xPAND

(*Perturbation Are Not Difficult*)

Cyril PITROU

xPanding

(*Perturbation Are Not Difficult In Newtonian Gauge*)

Cyril PITROU

Outline

Introduction

1) Why using a tensor algebra package?
2) How should it work? How does it work?

General applications with xAct

3) Perturbations in GR.
4) Action variation

Cosmological perturbations

5) A geometric approach
6) Implementation xPand

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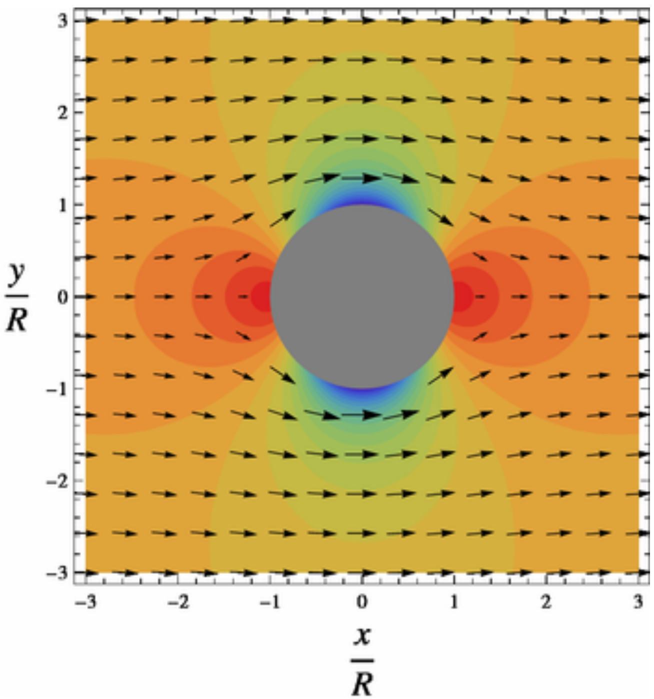
Tensor fields

All classical and continuous physics is expressed in terms of *tensor fields* on a *manifold*. Well known examples are

-*Fluid dynamics (Pressure P, Energy density* ρ *, velocity* $\vec{v}$
-*Electromagnetism (*$\vec{E}$ $\vec{B}$ *fields, but also V &* $\vec{A}$ *)*



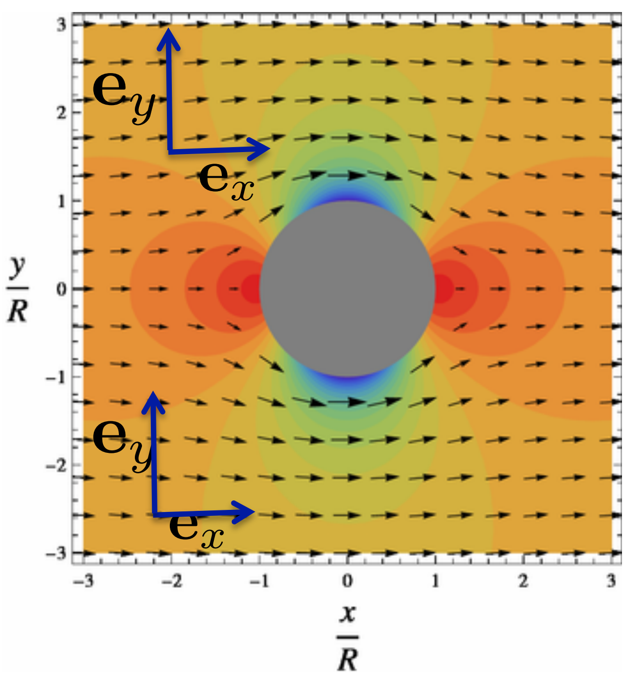
Manifold, coordinates, and vector basis

In general the mathematical structure is obvious and hidden

- The manifold is flat and trivial : $\mathbb{R}^3$ and time is a parameter

- Natural coordinates are (x,y,z) or (r,ϑ,φ) or (r,ϑ,z)
  This labels the point we are considering

- Natural basis $$\mathbf{e}_1 = \frac{\partial}{\partial x} = (1,0,0)$$



Temperature T(x,y,z)

Velocity (components)
$v^x(x,y,z)$, $v^y(x,y,z)$, $v^z(x,y,z)$

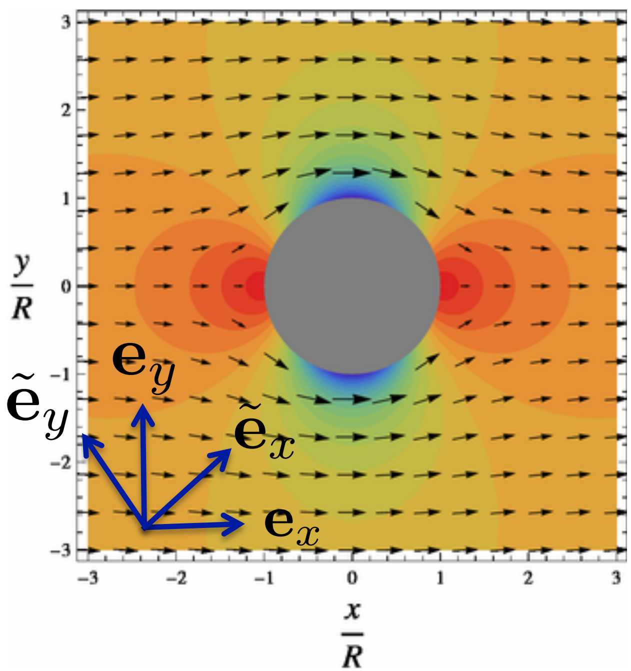
Vector basis and forms basis

We need a set of 1-forms to get the components of a vector.
Indeed a form associates a number to a vector

