Precision Cosmology and Detectability of Quantum Gravitational Effects in the Early Universe

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Based on:

- "*Scalar and tensor perturbations in loop quantum cosmology: High-order corrections*," JCAP (2015) [arXiv:1508.03239]
- "*Detecting quantum gravitational effects of loop quantum cosmology in the early universe*?" ApJL 807 (2015) L17 [arXiv:1503.06761]
- "*Power spectra and spectral indices of* k*-inflation: high-order corrections*," PRD90 (2014) 103517 [arXiv:1407.8011]
- "*Quantum effects on power spectra and spectral indices with higher-order corrections*," PRD90 (2014) 063503 [arXiv:1405.5301]
- "*Inflationary cosmology with nonlinear dispersion relations*," PRD89 (2014) 043507 [arXiv:1308.5708]
- "*Constructing analytical solutions of linear perturbations of inflation with modified dispersion relations*," IJMPA29 (2014) 1450142 [arXiv:1308.1104]

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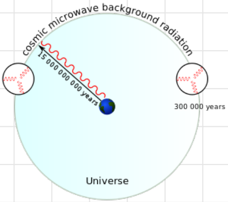
1.1 Why inflation?

(A. Guth, PRD23 (1981) 347)

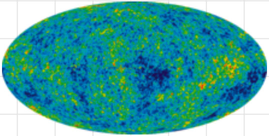


- It solves the problems of standard big bang cosmology: flatness, horizon, cosmic relics, etc.
- It provides a causal mechanism for generations of adiabatic, Gaussian, and nearly scale-invariant primordial fluctuations, which leads to
  - the formation of large scale structure of the universe;
  - the Cosmic Microwave Background (CMB) anisotropies.
- Inflation is remarkably successful and its predictions are matched to observations with astonishing precision.[1]

[1] Planck Collaboration, arXiv:1507.02704.

1.2 Quantum Fluctuations During Inflation

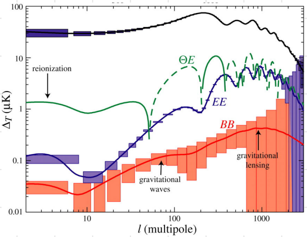
(D. Baumann, arXiv:0907.5424)



- According to the inflation paradigm, the large-scale structure of our universe and CMB all originated from *quantum fluctuations produced during Inflation, which can be decomposed into:*

  scalar, vector and tensor.

- But, because of the expansion of the universe and particular nature of the fluctuations, vector perturbations did not grow, and observationally can be safely ignored.



1.2 Quantum Fluctuations During Inflation (Cont.)

- Scalar and tensor perturbations are described by mode functions $\mu_k(\eta)$,

$$\mu_k'' + \left(\omega_k^2 - \frac{z''}{z}\right)\mu_k = 0, \quad z \equiv \begin{cases} \frac{a\phi'}{\mathcal{H}}, & \text{scalar} \\ a, & \text{tensor} \end{cases} \qquad (1)$$

  - ω_k^2: energy of the mode, and in general relativity (GR) is given by,

$$\omega_k^2 = k^2$$

  - k: comoving wavenumber
  - ϕ: the scalar field — the inflaton; and $\phi' \equiv d\phi/d\eta$
  - η: the conform time, $d\eta \equiv dt/a(t)$
  - $a(\eta)$: the expansion factor of the universe; and $\mathcal{H} \equiv a'/a$

1.2 Quantum Fluctuations During Inflation (Cont.)

- Power spectra, $\Delta_{\rm i}^2$, are defined as,

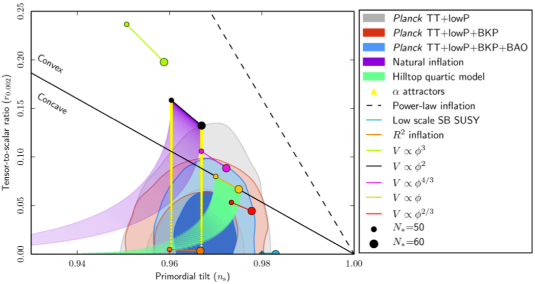
$$\Delta_{\rm i}^2 \equiv \frac{\rm k^3}{2\pi^2}\left|\frac{\mu_{\rm k}}{\rm z}\right|_{\rm i}^2, \ ({\rm i}={\rm S},{\rm T}).$$

- Spectral indexes are defined as

$$\rm n_S \equiv 1 + \frac{d\ln\Delta_S^2(k)}{d\ln k}, \quad n_T \equiv \frac{d\ln\Delta_T^2(k)}{d\ln k}.$$

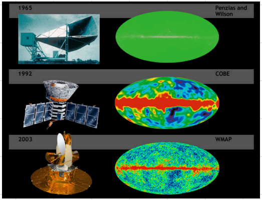
- The ratio $\rm r$ is defined as,

$$\rm r \equiv \frac{\Delta_T^2}{\Delta_S^2}.$$

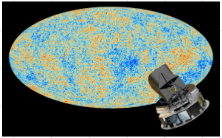


1.3 Precision Cosmological Measurments

- Since the first measurement of CMB in 1964 by Penzias and Wilson (PW), there have been a variety of experiments to measure its radiation anisotropies and polarization, such as WMAP, PLanck and BICEP2, with ever increasing precision.



PW, COBE, WMAP



Planck

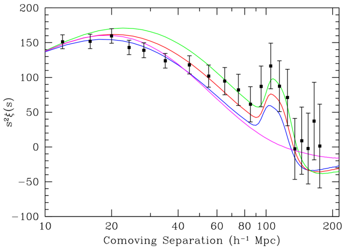
BICEP2

1.3 Precision Cosmological Measurments (cont.)

- In the coming decade, we anticipate that various new surveys will make even more accurate CMB measurements:
  - Balloon experiments: Balloon-borne Radiometers for Sky Polarisation Observations (BaR-SPoRT); The E and B Experiment (EBEX); ...
  - Ground experiments: Cosmology Large Angular Scale Surveyor (CLASS); Millimeter-Wave Bolometric Interferometer (MBI-B); Qubic; ...
  - Space experiments: Sky Polarization Observatory (SPOrt); ...

