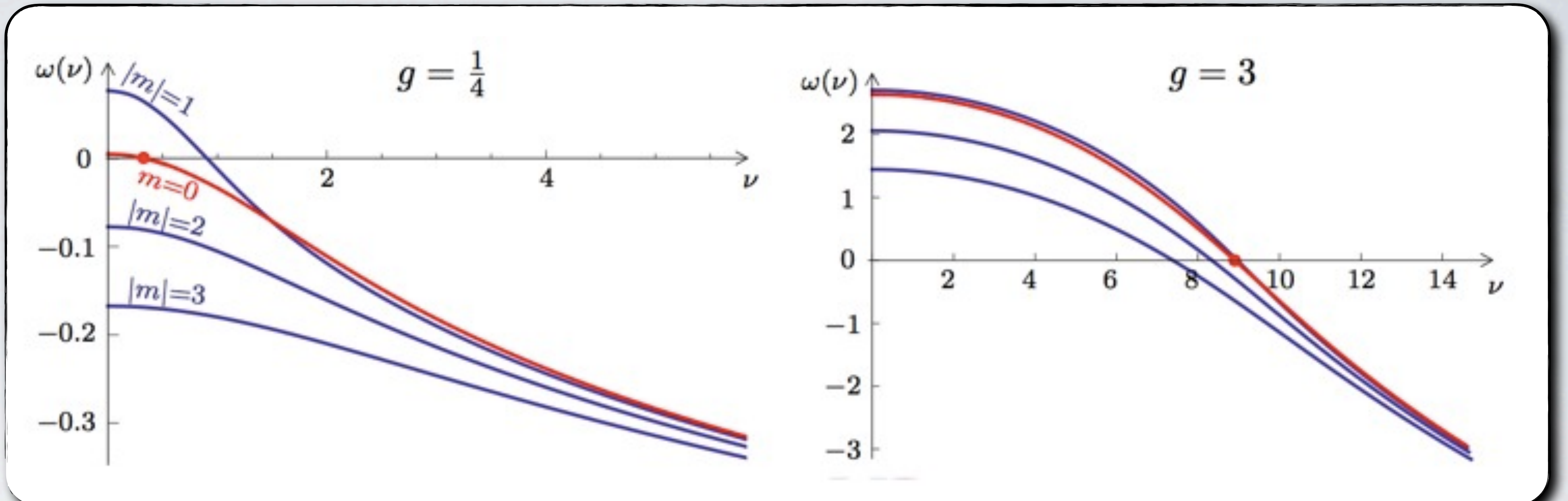
$$\omega(u,m) = \int_0^\infty \frac{dt}{t}\left(K(t) - \frac{K(-t)+K(t)}{2}\cos(ut)e^{-|m|t/2}\right)$$

$$\nu(u,m) = 2u + \int_0^\infty \frac{dt}{t}\frac{K(-t)-K(t)}{2}\sin(ut)e^{-|m|t/2}$$

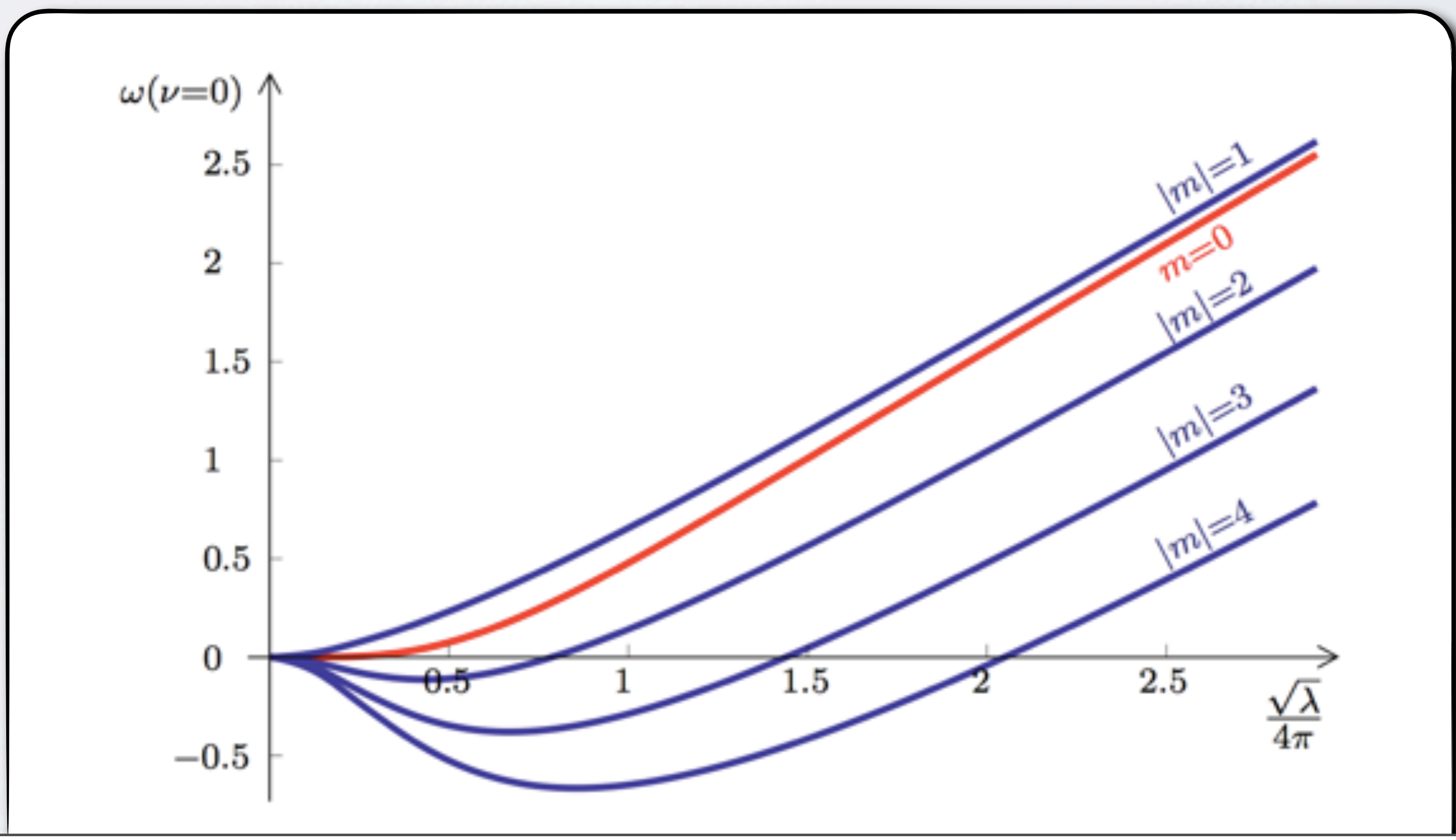


Adjoint eigenvalues at finite coupling

Eigenvalues :



Intercepts :



Other applications ?