The co-basis is a set of forms $\mathbf{e}^i$ i =1,2,3 (or i=x,y,z)

$$\mathbf{e}^i[\mathbf{e}_j] = \delta^i_j \quad \Rightarrow \mathbf{e}^i[\mathbf{V}] = \mathbf{e}^i[V^k \mathbf{e}_k] = V^i$$

Change of basis



$$\tilde{\mathbf{e}}_i = \mathbf{e}_j R^j{}_i \qquad \text{Co-variance}$$

$$\mathbf{e}^i = R^i{}_j \tilde{\mathbf{e}}^j \qquad \text{Contra-variance}$$

A vector field is defined, independently
from the basis used to measure its components.
It has a pure geometrical meaning

$$\mathbf{V} = V^i \mathbf{e}_i = \tilde{V}^i \tilde{\mathbf{e}}_i$$

Examples of tensors

We define the object $$\mathbf{R} = R^i{}_j \mathbf{e}_i \otimes \mathbf{e}^j$$

$$\tilde{\mathbf{e}}_j = \mathbf{R}[\mathbf{e}_j] = \mathbf{e}_i R^i{}_j$$

So very naturally, the quantities which appear in equations are more general than just scalar fields and vector vields.
But it appears more natural to work with coordinates.

e.g. In fluid dynamics we have the strain rate,
which comes from differences of velocity inside the fluid

$$\sigma = \sigma_{ij} \mathbf{e}^i \otimes \mathbf{e}^j \qquad \sigma_{ij} = \partial_i v_j + \partial_j v_i$$

and the stress tensor $\Sigma = \Sigma_{ij} \mathbf{e}^i \otimes \mathbf{e}^j$ which enters the Navier-Stokes

$$\Sigma = \mu[\sigma] \qquad \Sigma_{ij} = \mu_{ij}{}^{kl} \sigma_{kl}$$

Measuring the components of the viscosity tensor μ is the program of rheology

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Adapted notation for tensors

1) We want to keep the geometrical meaning

Relations are valid for any basis

We can specify to Cartesian or spherical coordinates only at the very end

2) When a tensor is applied to another tensor and is of complicated nature like μ we need to keep track of what 1-form is applied to which vector

The solution is to use *abstract indices* which are only here to remember the tensorial nature

- We can contract indices (apply a form to a vector) $A^{\mu\nu\sigma}B_{\nu\sigma}$
- We can take the tensor product of two tensors $v^{\mu}v^{\nu}$
- We can build new tensors with (covariant) derivatives $\nabla_{\mu}v_{\nu}$
- We can specify some symmetries, antisymmetries $T_{\mu\nu}-T_{\nu\mu}$

Usually we use a different set of indices to avoid confusion
Greek indices instead of i,j,k in general.

Components in a given basis can be recovered through $X^i = e^i[\mathbf{X}] = e^i{}_{\mu}X^{\mu}$

So why an abstract tensor calculus

1) Expressions can become quite large in the derivation manipulation of equations
   space ! We use the RAM

2) Simplifications can be complicated and take a very long time
   time ! We use the CPU

Specifications we need :

Input
-a notation for up and down abstract indices
-contraction of indices (Einstein convention of summation)
-covariant derivatives
-a metric and an inverse metric to raise or lower indices. *Not necessarily flat*.

Output
-A simplification routine which detects terms which are equivalent
-Nice display with indices placed in the correct position (up&down)

Several packages available : I know only *xAct* ...

http://www.xact.es/

Especially for cosmological perturbations…

1) The large-scale structure of the universe is very close to a homogeneous and isotropic solution.

2) The growth of structure is understood as the result of gravitational collapse of small fluctuations around this idealized background

3) Linearizing the theory of gravity (GR for the standard model) is the easiest
-it accounts very well for the growth of structure on large and intermediary scales
-it is rather computationally involved to get the equations due to the complexity of GR

4) Non-linear theory is needed to account for small scales (non-linear) effects
-non-Gaussianity induced by non-linear effects ?
-GR effects in the N-body simulations ?

The derivation of equations can take a very long time and is rather tedious

Loading the package xAct

```
In[1]:= << xAct/xTensor.m

------------------------------------------------------------

Package xAct`xPerm`  version 1.2.0, {2013, 1, 27}

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Connecting to external mac executable...

Connection established.

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  Disclaimer[]. This is free software, and you are welcome to redistribute
  it under certain conditions. See the General Public License for details.
------------------------------------------------------------

In[2]:= DefManifold[M4, 4, {a, b, c, d, e, f, g, h}]

  ** DefManifold: Defining manifold M4.

  ** DefVBundle: Defining vbundle TangentM4.

In[3]:= DefMetric[-1, metric[-a, -b], CD]

  ** DefTensor: Defining symmetric metric tensor metric[-a, -b].

  ** DefTensor: Defining antisymmetric tensor epsilonmetric[-a, -b, -c, -d].

  ** DefTensor: Defining tetrametric Tetrametric[-a, -b, -c, -d].

  ** DefTensor: Defining tetrametric Tetrametric†[-a, -b, -c, -d].

  ** DefCovD: Defining covariant derivative CD[-a].
```