1.3 Precision Cosmological Measurments (cont.)

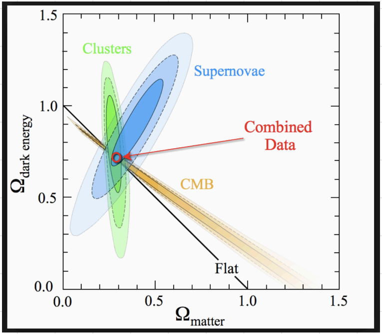
- In addition to CMB measurements, Large-scale structure surveys, measuring the galaxy power spectrum and the position of the baryon acoustic peak, have provided independently valuable information on the evolution of the universe.
- The first measurement of the kind started with the baryon acoustic oscillation (BAO) in the SDSS LRG and 2dF Galaxy surveys [2].



[2] D. J. Eisenstein, et al., ApJ 633 (2005) 560; S. Cole, et al., MNRAS362 (2005) 505.

1.3 Precision Cosmological Measurments (cont.)

- Since then, various large-scale structure surveys have been carried out [3], and provided sharp constraint on the budgets that made of the universe.



[3] Tegmark, M., et al. 2006, Phys. Rev. D, 74, 123507; Kazin, E. A., et al. 2010, ApJ, 710, 1444; Blake, C., Kazin, E., Beutler, F., et al. 2011, MNRAS, 418, 1707.

1.3 Precision Cosmological Measurments (cont.)

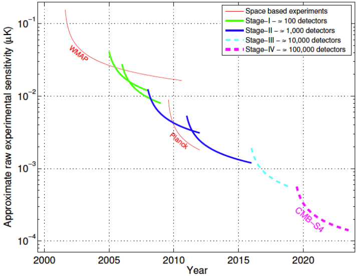
- Various new surveys will make even more accurate measurements of the galaxy power spectrum:
  - Ground-Based: the Prime Focus Spectrograph, Big BOSS, ...
  - Space-based: Euclid, WFIRST, ...
- Cosmology indeed enters its golden age:

Precision Cosmology!

1.3 Precision Cosmological Measurments (cont.)

- In particular, the Stage IV experiments [4] will measure the physical variables

$$\sigma(n_s, r) = 10^{-3} \sim 10^{-4}$$



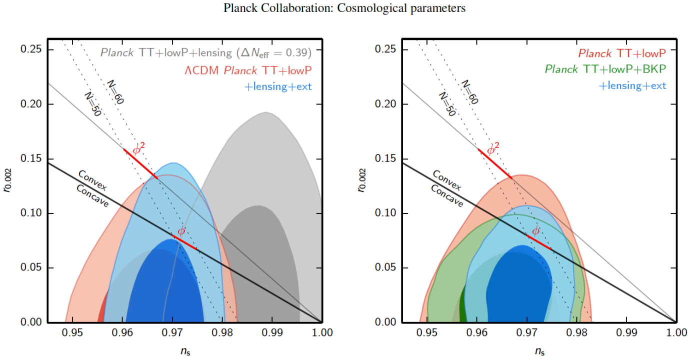
  - n_s: the spectral index of scalar perturbations
  - r: the ratio between tensor and scalar perturbations
- With this level of uncertainty, the Stage IV experiments will make a clear detection ($> 5\sigma$) of tensor modes from any inflationary model with $r \geq 0.01$.

[4] K. N. Abazajian et al., Astropart. Phys. 63, 55 (2015) [arXiv:1309.5381].

1.3 Precision Cosmological Measurements (cont.)

- Note that current measurements from Planck 2015 (Stage II) [Planck Collaboration, arXiv:1502.02114] are,

$$n_s = 0.968 \pm 0.006, \quad r_{0.002} < 0.11 \ (95\ \%\ \mathrm{CL})$$



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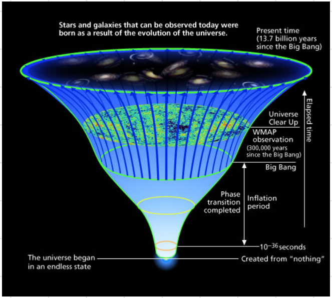
2.1 Trans-Planckian Problem

(Brandenberger and Martin, CGQ30 (2013) 113001)

But, inflation is very sensitive to Planck-scale physics, and its origin has still remained elusive.



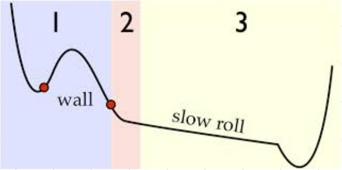
- In particular, during inflation the wavelengths, related to present observations, were exponentially stretched.
- To be consistent with observations, the inflationary period needs to be lasting long enough. If it is more than 70 e-folds, the wavelengths corresponding to present observations, should be smaller than the Planck length at the beginning of the inflation, and quantum gravitational effects become important — the trans-Planckian problem.

2.2 The η- problem:

- For a single inflaton field ϕ, its potential $V(\phi)$ must be very flat,

$$\epsilon_V \equiv M_{pl}^2 \left(\frac{V_{,\phi}}{V}\right)^2 \ll 1,$$

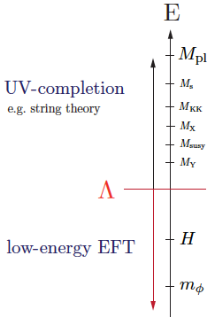
$$\eta_V \equiv \frac{M_{pl}^2}{2} \left(\frac{V_{,\phi\phi}}{V}\right) \ll 1,$$



in order for the universe to expand large enough to solve the problems of big bang model, mentioned above.

2.2 The η- problem:

- In effective field theory (EFT), the cutoff energy scale Λ is defined by the lightest particle that is not included in the spectrum of the low-energy theory. Integrating out particles of mass $\mathtt{M} \geq \Lambda$ gives rise to the operators of the form,



$$\frac{\mathcal{O}_\delta}{\Lambda^{\delta-4}},$$

δ: the mass dimension of the operator

$\Lambda \simeq \mathtt{M}_{\mathtt{pl}}$

2.2 The η- problem (Cont.):

- In inflation, the flatness of the potential in Planck units introduces sensitivity to $\delta \leq 6$ Planck-suppressed operators.

- In particular, in the absence of any specific symmetries to protect the inflation potential, contributions to the Lagrangian of the general form

$$\frac{\langle \mathcal{O}_6 \rangle}{\mathrm{M}_{\mathrm{pl}}^2} = \frac{\langle \mathcal{O}_4 \rangle}{\mathrm{M}_{\mathrm{pl}}^2} \phi^2,$$

are allowed.