- Can we get a systematic expansion in the Regge limit (i.e., understand the infinite tower of daughter trajectories controlling energy suppressed corrections)

-Can we plot amplitudes at finite coupling?

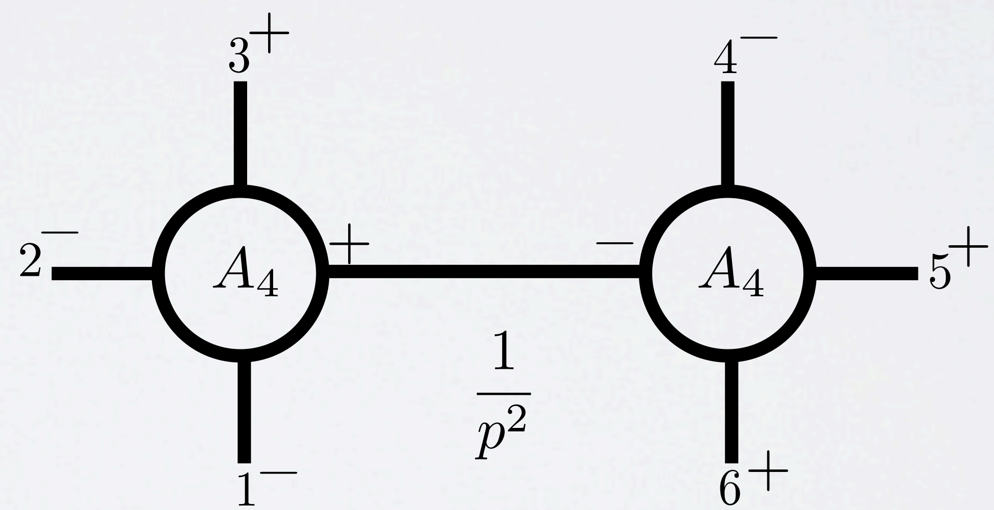
- Can we explore/learn something about amplitudes at non-perturbative level?

Example : factorization of scattering amplitudes

[work in progress with Sever and Vieira]

[Dixon,von Hippel'14], [Dixon,von Hippel,McLeod'15]

$$A_6^{\mathrm{NMHV}} \xrightarrow[p^2 \to 0]{}$$



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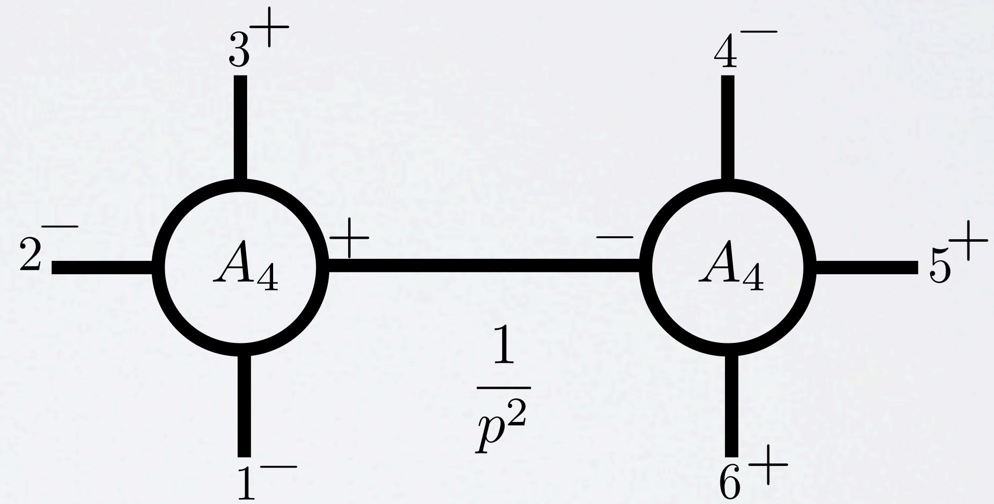

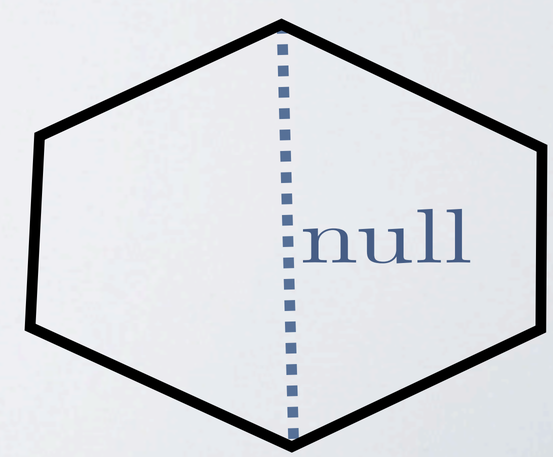

Dual limit for Wilson loops corresponds to two cusps becoming null separated (This limit is within radius of convergency of the OPE)

Other applications ?

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Example : factorization of scattering amplitudes

Toy model : $$I \equiv \int_0^\infty du\, e^{-u\, p^2 - \Gamma_{\rm cusp} \log^2 u}$$

1) At weak coupling $$I = \frac{1}{p^2} \sum_l g^{2l}\, \mathrm{Pol}_l(\log p^2)$$

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3) There is a smooth discontinuity $$\mathrm{Dis}\, A_6 \propto e^{-\Gamma_{\rm cusp} \log^2 (p^2)} \neq 0$$

Other applications ?