Basic syntax

```
In[4]:= DefTensor[v[a], M4]
        DefTensor[T[a, b], M4]

        ** DefTensor: Defining tensor v[a].

        ** DefTensor: Defining tensor T[a, b].

In[6]:= T[-a, b] v[-b]
```

Out[6]= $T_a{}^b\, v_b$

```
In[7]:= T[-a, b] v[-b] + T[-a, -b] v[b] - 2 metric[b, c] v[-c] T[-a, -b]
```

Out[7]= $T_a{}^b\, v_b + T_{ab}\, v^b - 2\, \mathrm{metric}^{bc}\, T_{ab}\, v_c$

```
In[8]:= ContractMetric[%]
```

Out[8]= $T_a{}^b\, v_b - T_{ab}\, v^b$

```
In[9]:= ToCanonical[%]
```

Out[9]= 0

```
In[10]:= CD[-a]@v[-b]
```

Out[10]= $\nabla_a\, v_b$

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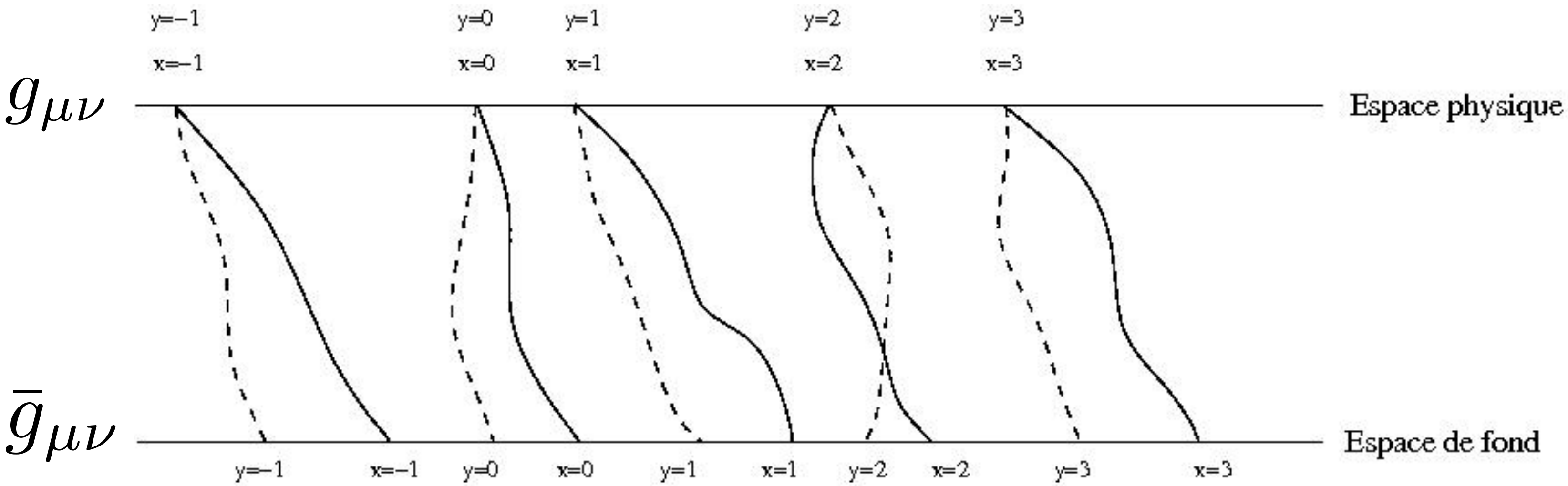
Perturbations in General



Perturbations are tensorial fields, living on the background spacetime.
They can be decomposed in orders of perturbations

$$\phi^*[g_{\mu\nu}] - \bar{g}_{\mu\nu} \equiv \Delta[g_{\mu\nu}]$$

$$\Delta[g_{\mu\nu}] = \sum_{n=1}^{\infty} \frac{{}^{(n)}h_{\mu\nu}}{n!}$$

Perturbations in General

Perturbation of the inverse metric $$\frac{1}{1+x} = 1 - x + x^2 - x^3 \ldots$$

$$\Delta^n \left[(\bar{g}^{-1})^{\mu\nu}\right] = \sum_{(k_i)} (-1)^m \frac{n!}{k_1! \ldots k_m!} \; {}^{\{k_m\}}h^{\mu\zeta_m} \; {}^{\{k_{m-1}\}}h_{\zeta_m}{}^{\zeta_{m-1}} \ldots {}^{\{k_2\}}h_{\zeta_3}{}^{\zeta_2} \; {}^{\{k_1\}}h_{\zeta_2}{}^{\nu},$$

Perturbation of the Christoffel

$$\Delta^n \left[\bar{\Gamma}^\rho{}_{\mu\nu}\right] = \sum_{(k_i)} (-1)^{m+1} \frac{n!}{k_1! \ldots k_m!} \; {}^{\{k_m\}}h^{\rho\zeta_m} \; {}^{\{k_{m-1}\}}h_{\zeta_m}{}^{\zeta_{m-1}} \ldots {}^{\{k_2\}}h_{\zeta_3}{}^{\zeta_2} \; {}^{\{k_1\}}h_{\zeta_2\mu\nu},$$

$${}^{\{n\}}h_{\rho\mu\nu} = \frac{1}{2}\left(\bar{\nabla}_\nu \, {}^{\{n\}}h_{\rho\mu} + \bar{\nabla}_\mu \, {}^{\{n\}}h_{\rho\nu} - \bar{\nabla}_\rho \, {}^{\{n\}}h_{\mu\nu}\right).$$