2.2 The η- problem (Cont.):

- In a generic effective theory, with cutoff Λ, the mass of a scalar field runs to the cutoff scale unless it is protected by some symmetry. Since $\Lambda \geq \mathrm{H}$, this implies that a small inflaton mass $\mathrm{m}_\phi \ll \mathrm{H}$ is radiatively unstable. Equivalently, the η parameter receives radiative corrections,

$$\Delta\eta_{\mathrm{V}} \equiv \frac{\Delta \mathrm{m}_\phi^2}{\mathrm{H}^2} \simeq \mathcal{O}(1),$$

which prevents a long-time exponential expansion — the η problem. It should be addressed only when the Planck physics is invoked.

2.2 The η- problem (Cont.):

- The difficulty here is analogous to the Higgs hierarchy problem, but supersymmetry does not suffice to stabilize the inflation mass, as the inflationary energy necessarily breaks supersymmetry, and the resulting splittings in supermultiplets are of order H, so that supersymmetry does not protect a small inflation mass $\mathrm{m}_\phi \ll \mathrm{H}$.

- One possibility is to protect the inflation potential via a shift symmetry [5],

$$\phi \to \phi + \phi_0,$$

ϕ_0: a constant.

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[5] K. Freese, J. A. Frieman, and A. V. Olinto, Phys. Rev. Lett. 65, 3233 (1990).

2.3 Initial Conditions:

- Many inflationary scenarios only work if the fields are initially very homogeneous and/or start with precise initial positions and velocities. Any physical understanding of this “fine-tuning” requires a more complete formulation with ever-higher energies, such as string theory.
- ...

Therefore: Inflation is very sensitive to Planck-scale physics, and quantum gravitational effects in the early universe are important [6].

---

[6] D. Baumann, TASI Lectures on Inflation, arXiv:0907.5424; C.P. Burgess, M. Cicoli, F. Quevedo, JCAP 1311 (2013) 003.

2.4 Experimental Tests for Quantum Gravity:

- On the other hand, quantization of gravity has been one of the main driving forces in physics in the past decades, and various approaches have been pursued, including String/M-Theory, loop quantum gravity, Horava-Lifshitz theory.

- However, it is fair to say that our understanding of them is still highly limited, and none of the aforementioned approaches is complete.

- One of the main reasons is the lack of experimental guides, due to the extreme weakness of gravitational fields.

2.4 Experimental Tests for Quantum Gravity (Cont.):

- This situation has been dramatically changed recently, with the arrival of the era of precision cosmology [7].
- One of our goals is to develop an ACCURATE and EFFECTIVE mathematical tool to study quantum gravitational effects in the early universe, whereby look for possible observational tests of quantum gravity.
- Although it is very ambitious, I am going to argue that this may become possible in the forthcoming cosmological observations with the rapid development of high technology.

---

[7] C. Kiefer, and M. Kramer, Phys. Rev. Lett. 108, 021301 (2012); L.M. Krauss and F. Wilczek, Phys. Rev. D89, 047501 (2014); R.P. Woodard, arXiv:1407.4748.

2.5 Quantum Gravitational Effects

- **String/M-Theory:** As the most promising candidate for a UV-completion of the Standard Model that unifies gauge and gravitational interaction in a consistent quantum theory, String/M theory can provide possibilities for an explicit realization of the inflationary scenario.
- String/M theory usually leads to a non-trivial time-dependent speed of sound for primordial perturbations [8],

$$\omega_k^2 = c_s^2(\eta) k^2, \tag{2}$$

  - $c_s^2(\eta)$: the speed of sound, and could be very close to zero in the far UV regime.

[8] L. McAllister and E. Silverstein, Gen. Rel. Grav. 40, 565 (2007); C. P. Burgess, M. Cicoli, and F. Quevedo, arXiv: 1306.3512.

2.5 Quantum Gravitational Effects (Cont.)

- Loop quantum cosmology (LQC): Offers a natural framework to address the trans-Planckian issue and initial singularity.
- In particular, because effects of its underlying quantum geometry dominates at the Planck scale, leading to singularity resolution in a variety of cosmological models, where the initial singularity is replaced by the big bounce.
- There are mainly two kinds of quantum corrections to the cosmological background and perturbations [9]: (a) holonomy, and (b) inverse-volume

[9] A. Barrau and J. Grain, arXiv:1410.1714; M. Bojowald, Rep. Prog. Phys. 78 (2015) 023901; A. Ashtekar and A. Barrau, arXiv:1504.07559.

2.5 Quantum Gravitational Effects (Cont.)

- Due to the holonomy corrections, the dispersion relation in the mode functions is modified to

$$\omega_{\mathrm{k}}^2 = \left(1 - 2\frac{\rho(\eta)}{\rho_{\mathrm{c}}}\right)\mathrm{k}^2 \tag{3}$$

  - ρ_{c}: the energy density at which the big bounce happens.

- Due to the inverse-volume corrections, the dispersion relation in the mode functions is modified to

$$\omega_{\mathrm{k}}^2 = \mathrm{k}^2 \times \begin{cases} 1 + \left[\frac{\sigma\nu_0}{3}\left(\frac{\sigma}{6}+1\right) + \frac{\alpha_0}{2}\left(5 - \frac{\sigma}{3}\right)\right]\delta_{\mathrm{PL}}(\eta), & \texttt{scalar} \\ 1 + 2\alpha_0\delta_{\mathrm{PL}}, & \texttt{tensor} \end{cases} \tag{4}$$

  - $\alpha_0,\ \nu_0,\ \sigma$: encode the specific features of the model
  - $\delta_{\mathrm{PL}}(\eta)$: time-dependent, given by $\delta_{\mathrm{PL}} = (\mathrm{a_{PL}}/\mathrm{a})^{\sigma} < 1,$ with $\sigma > 0$.

2.5 Quantum Gravitational Effects (Cont.)

- To quantize gravity using quantum field theory, in 2009 Horava proposed a theory - Horava-Lifshitz (HL) gravity [10], which is power-counting renormalizable, and has attracted lot of attention since then.
- In this theory, the dispersion relation is modified to [AW and R. Maartens, PRD81 (2010) 024009; AW, PRD82 (2010) 124063],

$$\omega_k^2(\eta) = k^2 - b_1 \frac{k^4}{a^2 M_*^2} + b_2 \frac{k^6}{a^4 M_*^4} \tag{5}$$

  - M_*: the energy scale of the HL gravity
  - b_1, b_2: depend on the coupling constants of the HL theory and the type of perturbations, scalar or tensor.