- Can we get a systematic expansion in the Regge limit (i.e., understand the infinite tower of daughter trajectories controlling energy suppressed corrections)

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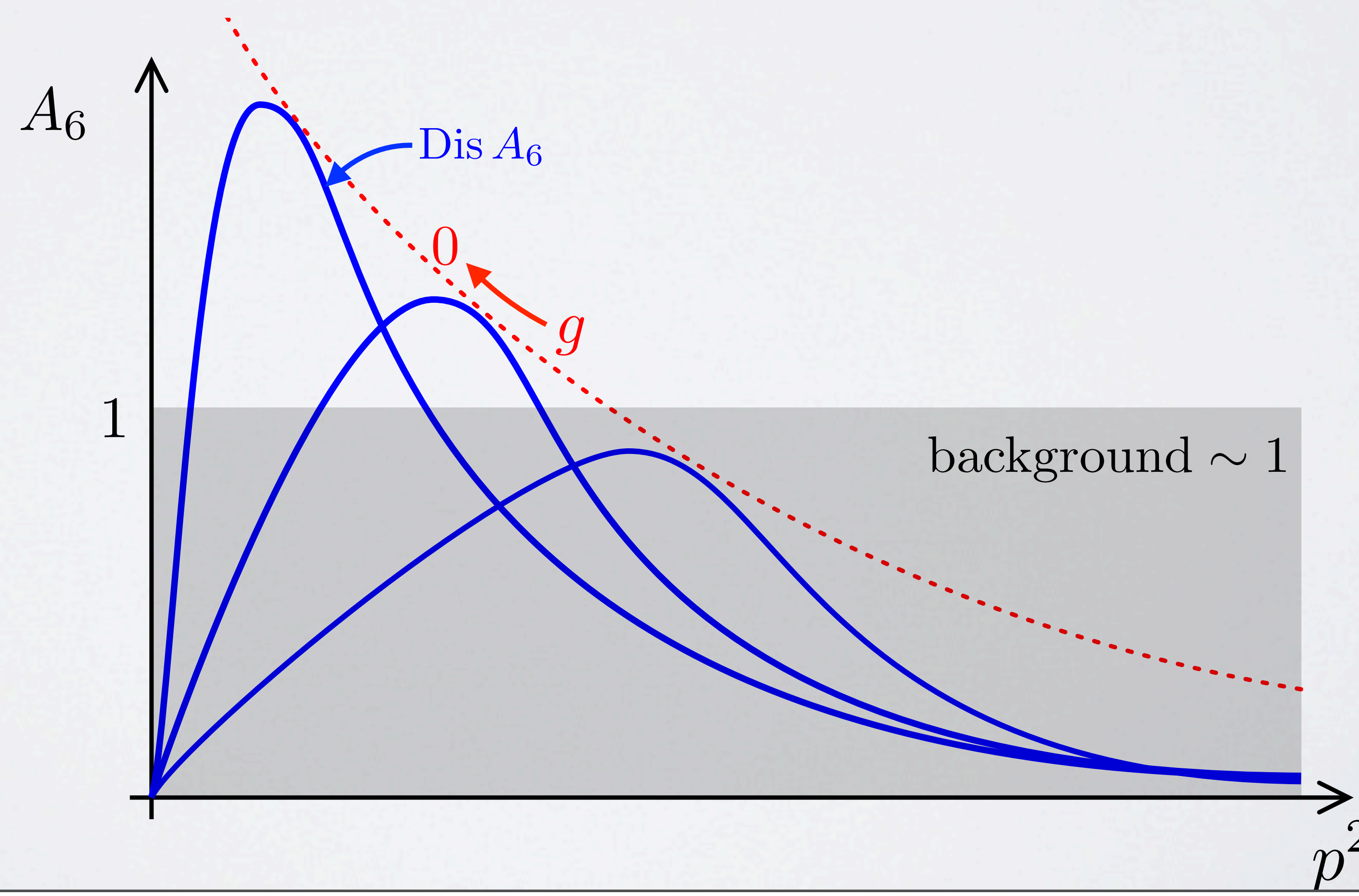
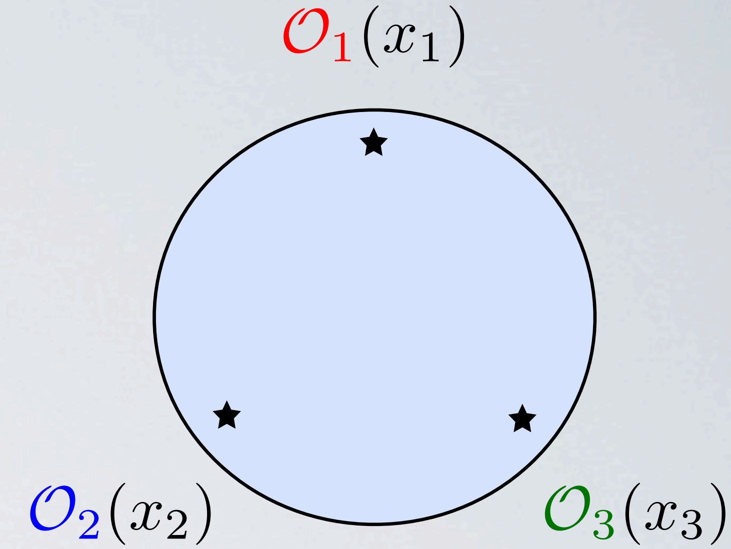
Example : factorization of scattering amplitudes