Perturbation of the Riemann tensor

$$\Delta^n \left[\bar{R}_{\mu\nu\rho}{}^\sigma\right] = \bar{\nabla}_\nu \left(\Delta^n \left[\bar{\Gamma}^\sigma{}_{\mu\rho}\right]\right) - \sum_{k=1}^{n-1} \binom{n}{k} \Delta^k \left[\bar{\Gamma}^\zeta{}_{\nu\rho}\right] \Delta^{n-k} \left[\bar{\Gamma}^\sigma{}_{\zeta\mu}\right] - (\mu \leftrightarrow \nu),$$

All perturbations of geometrical tensors are expressed in function of the perturbed metric and its background covariant derivative

$$ {}^{(n)}h_{\mu\nu} \qquad \bar{\nabla}_\alpha {}^{(n)}h_{\mu\nu} \qquad \bar{\nabla}_\alpha \bar{\nabla}_\beta {}^{(n)}h_{\mu\nu}$$

General perturbations : using *xPert* (D. Brizuela *et al* 2006-now)

```
In[1]:= << xAct`xPert`
  ------------------------------------------------------------
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  Connection established.
  ------------------------------------------------------------
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In[2]:= DefManifold[M, 4, {α, β, μ, ν, λ, σ}]
        DefMetric[-1, g[-α, -β], CD, {";", "∇"}, PrintAs → "g̅"];
  ** DefManifold: Defining manifold M.
  ** DefVBundle: Defining vbundle TangentM.
  ** DefTensor: Defining symmetric metric tensor g[-α, -β].
  ** DefTensor: Defining antisymmetric tensor epsilong[-α, -β, -λ, -μ].
  ** DefTensor: Defining tetrametric Tetrag[-α, -β, -λ, -μ].
  ** DefTensor: Defining tetrametric Tetrag†[-α, -β, -λ, -μ].
  ** DefCovD: Defining covariant derivative CD[-α].

In[4]:= DefMetricPerturbation[g, dg, ϵ];
  ** DefParameter: Defining parameter ϵ.
  ** DefTensor: Defining tensor dg[LI[order], -α, -β].

In[5]:= ExpandPerturbation@Perturbed[RicciScalarCD[], 1] // ContractMetric // ToCanonical
```

Out[5]= $-\epsilon\ dg^{1\alpha\beta}\ R[\nabla]_{\alpha\beta} + R[\nabla] + \epsilon\ dg^{1\alpha\beta}{}_{;\alpha;\beta} - \epsilon\ dg^{1\alpha}{}_{\alpha}{}^{;\beta}{}_{;\beta}$

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Variation of Actions

$$\int E_{\mu\nu}\delta g^{\mu\nu}\sqrt{g}\mathrm{d}^4x \Longrightarrow E_{\mu\nu}$$

$$\int E_{\mu\nu\alpha}\nabla^\alpha\delta g^{\mu\nu}\sqrt{g}\mathrm{d}^4x \Longrightarrow -\nabla^\alpha E_{\mu\nu\alpha}$$

```
L = RicciScalarCD[]
ExpandPerturbation@Perturbation[detg * L] /. Ruleh
ToCanonical@ContractMetric[ VarD[h[a, b], CD][%] / detg]
```

Out[51]= $R[\nabla]$

Out[52]= $-\dfrac{\tilde{\tilde{g}}\, h_a{}^a\, R[\nabla]}{2\sqrt{-\tilde{\tilde{g}}}} + \sqrt{-\tilde{\tilde{g}}}\left(-h^{ab}\, R[\nabla]_{ab} + g^{ab}\left(\dfrac{1}{2}\left(-\nabla_a\nabla_b h^c{}_c - \nabla_a\nabla_c h^c{}_b + \nabla_a\nabla^c h_{bc}\right) + \dfrac{1}{2}\left(\nabla_c\nabla_a h^c{}_b + \nabla_c\nabla\right.\right.\right.$

Out[53]= $-R[\nabla]_{ab} + \dfrac{1}{2}g_{ab}\,R[\nabla]$

In[35]:=
```
L = RicciScalarCD[]
VarAction[L, g[a, b]]
```

Out[35]= $R[\nabla]$

Out[36]= $R[\nabla]_{ab} - \dfrac{1}{2}g_{ab}\,R[\nabla]$

Variation of Actions

```
In[59]:= L = f[RicciScalarCD[]]
         VarAction[L, g[a, b]]
```

Out[59]= $f[R[\nabla]]$

Out[60]= $-\frac{1}{2} f[R[\nabla]]\, g_{ab} + R[\nabla]_{ab}\, f'[R[\nabla]] - \nabla_b \nabla_a R[\nabla]\, f''[R[\nabla]] + g_{ab}\, \nabla_c \nabla^c R[\nabla]\, f''[R[\nabla]] - \nabla_a R[\nabla]\, \nabla_b R[\nabla]\, f^{(3)}[R$

```
In[61]:= L = RiemannCD[a, b, c, d] RiemannCD[-a, -b, -c, -d]
         VarAction[L, g[a, b]]
```

Out[61]= $R[\nabla]_{abcd}\, R[\nabla]^{abcd}$

Out[62]= $2\, R[\nabla]_a{}^{cde}\, R[\nabla]_{bcde} - \frac{1}{2} g_{ab}\, R[\nabla]_{cdei}\, R[\nabla]^{cdei} + 2\, \nabla_c \nabla_d R[\nabla]_a{}^c{}_b{}^d + 2\, \nabla_d \nabla_c R[\nabla]_a{}^c{}_b{}^d$

```
In[63]:= L = F[a, b] F[-a, -b] / 4 + RicciScalarCD[]
         VarAction[L, g[a, b]]
         VarAction[L, A[a]]
```