[10] P. Horava, PRD79 (2009) 084008.

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3.1 Modified Equation of Mode Function

- Taking the quantum effects into account, either from string/ M-Theory, or loop Quantum Cosmology, or HL gravity, or any of other theories, the equation of motion for the mode function μ_k can be cast in the general form,

$$\frac{d^2\mu_k(y)}{dy^2} = \left[g(y) + q(y)\right]\mu_k(y), \tag{6}$$

  $g(y),\ q(y)$: functions of $y[\equiv -k\eta]$, to be determined by minimizing the errors.

- For example, in the HL gravity, we have

$$g(y) + q(y) \quad = \quad \frac{\nu^2 - 1/4}{y^2} - 1 + b_1\epsilon_*^2 y^2 - b_2\epsilon_*^4 y^4,$$

  with $\epsilon_* \equiv H/M_*, z''/z \equiv (\nu^2(\eta) - 1/4)/\eta^2$.

3.1 Modified Equation of μ_k (Cont.)

- In the following, I am going to present *the uniform asymptotic approximation method,* first introduced into GR by Habib et al [11], and recently developed systematically by us, in order to study quantum gravitational effects. It has the following advantages:
  - The error bounds are given explicitly. By properly choosing the functions $g(y)$ and $q(y)$, we can minimize the errors, so that analytical solutions with high accuracy can be constructed to high orders.
  - Up to the third-order approximations in terms of the parameter λ, introduced in the method, the errors are

$$\delta \lesssim 0.15\%$$

[11] S. Habib, K. Heitmann, G. Jungman, and C. Molina-Paris, PRL89 (2002) 281301.

3.2 Liouville Transformations

- The strategy is, following Olvier [12], to use the well-established Liouville transformations to introduce
  - a new variable ξ, instead of y,
  - a new function U, instead of μ_k,

$$y \rightarrow \xi,$$
$$\mu_k(y) \rightarrow U(\xi),$$

- so that the resulted equation can be solved:
  - analytically order by order in terms of $\epsilon \equiv 1/\lambda \ll 1$
  - the corresponding error bounds are well under control at each step, so the errors can be minimized

[12] F. W. J. Olver, Asymptotics and Special functions, (AKP Classics, Wellesley, MA 1997).

3.2 Liouville Transformations (Cont.)

- The Liouville Transformations are

$$
\begin{aligned}
U(\xi) &= \chi^{1/4}\mu_k(y), \qquad \chi \equiv \xi'^2 = \frac{|g(y)|}{f^{(1)}(\xi)^2}, \\
f(\xi) &= \int^{y} \sqrt{|g(y)|}\,dy, \qquad f^{(1)}(\xi) = \frac{df(\xi)}{d\xi}, \qquad (7)
\end{aligned}
$$

- χ must be regular and not vanish in the intervals of interest
- $g(y)$ & $q(y)$ are properly chosen to minimize the errors
- $f^{(1)}(\xi)$ must be chosen so that it has zeros and singularities of the same type as that of $g(y)$

3.2 Liouville Transformations (Cont.)

- The equation of motion for the mode function reduces to,

$$\frac{d^2U(\xi)}{d\xi^2} = \left[\pm f^{(1)}(\xi)^2 + \psi(\xi)\right] U(\xi), \qquad (8)$$

with

$$\psi(\xi) = \frac{q(y)}{\chi} - \chi^{-3/4}\frac{d^2(\chi^{-1/4})}{dy^2}, \qquad (9)$$

in the above "$+$" for $g(y) > 0$, and "$-$" for $g(y) < 0$.

- Neglecting $\psi(\xi)$ we obtain solutions to the first-order approximation
- Choosing properly $f^{(1)}(\xi)$ in order for the equation to be:
  (a) solved analytically, and
  (b) minimizing the error bounds.

3.2 Liouville Transformations (Cont.)

- To solve the differential equation for the new function $U(\xi)$ analytically, we divide the task into three steps:
  - Solve it near some singular points,

$$g(y) + q(y) = \pm\infty,$$

    often called poles.
  - Solve it near the turning points,

$$g(y) = 0.$$

  - Then, matching all the solutions together with the initial conditions, which are normally taken as the Bunch-Davies (adiabatic) vacuum.

3.3 Solutions near poles

- To illustrate our method, in the following we shall restrict ourselves to the case,

$$\begin{aligned} g(y) + q(y) &\equiv \left(\frac{z''}{z} - \omega_k^2\right) k^{-2} \\ &= \frac{\nu^2 - 1/4}{y^2} - 1 + b_1 \epsilon_*^2 y^2 - b_2 \epsilon_*^4 y^4 \end{aligned}$$

although our method can be used for any given dispersion relation ω_k^2 and background z''/z.

- Eq.(10) has two poles (singularities), $y = 0, \infty$, and three turning points (roots of $g(y) = 0$).

3.4 Solutions near turning points

- Error Control Function and Choice of $g(y)$ and $q(y)$:
  The convergence of the error control function $F(y)$ requires that one must choose [Zhu et al, PRD89 (2014) 043507],

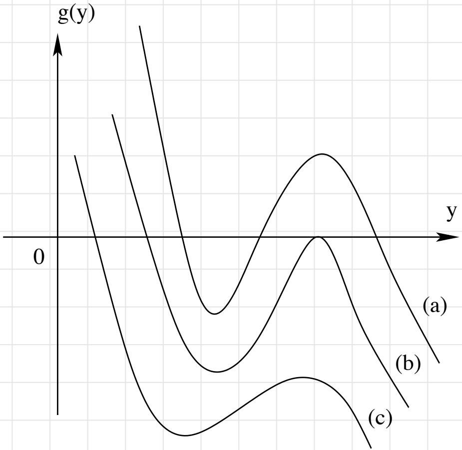
$$
\begin{aligned}
g(y) &= \frac{\nu^2}{y^2} - 1 + b_1\epsilon_*^2 y^2 - b_2\epsilon_*^4 y^4, \\
q(y) &= -\frac{1}{4y^2}.
\end{aligned}
$$

- Then, $g(y)$, usually has three roots, (y_0, y_1, y_2), where
  - we assume $0 < y_0 < \mathrm{Re}(y_1) \leq \mathrm{Re}(y_2)$
  - y_0 is always real and positive
  - when y_1, y_2 real, we assume $y_1 \leq y_2$
  - When y_1, y_2 are complex, we have $y_1 = y_2^*$.