Cartoon of what is happening





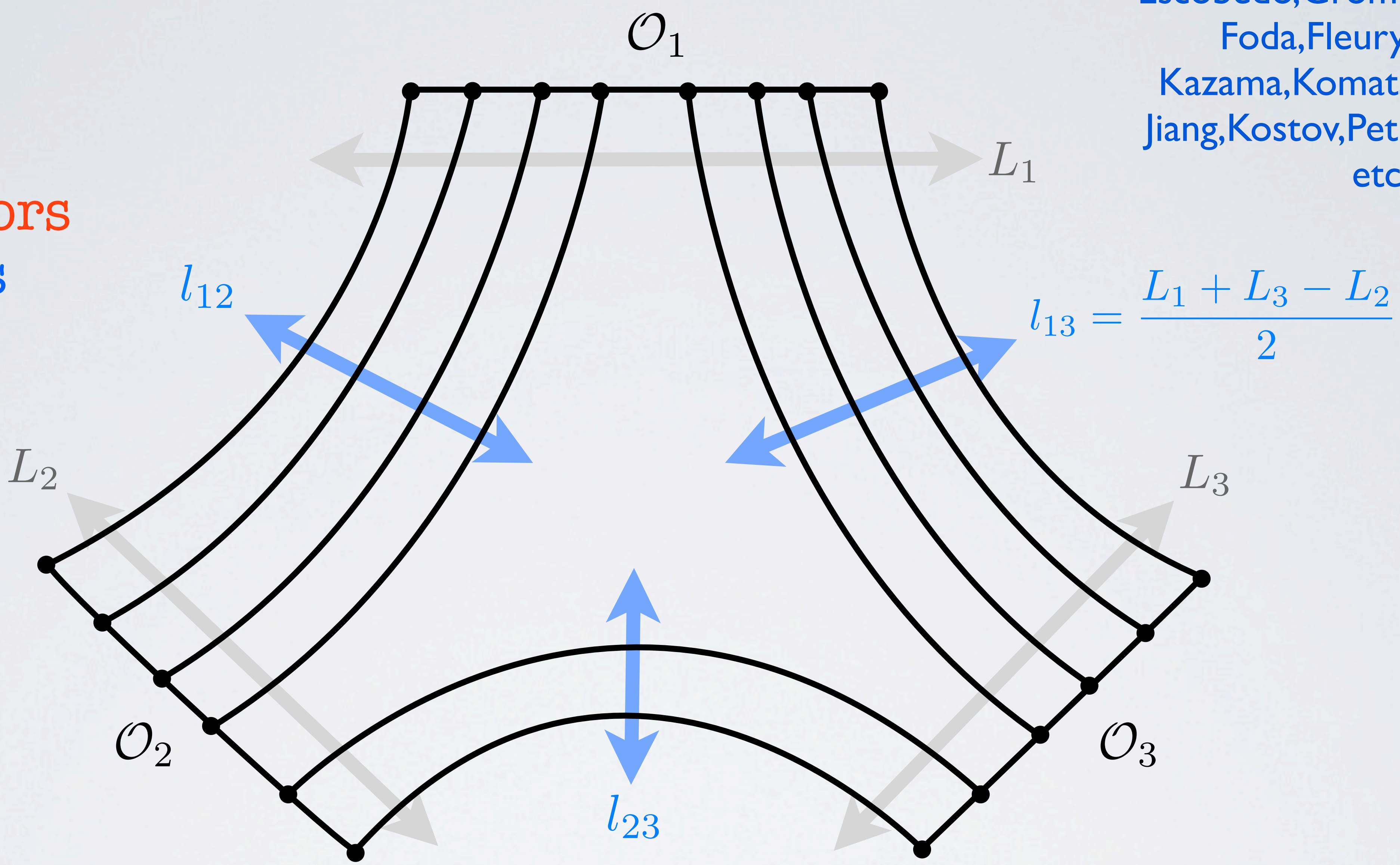
Structure constants and string splitting/joining

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_{12}}x_{23}^{\Delta_{23}}x_{13}^{\Delta_{13}}}$$

Spin chain tayloring

[Many people, see e.g. Escobedo,Gromov,Sever,Vieira, Foda,Fleury,Caetano, Kazama,Komatsu,Nishimura, Jiang,Kostov,Petrovskii,Serban, etc.]

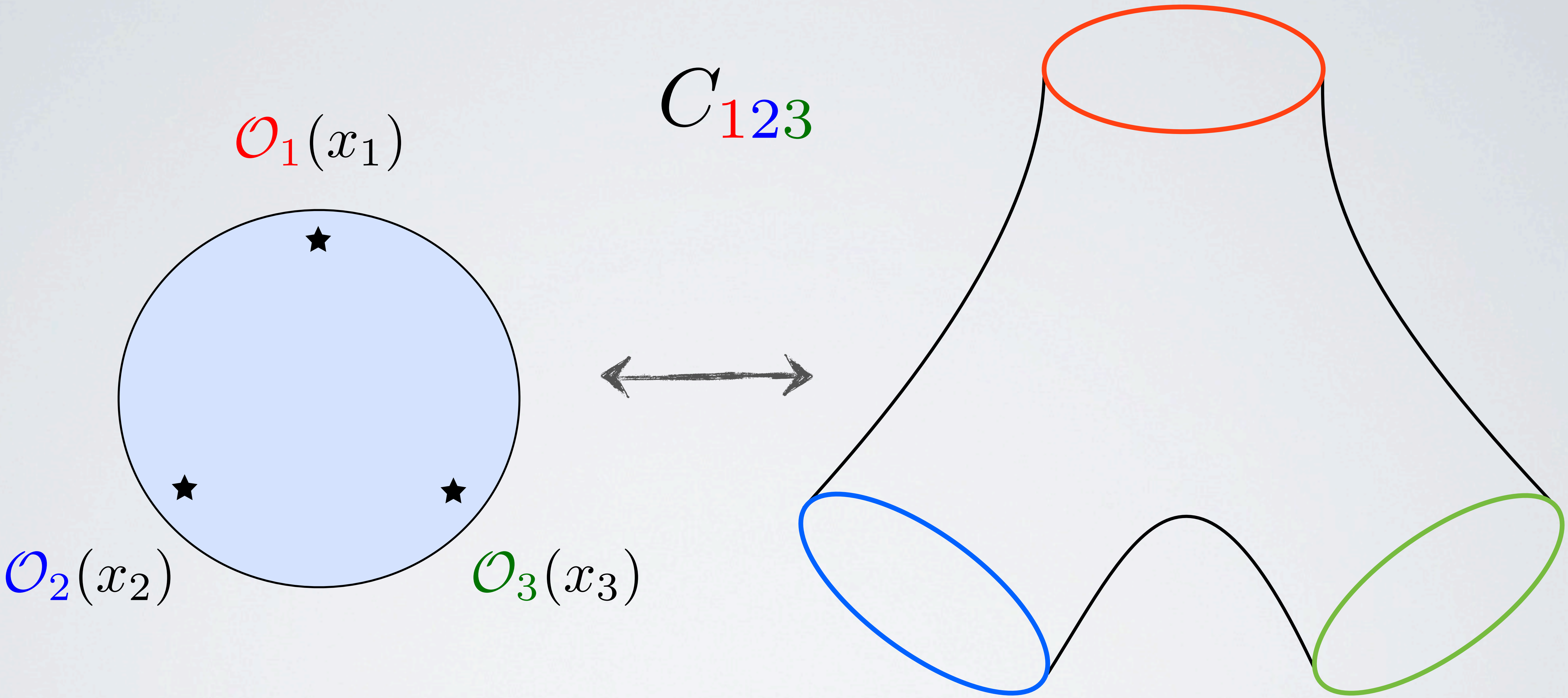
topology :
- 3 operators
- 3 bridges



Recipe : cut spin chain states and compute their overlap following the Wick contractions

How to go to higher loops? (spin chain wave functions are unknown, as well as correction to splitting vertex)