Out[63]= $R[\nabla] + \frac{1}{4} \left(\nabla_a A_b - \nabla_b A_a\right) \left(\nabla^a A^b - \nabla^b A^a\right)$

Out[64]= $R[\nabla]_{ab} - \frac{1}{2} g_{ab}\, R[\nabla] + \frac{1}{2} \nabla_a A^c\, \nabla_b A_c - \frac{1}{2} \nabla_b A_c\, \nabla^c A_a + \frac{1}{2} \nabla_c A_b\, \nabla^c A_a - \frac{1}{2} \nabla_a A_c\, \nabla^c A_b + \frac{1}{4} g_{ab}\, \nabla_c A_d\, \nabla^d A^c - \frac{1}{4} g_{ab}\, \nabla_d A_c$

Out[65]= $\nabla_b \nabla_a A^b - \nabla_b \nabla^b A_a$

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Cosmological Perturbations.

When cosmologists write the simplest perturbed metric, they write

$$\mathrm{d}s^2 = a(\eta)^2[-(1+2\Phi)\mathrm{d}\eta^2) + (1-2\Psi)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j]$$

What they mean

$$\phi^*[g_{\mu\nu}] - \bar{g}_{\mu\nu} = a^2[-2\Phi\mathrm{d}\eta\otimes\mathrm{d}\eta - 2\Psi\mathrm{d}x^i\otimes\mathrm{d}x^j]$$

Roadmap :

-Find the general expression for a perturbed tensor
(perturbation of Einstein tensor typically)

-Perform a conformal transformation to account for the scale factor

-Restrict to a given order (first order for simplicity)

-Replace the general first order metric perturbation by its parameterization ?

-Read the result ?

Conformal transformation

In[7]:= `Conformal[g, ga2][RiemannCD[-α, -β, -μ, ν]]`

`** DefTensor: Defining tensor ChristoffelCDCDa2[α, -β, -λ].`

Out[7]=

$$R[\nabla]_{\alpha\beta\mu}{}^{\nu} - \frac{\delta_\beta{}^\nu\, g_{\alpha\mu}\, (\nabla_\lambda a)\, (\nabla^\lambda a)}{(a)^2} + \frac{\delta_\alpha{}^\nu\, g_{\beta\mu}\, (\nabla_\lambda a)\, (\nabla^\lambda a)}{(a)^2} + \frac{2\, \delta_\beta{}^\nu\, (\nabla_\alpha a)\, (\nabla_\mu a)}{(a)^2} - \frac{2\, \delta_\alpha{}^\nu\, (\nabla_\beta a)\, (\nabla_\mu a)}{(a)^2} - \frac{\delta_\beta{}^\nu\, (\nabla_\mu \nabla_\alpha a)}{(a)} + \frac{\delta_\alpha{}^\nu\, (\nabla_\mu \nabla_\beta a)}{(a)} - \frac{2\, g_{\beta\mu}\, (\nabla_\alpha a)\, (\nabla^\nu a)}{(a)^2} + \frac{2\, g_{\alpha\mu}\, (\nabla_\beta a)\, (\nabla^\nu a)}{(a)^2} + \frac{g_{\beta\mu}\, (\nabla^\nu \nabla_\alpha a)}{(a)} - \frac{g_{\alpha\mu}\, (\nabla^\nu \nabla_\beta a)}{(a)}$$

Roadmap

-Find the general expression for a perturbed tensor
(perturbation of Einstein tensor typically)

-Perform a conformal transformation to account for the scale factor

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-Read the result ?

Parameterisation of the perturbed metric

So when the parameterisation of the perturbed metric is

$$a^2[-2\Phi \mathrm{d}\eta \otimes \mathrm{d}\eta - 2\Psi \gamma_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j]$$

we replace the first order perturbed metric by

$$^{(1)}g_{\mu\nu} = -2\Phi \bar{n}_\mu \bar{n}_\nu - 2\Psi \bar{h}_{\mu\nu}$$

Roadmap

-Find the general expression for a perturbed tensor
(perturbation of Einstein tensor typically)

-Perform a conformal transformation to account for the scale factor

-Restrict to a given order (first order for simplicity)

-Replace the general first order metric perturbation by its parameterization ?

-Read the result ?

Reading the result: 1+3 splitting of background

See Eric Gourgoulhon's review on 1+3/ADM

The background cosmological solution (Friedmann-Lemaître) :

-The expansion is contained entirely in the scale factor $\tilde{\bar{g}}_{\mu\nu} = a^2 \bar{g}_{\mu\nu}$

-A class of free falling observers define the cosmic time $\bar{n}^\mu$

-The spatial sections have (conformal) metric $\bar{h}_{\mu\nu} = \bar{g}_{\mu\nu} + \bar{n}_\mu \bar{n}_\nu$

-The spatial sections are homogeneous (invariant under translation)

This means that we have a natural 1+3 slicing of the background manifold

-The scale factor contains all the *extrinsic curvature* of the slicing. $\bar{\nabla}_\mu \bar{n}_\nu = 0$

-There is no acceleration of the vector normal to the slices $\bar{n}^\mu \bar{\nabla}_\mu \bar{n}_\nu = 0$