3.4 Solutions near turning points (Cont.)

The turning points are defined as the roots of

$$g(y) = \frac{\nu^2}{y^2} - 1 + b_1\epsilon_*^2 y^2 - b_2\epsilon_*^4 y^4 = 0.$$



3.5 Matching to the Initial solution

- We assume the universe was initially at the adiabatic (Bunch-Davies) vacuum,

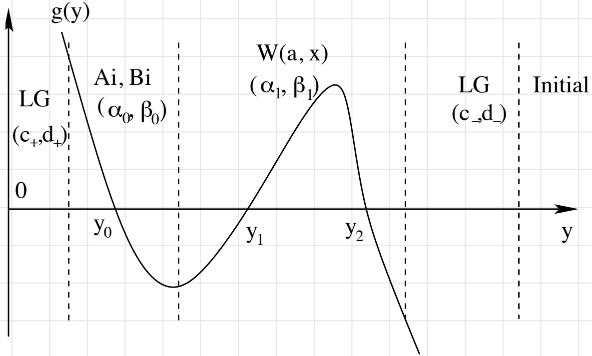
$$\begin{aligned}\lim_{y \to +\infty} \mu_k(y) &= \frac{1}{\sqrt{2\omega}} e^{-i \int \omega d\eta} \\ &\simeq \sqrt{\frac{k}{2}} \frac{1}{(-g)^{1/4}} \exp\left(-i \int_{y_i}^{y} \sqrt{-g} dy\right).\end{aligned}$$

- Since the equation of the mode function is second-order, we need one more condition to completely fix the free parameters in the solutions. We choose the second one as the Wronskian condition

$$\mu_k(y)\mu_k^*(y)' - \mu_k^*(y)\mu_k(y)' = i.$$

3.5 Matching to the Initial solution (Cont.)

- We found four sets of solutions:
  - LG solution near the pole $y = \infty$ with (c_-, d_-)
  - Cylindrical function solution near the two turning points $y_{1,2}$ with (α_1, β_1)
  - Airy function solution near the turning point y_0 with (α_0, β_0)
  - LG solution near the pole $y = 0$ with (c_+, d_+)



3.5 Matching to the Initial solution (Cont.)

- Using the initial conditions to the LG solution near the pole $y = \infty$, we find that

$$C_{-} = 0, \quad d_{-} = \sqrt{\frac{1}{2k}}.$$

- Matching the LG solution with the cylindrical function solutions at $y \gg y_2$ we find that

$$\begin{aligned} \alpha_1 &= 2^{-3/4} k^{-1/2} j^{-1/2}(\xi_0), \\ \beta_1 &= -i 2^{-3/4} k^{-1/2} j^{1/2}(\xi_0), \end{aligned}$$

with $j(\xi_0) \equiv \sqrt{1 + e^{\pi \xi_0^2}} - e^{\pi \xi_0^2 / 2}$.

3.5 Matching to the Initial solution (Cont.)

- Matching the cylindrical function solution with the Airy function one in their common region $y \in (y_0, y_1)$, we find

$$\alpha_0 = \sqrt{\frac{\pi}{2k}} \left[j^{-1}(\xi_0) \sin \mathfrak{B} - i j(\xi_0) \cos \mathfrak{B} \right],$$

$$\beta_0 = \sqrt{\frac{\pi}{2k}} \left[j^{-1}(\xi_0) \cos \mathfrak{B} + i j(\xi_0) \sin \mathfrak{B} \right],$$

where

$$\mathfrak{B} \equiv \int_{y_0}^{y_1} \sqrt{-g}\, dy + \phi(\xi_0^2/2),$$

$$\phi(x) \equiv \frac{x}{2} - \frac{x}{4} \ln x^2 + \frac{1}{2} \mathrm{ph}\Gamma\left(\frac{1}{2} + i x\right).$$

3.5 Matching to the Initial solution (Cont.)

- Finally, matching the LG solution near the pole $y = 0$ with the Airy function one in their common region $y \in (0, y_0)$, we find

$$
\begin{aligned}
d_+ &= \frac{\alpha_0}{2\sqrt{\pi}} \exp\left(-\int_{0^+}^{y_0} \sqrt{g}\,dy\right), \\
c_+ &= \frac{\beta_0}{\sqrt{\pi}} \exp\left(\int_{0^+}^{y_0} \sqrt{g}\,dy\right).
\end{aligned}
$$

3.6 Comparing with numerical (exact) solutions

- When $y_{1,2}$ are real and $y_1 \neq y_2$:

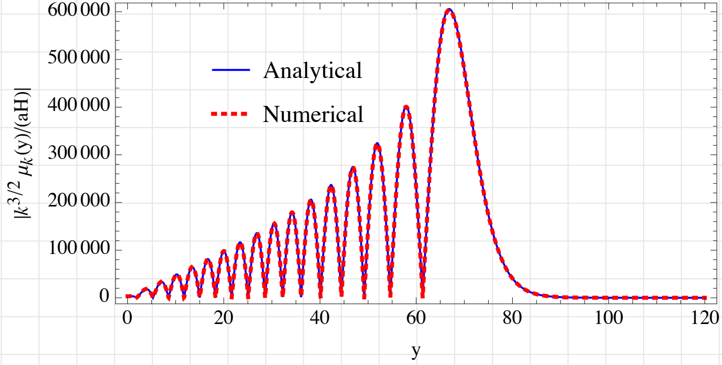


Figure: The numerical (exact) (red dotted curves) and analytical (blue solid curves) solutions with $b_1 = 3$, $b_2 = 2$, $\nu = 3/2$, and $\epsilon_* = 0.01$.