Inspiration from string



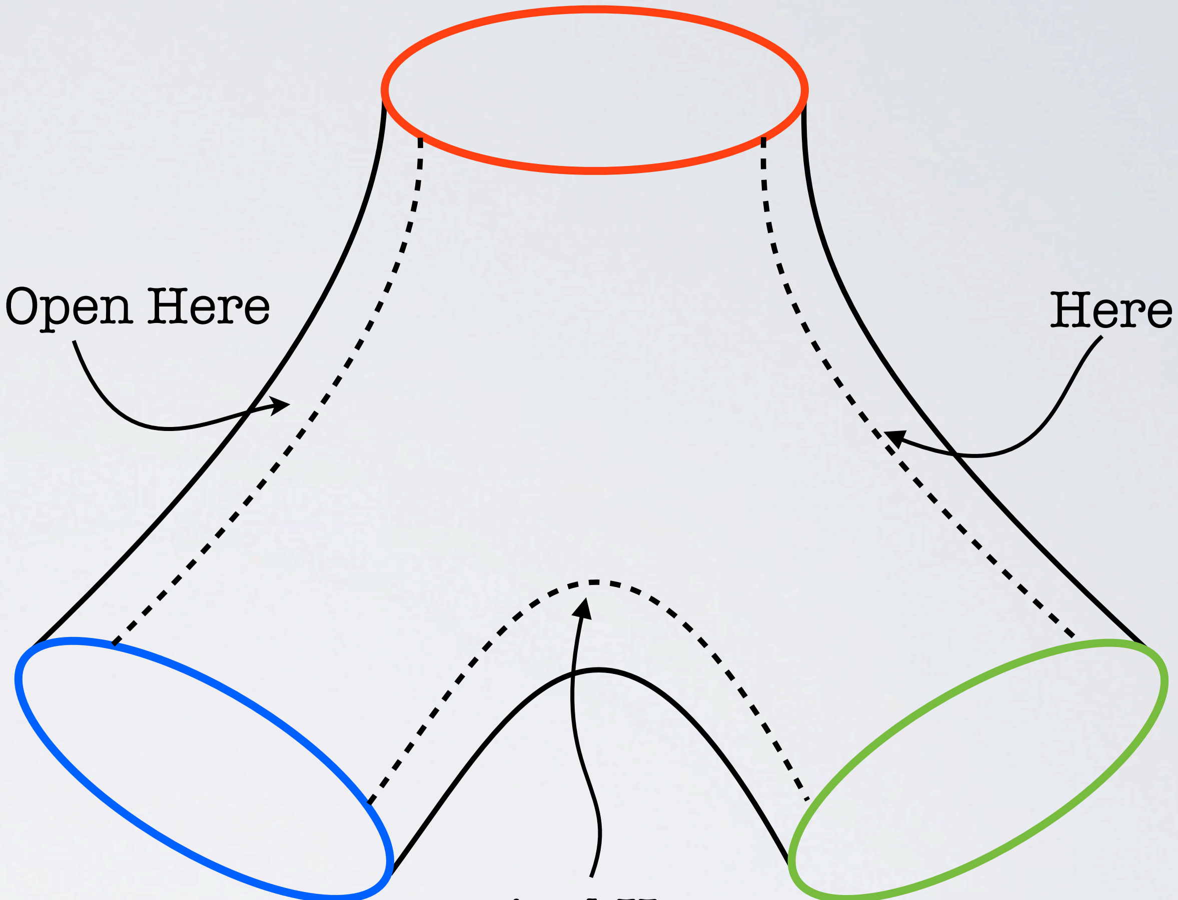
3-punctured sphere

pair of pants

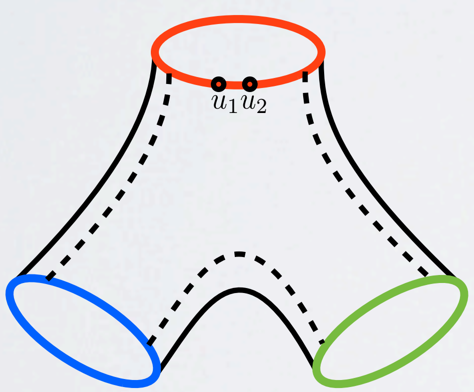
Cutting procedure

Cut in smaller pieces

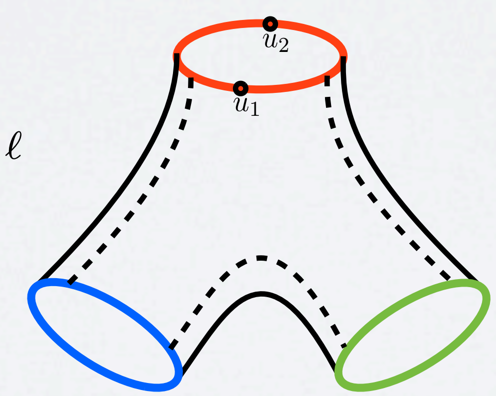
1 pair of pants = 2 hexagons



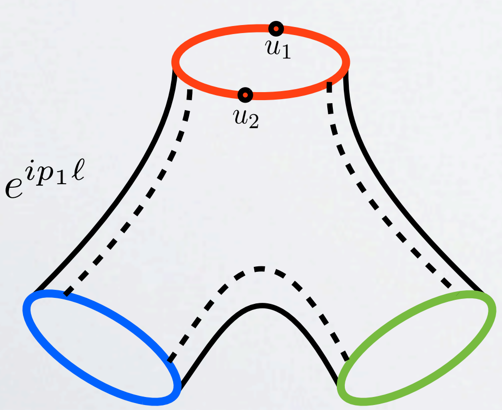
Same with magnons