-The curvature of the spatial section is the curvature of $\bar{h}_{\mu\nu}$

Gauss-Codacci relates ${}^4\bar{R}_{\alpha\beta\mu\nu}$ to ${}^3\bar{R}_{\alpha\beta\mu\nu}$

$${}^3\bar{R}_{\mu\nu\rho\sigma} = 2k\,\bar{h}_{\rho[\mu}\,\bar{h}_{\nu]\sigma}\,, \qquad {}^3\bar{R}_{\mu\nu} = 2k\,\bar{h}_{\mu\nu}\,, \qquad {}^3\bar{R} = 6k$$

$$\tilde{\bar{g}}_{\mu\nu} = a^2 \bar{g}_{\mu\nu} \qquad \bar{g}_{\mu\nu} = -\bar{n}_\mu \otimes \bar{n}_\nu + \bar{h}_{\mu\nu} \qquad {}^3\bar{R}_{\alpha\beta\mu\nu}$$

Splitting perturbation equation

The equations obtained by hand (Einstein equations) are typically of the form

$$\Phi'' + \frac{a'}{a}\Phi' + \partial_i\partial^i\Psi = 0$$

What is the geometrical meaning of

- Time derivative ? Answer : $\mathcal{L}_{\bar{n}}\Phi = \Phi'$
- Partial derivative ? Answer : an induced covariant derivative

$$D_\mu\Phi \equiv h^\nu_\mu\nabla_\nu\Phi$$

$$\boxed{\nabla_\mu\Phi = -\bar{n}^\mu\mathcal{L}_{\bar{n}}\Phi + D_\mu\Phi}$$

$$\mathcal{L}^2_{\bar{n}}\Phi + \frac{a'}{a}\mathcal{L}_{\bar{n}}\Phi + D_\mu D^\mu\Phi = 0$$

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Examples of implementation in xPand

http://www2.iap.fr/users/pitrou/xpand.htm

```
In[1]:= << xAct/xPand.m;

------------------------------------------------------------

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Connecting to external mac executable...

Connection established
```

```
In[2]:= DefManifold[M, 4, {α, β, μ, ν}];
DefMetric[-1, g[-α, -β], CD, {";", "∇"}];

** DefManifold: Defining manifold M.

** DefVBundle: Defining vbundle TangentM.

** DefTensor: Defining symmetric metric tensor g[-α, -β].

** DefTensor: Defining antisymmetric tensor epsilong[-α, -β, -μ, -ν].

** DefTensor: Defining tetrametric Tetrag[-α, -β, -μ, -ν].
```

```
In[4]:= DefMetricPerturbation[g, dg, ε];

** DefParameter: Defining parameter ε.

** DefTensor: Defining tensor dg[LI[order], -α, -β].
```

```
In[5]:= SetSlicing[g, n, h, cd, {"|", "D"}, "FLCurved"];

** DefTensor: Defining tensor n[ν$1095].

** DefTensor: Defining symmetric metric tensor h[-ν$1095, -ν$1096].

** DefTensor: Defining antisymmetric tensor epsilonh[-α, -β, -μ].

** DefTensor: Defining tetrametric Tetrah[-α, -β, -μ, -ν].
```

Examples of Implementation in xPand

```
In[6]:= order = 1;
        DefProjectedTensor[φ[], h];
        DefProjectedTensor[ψ[], h];

        ** DefTensor: Defining tensor
         φ[LI[xAct`xPand`Private`p$3488], LI[xAct`xPand`Private`q$3488]].

        ** DefTensor: Defining tensor
         ψ[LI[xAct`xPand`Private`p$3493], LI[xAct`xPand`Private`q$3493]].

MyRicciScalar =
 ExpandPerturbation@Perturbed[Conformal[g, gah2][RicciScalarCD[]], order]

** DefTensor: Defining tensor ChristoffelCDCDah2[α, -β, -μ].
```

$$\frac{R[\nabla]}{(a)^2} - \frac{6\,\nabla_\alpha \nabla^\alpha a}{(a)^3} + \epsilon \left(-\frac{6\left(-dg^{1\alpha\beta}\,\nabla_\alpha\nabla_\beta a - \frac{1}{2}\,g^{\alpha\beta}\,\nabla_\mu a\left(\nabla_\alpha dg^{1\mu}{}_\beta + \nabla_\beta dg^{1\mu}{}_\alpha - \nabla^\mu dg^1{}_{\alpha\beta}\right)\right)}{(a)^3} + \frac{1}{(a)^2} \left(-dg^{1\nu\nu1}\,R[\nabla]_{\nu\nu1} + g^{\nu\nu1}\left(\frac{1}{2}\left(-\nabla_\nu\nabla_{\nu1} dg^{1\nu2}{}_{\nu2} - \nabla_\nu\nabla_{\nu2} dg^{1\nu2}{}_{\nu1} + \nabla_\nu\nabla^{\nu2} dg^1{}_{\nu1\nu2}\right) + \frac{1}{2}\left(\nabla_{\nu2}\nabla_\nu dg^{1\nu2}{}_{\nu1} + \nabla_{\nu2}\nabla_{\nu1} dg^{1\nu2}{}_\nu - \nabla_{\nu2}\nabla^{\nu2} dg^1{}_{\nu1\nu}\right)\right)\right)\right)$$