3.6 Comparing with numerical (exact) solutions (Cont.)

- When $y_{1,2}$ are real and $y_1 = y_2$:

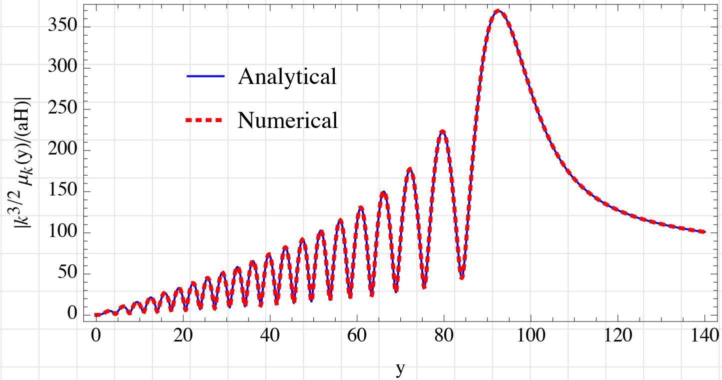


Figure: The numerical (exact) (red dotted curves) and analytical (blue solid curves) solutions with $b_1 = 2$, $b_2 = 1.00023$, $\nu = 3/2$, and $\epsilon_* = 0.01$.

3.6 Comparing with numerical (exact) solutions (Cont.)

- When $y_{1,2}$ are complex:

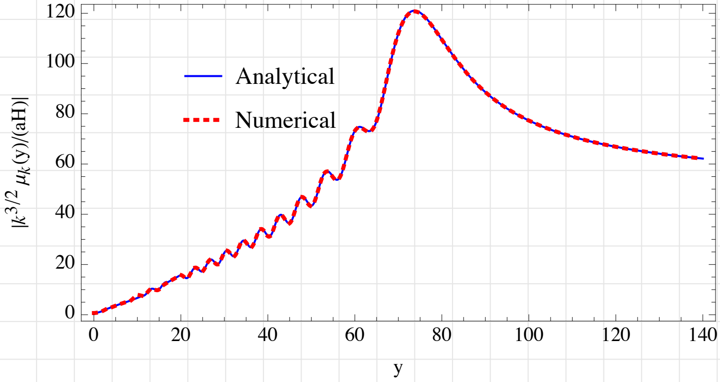


Figure: The numerical (exact) (red dotted curves) and analytical (blue solid curves) solutions with $b_1 = 3.5$, $b_2 = 3.2$, $\nu = 3/2$, and $\epsilon_* = 0.01$.

3.7 High-order Corrections

- To generalize the above to high-order approximation is challenging, mainly because of the matching. So, in the following we consider the case where $g(y) = 0$ has only one real root, y_0.
- Then, we can choose

$$f^{(1)}(\xi) = \pm\xi,$$

$\xi = \xi(y)$: a monotone decreasing function
"+": corresponds to $g(y) \geq 0$
"-": corresponds to $g(y) \leq 0$.

3.7 High-order Corrections (Cont.)

- Then, the mode function $U(\xi)$ is given by

$$
\begin{aligned}
U(\xi) \;\; = \;\; & \alpha_0 \left[\text{Ai}(\lambda^{2/3}\xi) \sum_{s=0}^{n} \frac{A_s(\xi)}{\lambda^{2s}} \right. \\
& \left. + \frac{\text{Ai}'(\lambda^{2/3}\xi)}{\lambda^{4/3}} \sum_{s=0}^{n-1} \frac{B_s(\xi)}{\lambda^{2s}} + \epsilon_3^{(2n+1)} \right] \\
& + \beta_0 \left[\text{Bi}(\lambda^{2/3}\xi) \sum_{s=0}^{n} \frac{A_s(\xi)}{\lambda^{2s}} \right. \\
& \left. + \frac{\text{Bi}'(\lambda^{2/3}\xi)}{\lambda^{4/3}} \sum_{s=0}^{n-1} \frac{B_s(\xi)}{\lambda^{2s}} + \epsilon_4^{(2n+1)} \right], \quad (10)
\end{aligned}
$$

where α_0 and β_0 are two integration constants.

3.7 High-order Corrections (Cont.)

- The coefficients A_s and B_s are given by

$$A_0(\xi) = 1,$$

$$B_s = \frac{\pm 1}{2(\pm\xi)^{1/2}} \int_0^{\xi} \{\psi(v)A_s(v) - A_s''(v)\} \frac{dv}{(\pm v)^{1/2}},$$

$$A_{s+1}(\xi) = -\frac{1}{2}B_s'(\xi) + \frac{1}{2}\int \psi(v)B_s(v)dv, (s = 0, 1, 2, ...) \quad (11)$$

where "+" corresponds to $\xi \geq 0$, and "-" to $\xi \leq 0$.

3.7 High-order Corrections (Cont.)

- The error bounds of $\epsilon_3^{(2n+1)}$ and $\epsilon_4^{(2n+1)}$ are given by,

$$\frac{\epsilon_3^{(2n+1)}}{M(\lambda^{2/3}\xi)}, \quad \frac{\partial\epsilon_3^{(2n+1)}/\partial\xi}{\lambda^{2/3}N(\lambda^{2/3}\xi)} \le 2E^{-1}(\lambda^{2/3}\xi)\exp\left[\frac{2\kappa_0\mathcal{V}_{\alpha,\xi}(|\xi^{1/2}|B_0)}{\lambda}\right] \times \frac{\mathcal{V}_{\alpha,\xi}(|\xi^{1/2}|B_n)}{\lambda^{2n+1}},$$

$$\frac{\epsilon_4^{(2n+1)}}{M(\lambda^{2/3}\xi)}, \quad \frac{\partial\epsilon_4^{(2n+1)}/\partial\xi}{\lambda^{2/3}N(\lambda^{2/3}\xi)} \le 2E(\lambda^{2/3}\xi)\exp\left[\frac{2\kappa_0\mathcal{V}_{\xi,\beta}(|\xi^{1/2}|B_0)}{\lambda}\right] \times \frac{\mathcal{V}_{\xi,\beta}(|\xi^{1/2}|B_n)}{\lambda^{2n+1}}.$$