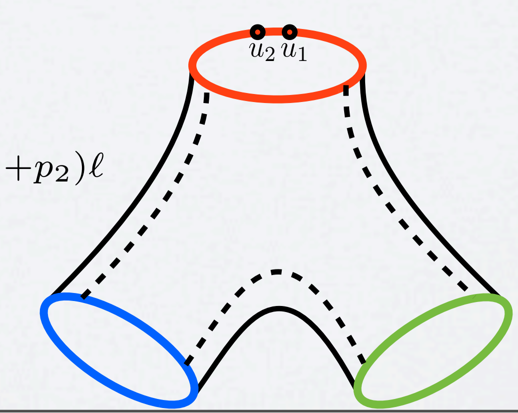
$+ \quad e^{ip_2 \ell}$



$+ \quad S(u_1, u_2) e^{ip_1 \ell}$



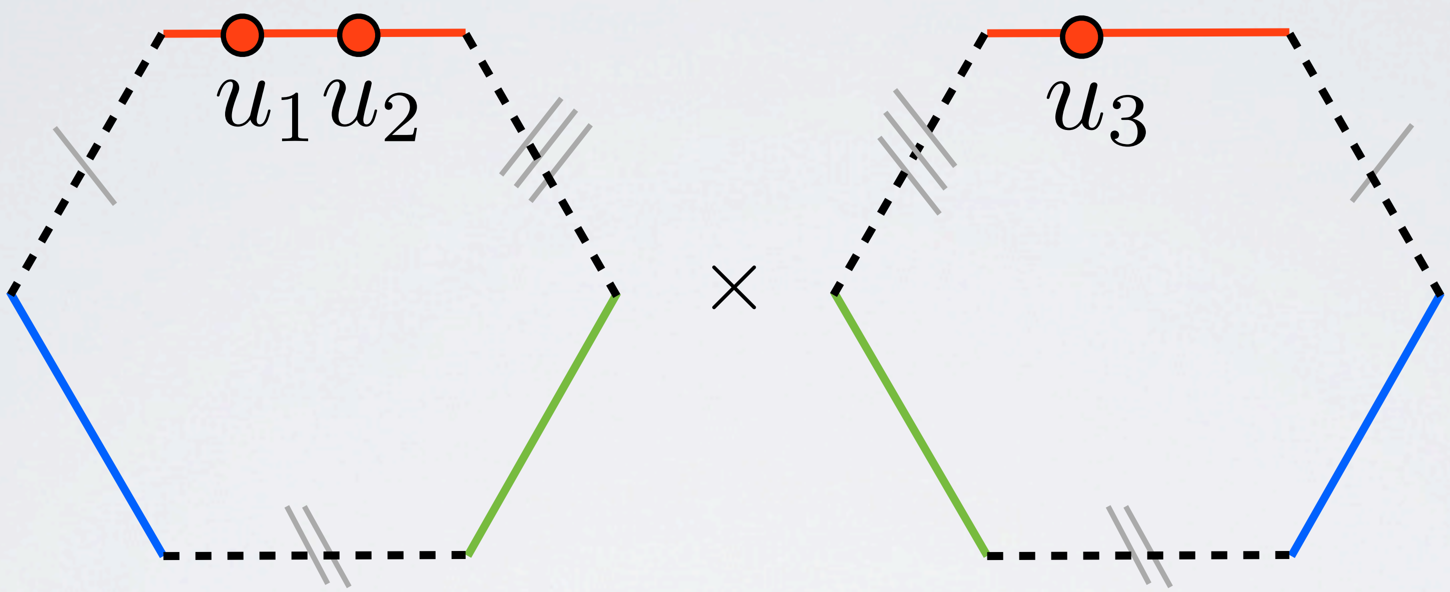
$+ \quad e^{i(p_1+p_2)\ell}$



Hexagon factorization

Elementary block

[BB,Komatsu,Vieira'15]



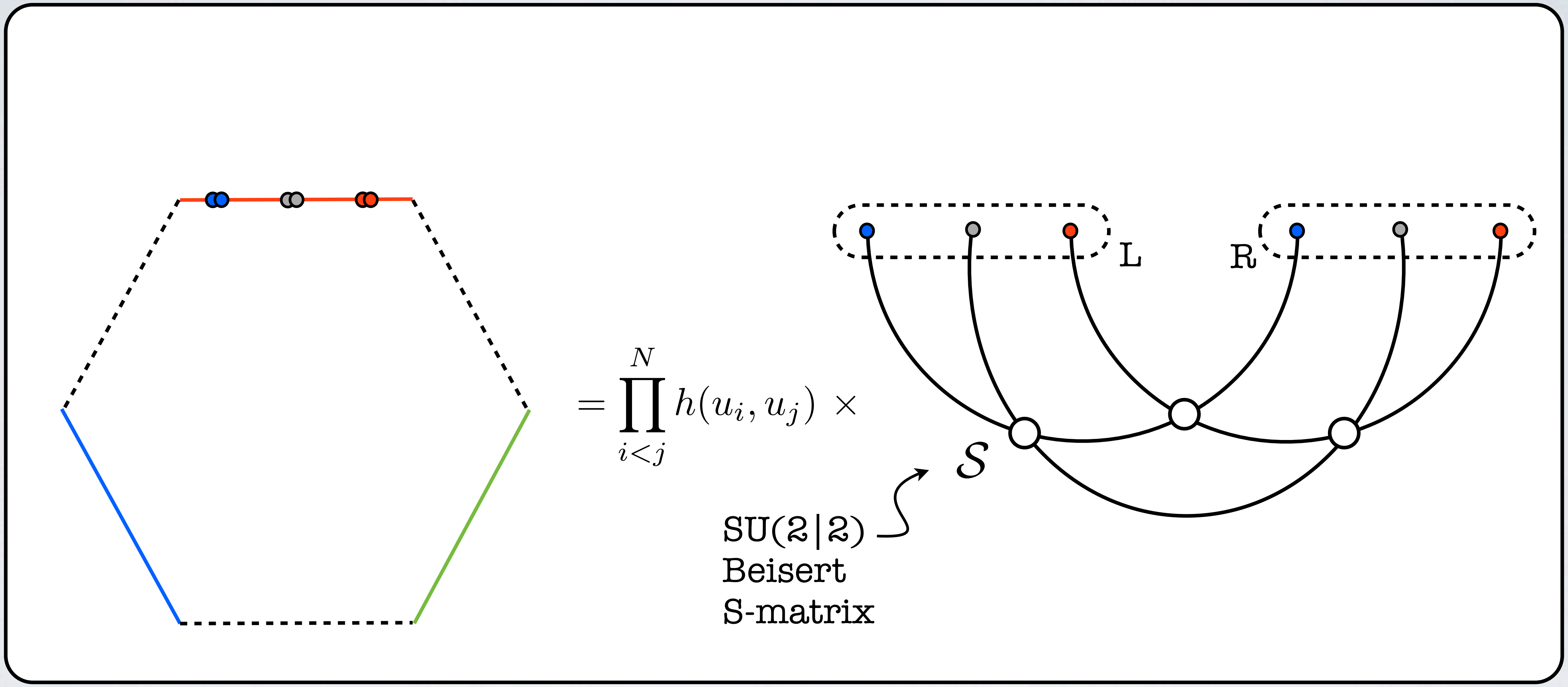
Hexagon form factor :