Examples of Implementation in xPand

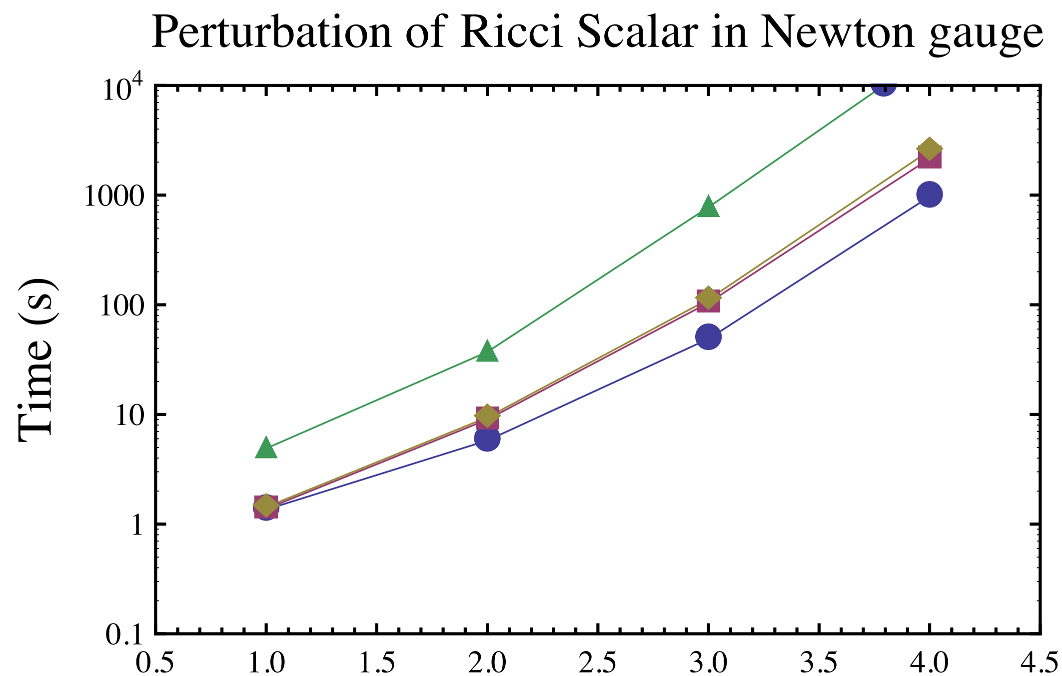
```
In[10]:= MyGauge =
   {dg[LI[order], α_, β_] :> -2 n[α] n[β] φ[LI[order]] - 2 h[α, β] ψ[LI[order]]};

In[12]:= SplitPerturbations[ah[]^2 MyRicciScalar, MyGauge, h]

The Splitting of a² +... was performed in 0.721687 seconds.
```

Out[12]= $6\,\mathcal{H}^2 + 6\,\mathcal{H}' + 6\,k + \varepsilon\left(-12\,\mathcal{H}^2\left({}^{(1)}\varphi\right) - 12\,\mathcal{H}'\left({}^{(1)}\varphi\right) - 6\,\mathcal{H}\left({}^{(1)}\varphi'\right) + 12\,k\left({}^{(1)}\psi\right) - 18\,\mathcal{H}\left({}^{(1)}\psi'\right) - 6\left({}^{(1)}\psi''\right) - 2\left(D_\alpha D^\alpha\,{}^{(1)}\varphi\right) + 4\left(D_\alpha D^\alpha\,{}^{(1)}\psi\right)\right)$

Benchmarking, and press review



If that hasn't impressed you, then you won't be happy until you own a personal army of elves that write your papers for you.

Jolyon Bloomfield, MIT (xAct tutorial)

Extensions

-We can have non scalar perturbations

-Can be extended to anisotropic spacetimes.

-Perturbation of fluids possible
(velocity constrained by normalization)

$$-3H^2 - 3\dot{H} + \epsilon\left(3H\left({}^{(1)}\dot{\phi}\right) + 6H\left({}^{(1)}\dot{\psi}\right) + 3\left({}^{(1)}\ddot{\psi}\right) + \frac{D_d D^d\,{}^{(1)}\phi}{a^2}\right) +$$

$$\epsilon^2\Bigg(\left({}^{(1)}\dot{E}_{de}\right)\left({}^{(1)}\dot{E}^{de}\right) + 2\left({}^{(1)}E^{de}\right)\left({}^{(1)}\ddot{E}_{de}\right) +$$

$$4\left({}^{(1)}E^{de}\right)\left({}^{(1)}\dot{E}_{de}\right)H - 6H\left({}^{(1)}\phi\right)\left({}^{(1)}\dot{\phi}\right) - 3\left({}^{(1)}\dot{\phi}\right)\left({}^{(1)}\dot{\psi}\right) +$$

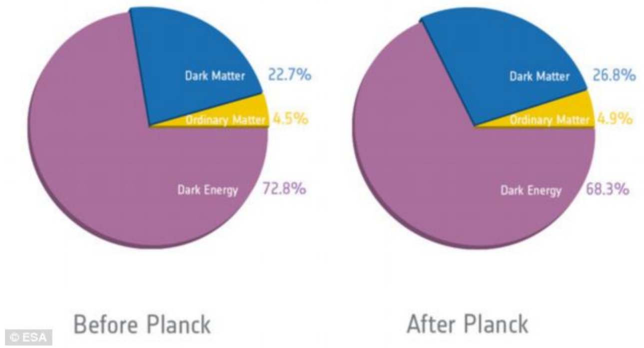
$$12H\left({}^{(1)}\psi\right)\left({}^{(1)}\dot{\psi}\right) + 3\left({}^{(1)}\dot{\psi}\right)^2 + 6\left({}^{(1)}\psi\right)\left({}^{(1)}\ddot{\psi}\right) +$$

$$\frac{2\left({}^{(1)}\psi\right)\left(D_d D^d\,{}^{(1)}\phi\right)}{a^2} - \frac{\left(D_d\,{}^{(1)}\phi\right)\left(D^d\,{}^{(1)}\phi\right)}{a^2} -$$