3.7 High-order Corrections (Cont.)

- To determine α_0 and β_0, we assume the same initial conditions as in the firs-order approximations, that is, the universe initially was in the Bunch-Davies vacuum, and μ_k satisfies the Wronskian condition,

$$\lim_{y \to +\infty} \mu_k(y) = \frac{1}{\sqrt{2\omega_k}} e^{-i \int \omega_k d\eta},$$

$$\mu_k(y)\mu_k^*(y)' - \mu_k^*(y)\mu_k(y)' = i.$$

- After tedious calculations, we surprisingly find a very simple result,

$$\alpha_0 = \sqrt{\frac{\pi}{2k}}, \quad \beta_0 = i\sqrt{\frac{\pi}{2k}}.$$

3.8 Power spectra and spectral indexes with High-order Corrections

- To the third-order approximations, the power spectrum is given by

$$\begin{aligned} \Delta^2(k) &\equiv \frac{k^3}{2\pi^2}\left|\frac{\mu_k(y)}{z}\right|^2_{y\to 0^+} \\ &= \frac{k^2}{4\pi^2}\frac{-k\eta}{z^2(\eta)\nu^2(\eta)}\exp\left(2\int_y^{\nu_0}\sqrt{\hat{g}(\hat{y})}d\hat{y}\right) \\ &\times\left[1+\frac{H(+\infty)}{\lambda}+\frac{H^2(+\infty)}{2\lambda^2}+\mathcal{O}(1/\lambda^3)\right]. \end{aligned} \tag{12}$$

3.8 Power spectra and spectral indexes with High-order Corrections (Cont.)

- To the third-order approximations, the spectral index is given by

$$
\begin{aligned}
n - 1 \equiv \; & \frac{d \ln \Delta^2(k)}{d \ln k} = 3 \\
& + 2 \int_{y}^{\nu_0} \frac{d\hat{y}}{\sqrt{\hat{g}(\hat{y})}} + \frac{1}{\lambda} \frac{dH(+\infty)}{d \ln k} + \mathcal{O}\left(\frac{1}{\lambda^3}\right). \quad (13)
\end{aligned}
$$

- Note that the order of the approximations is referred to $1/\lambda$, and we have not impose the slow-roll conditions, normally denoted by ϵ_n, adopted in inflation.
- ϵ_n and λ^{-1} are two set of independent parameters. So, our method can be equally applied to non−slow−roll cases.

3.9 Applications to Special Cases

- High-order power spectra and spectral indices have been calculated so far up to the second-order of the slow-roll parameters ϵ_{n} in two cases:

  (a) GR ($\omega_{k}^{2} = k^{2}$), first by using the Green function method and later confirmed by the improved WKB method[13],

  (b) k-inflation ($\omega_{k}^{2} = c_{s}^{2}(\eta)k^{2}$), by using the uniform approximation method but to the firs-order of $(1/\lambda)$[14].

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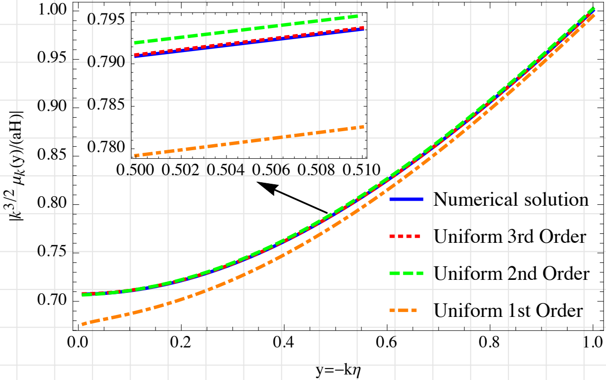
[13] J.-O. Gong and E.D. Stewart, PLB510 (2001) 1; S.M. Leach, A, Liddle, J. Martin and D. Schwarz, PRD66 (2002) 023515; J.-O. Gong, CQG21 (2004) 5555; R. Casadio, et al, PRD71 (2005) 043517; PLB625 (2005) 1.

[14] J. Martin, C. Ringeval and V. Vennin, JCAP06 (2013) 021.

3.9 Applications to Special Cases (Cont.)

- Applying our method to GR up to the second-order in terms of the slow-roll parameters ϵ_n and the third-order in terms of λ, we find that the exact (numerical) solutions are extremely well approximated by our analytical solutions [Zhu, et al, PRD90 (2014) 063503, arXiv:1405.5301].



3.9 Applications to Special Cases (Cont.)

- The resulted power spectra and spectral indexes of both scalar and tensor perturbations are consistent with the ones obtained by the Green function and improved WKB methods[15], within the allowed errors [Zhu, et al, PRD90 (2014) 063503, arXiv:1405.5301].
- Applying our method to k-inflation and Loop Quantum Cosmology, we obtained the power spectra, spectral indexes and runnings of both scalar and tensor perturbations with the highest accuracy existing in the literature so far [Zhu, et al, PRD90 (2014) 103517; ApJL 807 (2015) L17; JCAP (2015) [arXiv:1508.03239]].

[15] J.-O. Gong and E.D. Stewart, PLB510 (2001) 1; S.M. Leach, A, Liddle, J. Martin and D. Schwarz, PRD66 (2002) 023515; J.-O. Gong, CQG21 (2004) 5555; R. Casadio, et al, PRD71 (2005) 043517; PLB625 (2005) 1.

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4. Detecting Quantum Gravitational Effects in the Early Universe

- With the arrival of the era of the precision cosmology, it would be extremely important and interesting to see if any of these quantum gravitational effects in the early universe are within the range of the current and forthcoming experiments.
- In the following, we shall show that this might be possible. To be more specific, we concentrate ourselves on the inverse-volume corrections from LQC [16], for which the dispersion relations are,

[16] M. Bojowald and G. Calcagni, JCAP 03 (2011) 032; M. Bojowald, G. Calcagni, and S. Tsujikawa, Phys. Rev. Lett. 107, 211302 (2011); JCAP 11 (2011) 046.

4. Detecting Quantum Gravitational Effects in the Early Universe (Cont.)

$$\omega_k^2 = k^2 \times \begin{cases} 1 + \left[\frac{\sigma\nu_0}{3}\left(\frac{\sigma}{6}+1\right) + \frac{\alpha_0}{2}\left(5 - \frac{\sigma}{3}\right)\right]\delta_{PL}(\eta), & \texttt{scalar} \\ 1 + 2\alpha_0\delta_{PL}, & \texttt{tensor} \end{cases}$$

- $\alpha_0,\ \nu_0,\ \sigma$: encode the specific features of the model, with

$$0 < \sigma \leq 6,$$

- $\delta_{PL}(\eta)$: time-dependent,

$$\delta_{PL} = \left(\frac{a_{PL}}{a}\right)^{\sigma} < 1,$$

a_{PL}: a constant.