- Contribution of an hexagon decorated with magnons on its edges

- Apply integrable bootstrap again to determine it at finite coupling

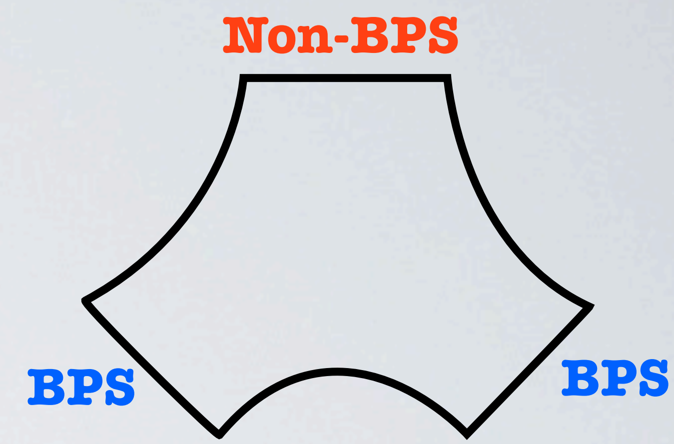
N-magnon hexagon

Conjecture (one can actually prove it for low number of magnons):



$$\mathfrak{h}^{A_1 \dot{A}_1 \cdots A_N \dot{A}_N} = (-1)^{\mathfrak{f}} \prod_{i<j}^{N} h_{ij} \langle \chi_N^{\dot{A}_N} \ldots \chi_1^{\dot{A}_1} | \mathcal{S} | \chi_1^{A_1} \ldots \chi_N^{A_N} \rangle$$

Concrete formula



Hexagon prediction :

$$\left(\frac{C^{\bullet\circ\circ}_{123}}{C^{\circ\circ\circ}_{123}}\right)^2 = \frac{\prod_{k=1}^{S}\mu(u_k)}{\det\ \partial_{u_i}\phi_j \prod_{i<j} S(u_i,u_j)} \times \mathcal{A}^2$$

e.g. $\mathcal{O}_1 = \mathrm{tr} D^S Z^{L_1}$

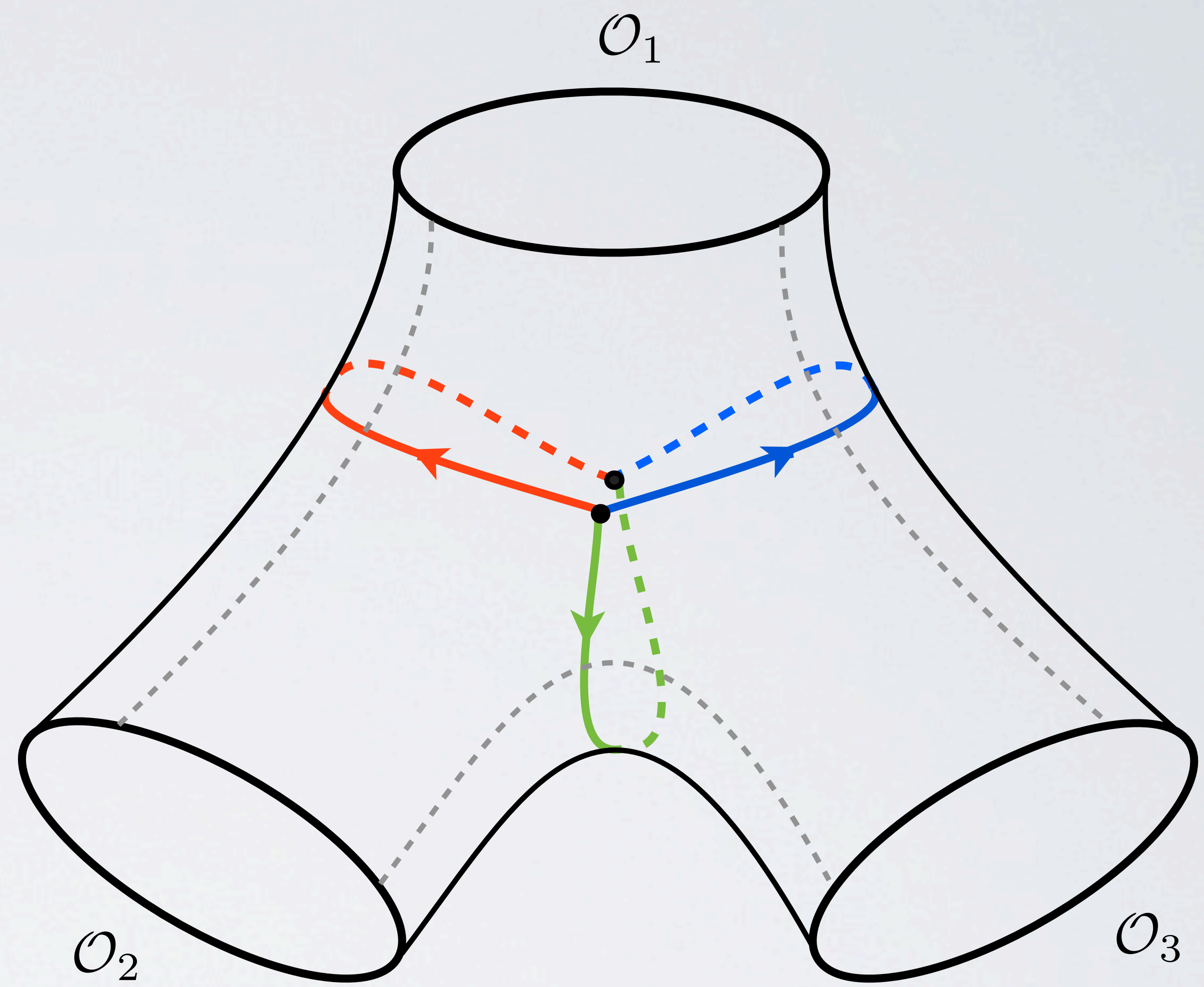
Hexagon part

sum over partitions of Bethe Roots

$$\mathcal{A} = \prod_{i<j} h(u_i,u_j) \sum_{\alpha\cup\bar{\alpha}=\mathbf{u}} (-1)^{|\bar{\alpha}|} \prod_{j\in\bar{\alpha}} e^{ip_j\ell} \prod_{i\in\alpha, j\in\bar{\alpha}} \frac{1}{h(u_i,u_j)}$$