Conclusion

-Tensor algebra is helpful whenever your brain saturates

-Action variation is piece of cake. Nobody should ever sweat on it…



-General method for cosmological perturbations

Can be implemented in any package for abstract tensor manipulations
Not restricted to GR

Extension needed: null geodesics
(geodesic equation, geodesic deviation equation) 1+3 -> 1+1+2

Thanks a lot for your attention

Blank Slide

Thanks a lot for your attention

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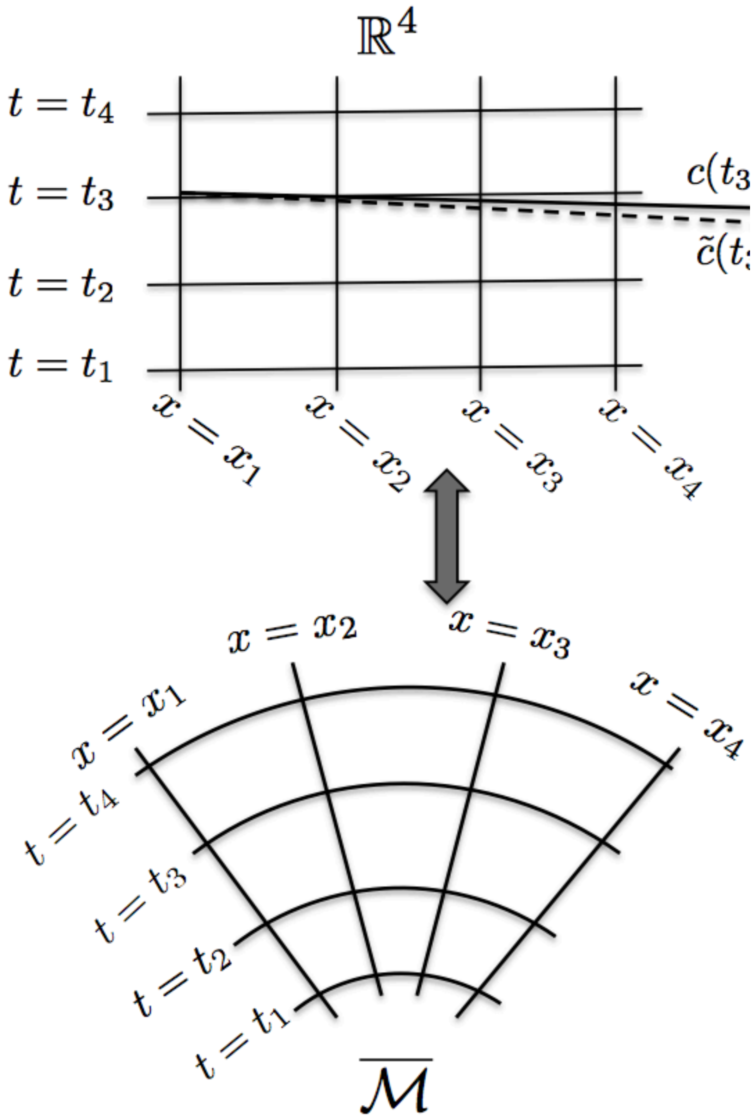
1+3 Background slicing made visual



More subtle details if there is time

-We have non scalar perturbations $B_\mu, E_\mu \qquad H_{\mu\nu}$

$$\bar{\nabla}_\rho T_{\mu_1\ldots\mu_p} = -\bar{n}_\rho \mathcal{L}_{\bar{n}} T_{\mu_1\ldots\mu_p} + \bar{D}_\rho T_{\mu_1\ldots\mu_p} + 2\sum_{i=1}^{p} \bar{n}_{(\mu_i} \bar{K}^\sigma{}_{\rho)} T_{\mu_1\ldots\mu_{i-1}\sigma\mu_{i+1}\ldots\mu_p} .$$

-General equations are second order. We need to commute induced derivatives and spatial derivatives (time and space derivative).

$$\mathcal{L}_{\bar{n}}\left(\bar{D}_\rho T_{\mu_1\ldots\mu_p}\right) = \bar{D}_\rho\left(\mathcal{L}_{\bar{n}} T_{\mu_1\ldots\mu_p}\right) + \sum_{i=1}^{p}\left(\bar{h}^{\sigma\zeta}\bar{D}_\zeta \bar{K}_{\rho\mu_i} - \bar{D}_\rho \bar{K}_{\mu_i}{}^\sigma - \bar{D}_{\mu_i}\bar{K}_\rho{}^\sigma\right) T_{\mu_1\ldots\mu_{i-1}\sigma\mu_{i+1}\ldots\mu_p} ,$$

-Can be extended to anisotropic spacetimes.
  Extrinsic curvature non-vanishing (symmetric trace-free part) $\bar{\nabla}_\mu \bar{n}_\nu = \sigma_{\mu\nu}$
  Meaning of homogeneity from Killing vector fields

-Perturbation of fluids require to ensure the norm of relativistic velocity is -1. Only three degrees of freedom.