4. Detecting Quantum Gravitational Effects in the Early Universe (Cont.)

Then, for $V(\phi) = \lambda_p \phi^p$, we find that

$$\begin{aligned} \Gamma &\equiv (n_s - 1) + \frac{p+2}{8p} r + \gamma_1 (n_s - 1)^2 \\ &= \mathcal{F}(\sigma) \frac{\delta(k_\star)}{\epsilon_V}, \end{aligned} \tag{14}$$

where $\delta(k) \equiv \alpha_0 \delta_{PL}(k)$,

$$\begin{aligned} \mathcal{F}(\sigma) &\sim \mathcal{O}(1), \quad \epsilon_V \equiv M_{pl}^2 V_\phi^2/(2V^2), \\ \gamma_1 &= [3p^2 + (18 - 12D_p + 12D_n)p + 24D_p \\ &\quad -24D_n - 4]/[6(p+1)^2]. \end{aligned}$$

$\mathcal{F}(\sigma)\frac{\delta(k_\star)}{\epsilon_V}$: the quantum gravitational effects from LQC.

4. Detecting Quantum Gravitational Effects in the Early Universe (Cont.)

Since $\sigma\left(\mathrm{n_s}\right) \simeq \sigma\left(\mathrm{r}\right) \simeq 10^{-3}$, we have

$$\sigma\left(\Gamma\right) \simeq 10^{-3}.$$

Therefore, if

$$\mathcal{F}(\sigma)\frac{\delta(\mathrm{k}_\star)}{\epsilon_\mathrm{V}} \gtrsim \mathcal{O}\left(10^{-3}\right),$$

the quantum gravitational effects from LQC is within the range of detection!

4. Detecting Quantum Gravitational Effects in the Early Universe (Cont.)

- Using our Cosmological Monte Carlo (CosmoMC) code[17] with the Planck, BAO, and Supernova Legacy Survey data[18], we first carry out our CMB likelihood analysis.
- In particular, we assume the flat cold dark matter model with effective number of neutrinos $N_{eff} = 3.046$ and fix the total neutrino mass $\Sigma m_\nu = 0.06 eV$.

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[17] Y.-G. Gong, Q. Wu, and A. Wang, Astrophys. J. 681, 27-39 (2008); http://cosmologist.info/cosmomc

[18] P. A. R. Ade (Planck Collaboration), arXiv: 1502.02114; L. Anderson et al., Mon. Not. R. Astron. Soc. 427, 3435 (2013); A. Conley, J. Guy, M. Sullivan, N. Regnault, P. Astier, C. Balland, S. Basa and R. G. Carlberg et al., Astrophys. J. Suppl. 192, 1 (2011).

4. Detecting Quantum Gravitational Effects in the Early Universe (Cont.)

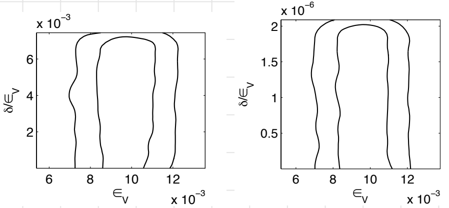
- We vary the seven parameters:
  (i) baryon density parameter, $\Omega_b h^2$
  (ii) dark matter density parameter, $\Omega_c h^2$
  (iii) the ratio of the sound horiozn to the angular diameter, θ
  (iv) the reionization optical depth, τ
  (v) $\delta(k_0)/\epsilon_V$
  (vi) ϵ_V
  (vii) $\Delta_s^2(k_0)$
- We take the pivot wave number $k_0 = 0.05$ Mpc^{-1}, used in Planck, to constrain $\delta(k_0)$ and ϵ_V.

4. Detecting Quantum Gravitational Effects in the Early Universe (Cont.)

- The following figures show the constraints:

$$\delta(\mathrm{k}_0) \lesssim 6.8 \times 10^{-5}, \qquad \sigma = 1,$$
$$\delta(\mathrm{k}_0) \lesssim 1.9 \times 10^{-8}, \qquad \sigma = 2,$$

at 1σ $(68\%$ CL) level.



4. Detecting Quantum Gravitational Effects in the Early Universe (Cont.)

- The upper bound for $\delta(\mathrm{k}_0)$ decreases dramatically as σ increases.
- However, despite the tight constraints on $\delta(\mathrm{k}_0)$, because of the $\epsilon_{\mathrm{V}}^{-1}$ enhancement in Eq.(14), such effects could be still well within the range of the detection of the forthcoming cosmological experiments[19] for

$$\sigma \lesssim 1,$$

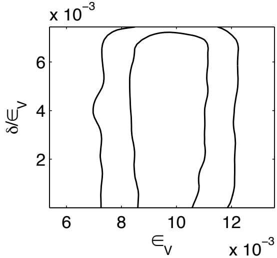
which is favorable even theoretically[20].

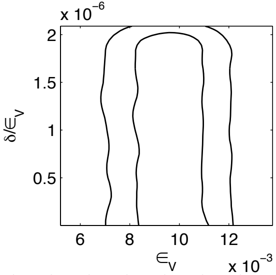
[19] K.N. Abazajian et al., "Inflation physics from the cosmic microwave background and large scale structure", Astropart. Phys. 63, 55 (2015) [arXiv:1309.5381].

[20] M. Bojowald and G. Calcagni, JCAP 03 (2011) 032.

4. Detecting Quantum Gravitational Effects in the Early Universe (Cont.)

- It is remarkable to note that, for any given σ, the best fitting value of ϵ_V is about 10^{-2}, which is rather robust in comparing with the case without the gravitational quantum effects.





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Conclusions

- Quantum gravitational effects in the early universe are important and need to be taken into account with the arrival of the era of precision cosmology.
- The uniform approximation method is designed to study analytically the evolution of the mode functions of perturbations generated in the early universe with such effects.
- The analytical results of power spectra and spectral indices are explicitly obtained in general case to the first-order approximations with the error bounds $\lesssim 15\%$.

Conclusions (Cont.)

- To the third-order, the analytical results of power spectra and spectral indices are explicitly obtained in the case with only one-turning point with the error bounds $\lesssim 0.15\%$.

- Applying them to the k-inflation, we obtained the most accurate results for power spectra, spectral indices and runnings, existing so far in the literature.

- Applying them to LQC, we found that the quantum gravitational effects from inverse-volume corrections might be within the detection of the next (Stage IV) experiments [21].

[21] K.N. Abazajian et al., arXiv:1309.5381.

Thank You!