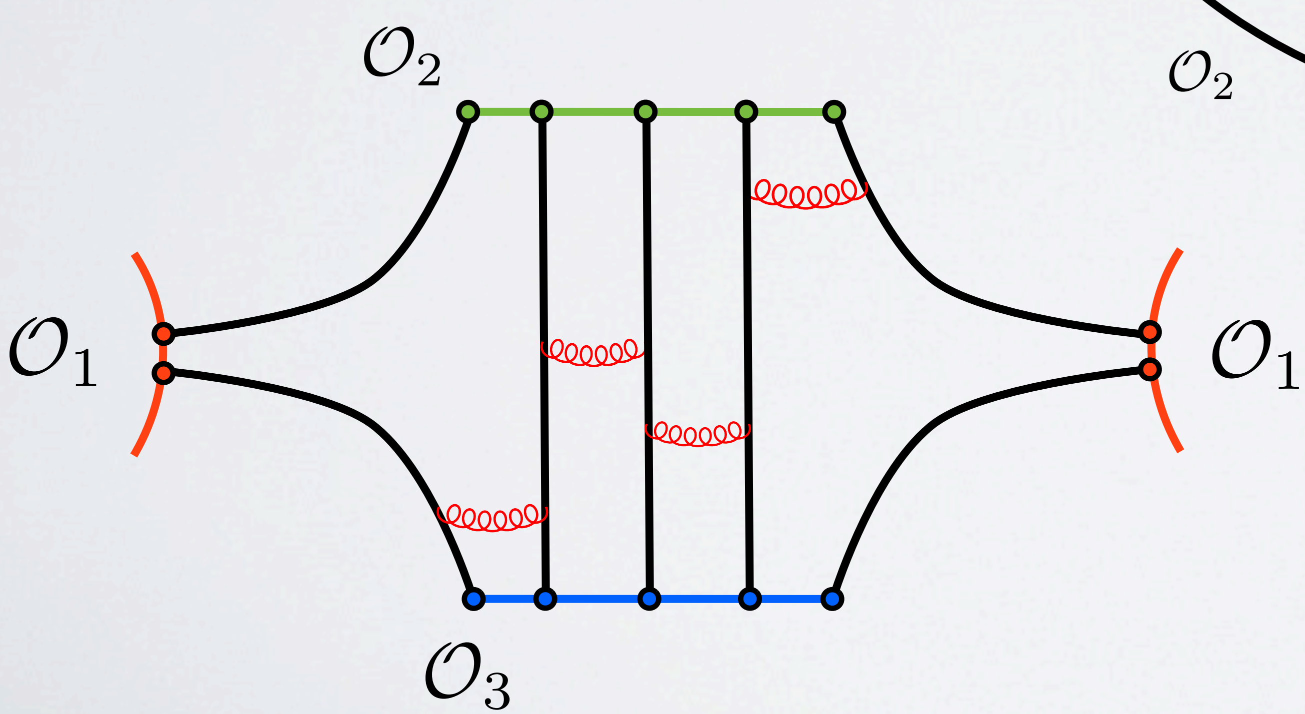
Valid to all loops up to finite size effects

Finite size effects



New virtual effects :

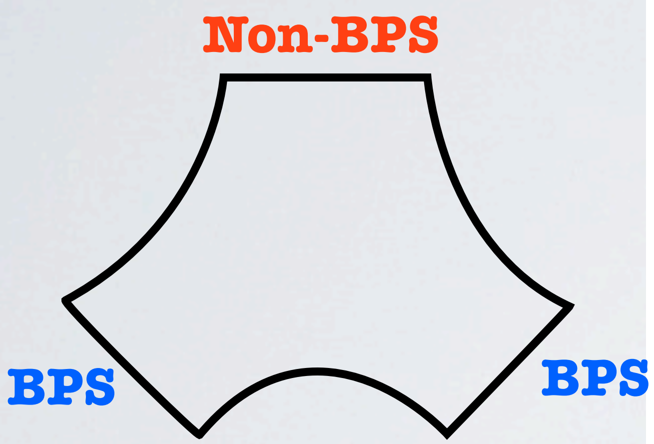
Exchange of (mirror) particles between the two hexagons



here a gluon is passing through the bridge (small effect at weak coupling)

Comparison with data

$$\left(\frac{C^{\bullet\circ\circ}_{123}}{C^{\circ\circ\circ}_{123}}\right)^2$$



$Spin$	“$Long$” $Bridge$ $i.e.$ $length$ $\ell = 2$
2	$\frac{1}{6} - 2g^2 + 28g^4 + \dots$
4	$\frac{1}{70} - \frac{205}{882}g^2 + \frac{36653}{9261}g^4 + \dots$
6	$\frac{1}{924} - \frac{553}{27225}g^2 + \frac{826643623}{2156220000}g^4 + \dots$
8	$\frac{1}{12870} - \frac{14380057}{9018009000}g^2 + \frac{2748342985341731}{85305405235050000}g^4 + \dots$
10	$\frac{1}{184756} - \frac{3313402433}{27991929747600}g^2 + \frac{156422034186391633909}{6220116940498323408000}g^4 + \dots$

$Spin$	“$Short$” $Bridge$ $i.e.$ $length$ $\ell = 1$
2	$\frac{1}{6} - 2g^2 + (28 + 12\zeta_3)g^4 + \dots$
4	$\frac{1}{70} - \frac{205}{882}g^2 + \left(\frac{76393}{18522} + \frac{10}{7}\zeta_3\right) g^4 + \dots$
6	$\frac{1}{924} - \frac{553}{27225}g^2 + \left(\frac{880821373}{2156220000} + \frac{7}{55}\zeta_3\right) g^4 + \dots$
8	$\frac{1}{12870} - \frac{14380057}{9018009000}g^2 + \left(\frac{5944825782678337}{170610810470100000} + \frac{761}{75075}\zeta_3\right)g^4 + \dots$
10	$\frac{1}{184756} - \frac{3313402433}{27991929747600}g^2 + \left(\frac{171050793565932326659}{6220116940498323408000} + \frac{671}{881790}\zeta_3\right) g^4 + \dots$

2-loop finite size effect

perfect agreement
(including zeta’s coming from finite size corrections)

Conclusions

Integrability comes with powerful new strategies for computing quantities at any value of the coupling in planar N=4 SYM theory

It allows us to attack increasingly complicated objects and find all-loop expressions (conjectures) for them,
like for amplitudes, structure constants, etc.

How far can we go? Can we bootstrap string loops? Can we solve to any order in the 1/N expansion?

How can we prove all these conjectures? Can one understand why is this theory integrable after all?

THANK YOU!