Consistent massive graviton on arbitrary backgrounds

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2nd mini-workshop on gravity and cosmology

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Plan

Brief introduction to massive gravity

How to obtain the linearized field equations and the constraints

Results in different cases

Fierz-Pauli theory (1939)

$$S_{h,m} = -\frac{1}{2}\bar{M}_h^2 \int d^4x \; h_{\mu\nu} \Big[\mathcal{E}^{\mu\nu\rho\sigma} + \frac{\bar{m}^2}{2}\left(\eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\mu\nu}\eta^{\rho\sigma}\right)\Big] h_{\rho\sigma}$$

$$\mathcal{E}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} \equiv -\tfrac{1}{2}\left[\delta^\rho_\mu \delta^\sigma_\nu \Box + \eta^{\rho\sigma}\partial_\mu\partial_\nu - \delta^\rho_\mu \partial^\sigma\partial_\nu - \delta^\rho_\nu \partial^\sigma\partial_\mu - \eta_{\mu\nu}\eta^{\rho\sigma}\Box + \eta_{\mu\nu}\partial^\rho\partial^\sigma\right] h_{\rho\sigma}$$

$$\boxed{\delta\bar{E}_{\mu\nu} \equiv \mathcal{E}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} + \frac{\bar{m}^2}{2}\Big(h_{\mu\nu} - h\,\eta_{\mu\nu}\Big) = 0}$$

- Field eqs. for a massive graviton that has 5 degrees of freedom.
  - $\partial^\nu \delta\bar{E}_{\mu\nu} \implies$ 4 vector constraints : $\partial^\mu h_{\mu\nu} - \partial_\nu h = 0.$
  - Taking another divergence : $2\partial^\mu\partial^\nu\delta\bar{E}_{\mu\nu} + \bar{m}^2\eta^{\mu\nu}\delta\bar{E}_{\mu\nu} = -\frac{3}{2}\bar{m}^4 h.$
  - Scalar constraint $\boxed{h = 0.}$
- **It is the only linear massive gravity theory free of ghost.**
- **But it needs to be generalized to a non-linear theory.**

The dRGT massive gravity theory

$$\boxed{S = M_g^2 \int \mathrm{d}^4x \sqrt{|g|}\Big[R(g) - 2m^2 V\left(S;\beta_n\right)\Big],}$$

$$V(S;\beta_n) = \sum_{n=0}^{3} \beta_n e_n(S)\,,$$

- Square-root matrix $S^{\mu}{}_{\rho} S^{\rho}{}_{\nu} = g^{\mu\rho} f_{\rho\nu}$,
- $e_n(S)$ elementary symmetric polynomials :

$$e_0(S) = 1\,, \quad e_1(S) = \mathrm{Tr}[S]\,, \quad e_2(S) = \frac{1}{2}\Big(\mathrm{Tr}[S]^2 - \mathrm{Tr}[S^2]\Big)\,,$$
$$e_3(S) = \frac{1}{6}\Big(\mathrm{Tr}[S]^3 - 3\mathrm{Tr}[S]\mathrm{Tr}[S^2] + 2\mathrm{Tr}[S^3]\Big)$$

- No BD ghost.

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Field equations

$$\boxed{E_{\mu\nu} \equiv \mathcal{G}_{\mu\nu} + m^2 \, V_{\mu\nu} = 0 \,,}$$

$$\mathcal{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \,, \qquad V_{\mu\nu} \equiv \frac{-2}{\sqrt{|g|}} \frac{\delta(\sqrt{|g|}V)}{\delta g^{\mu\nu}} \,.$$

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Linearized field equations around a background solution

$$\boxed{\delta E_{\mu\nu} \equiv \delta\mathcal{G}_{\mu\nu} + m^2 \delta V_{\mu\nu} \equiv \left[\tilde{\mathcal{E}}_{\mu\nu}{}^{\rho\sigma} + m^2\, \mathcal{M}_{\mu\nu}{}^{\rho\sigma} \right] h_{\rho\sigma} = 0\,,}$$

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$$\begin{aligned} \delta V_{\mu\nu} =\ & g^{\rho\sigma} V_{\sigma\nu} \delta g_{\mu\rho} \\ & - g_{\mu\rho} \sum_{n=1}^{3} (-1)^n \beta_n \sum_{k=1}^{n} (-1)^k \left\{ \frac{1}{2} \left[S^{n-k}\right]^{\rho}{}_{\nu} \sum_{m=1}^{k} (-1)^m e_{k-m}(S) \left[S^{m-2} \delta S^2\right]^{\sigma}{}_{\sigma} \right. \\ & \left. \qquad\qquad + e_{k-1}(S) \sum_{m=0}^{n-k} \left[S^m \delta S S^{n-k-m}\right]^{\rho}{}_{\nu} \right\}. \end{aligned}$$

Method 1: Variation of the matrix S

To linearized the field equations we first need to obtain the perturbed matrix S.
It can be done using 2 different methods.

Sylvester equation: $AX - XB = C$

$$S^{\mu}{}_{\nu}\,(\delta S)^{\nu}{}_{\sigma} + (\delta S)^{\mu}{}_{\nu}\,S^{\nu}{}_{\sigma} = \delta[S^2]^{\mu}{}_{\sigma}\,.$$

- Unique explicit solution for δS iff S **and** $-S$ **do not have common eigenvalues** $\iff \det\big(\mathbb{X}\big) \neq 0$.

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Cayley-Hamilton theorem

$$S^4 - e_1 S^3 + e_2 S^2 - e_3 S + e_4 \mathbb{1} = 0 \, .$$

$$\Big[e_3 \mathbb{1} + e_1 S^2\Big] \delta S = F\big(\delta S^2\big) \, .$$

- Solution for δS iff $\mathbb{X} \equiv e_3 \mathbb{1} + e_1 S^2$ is invertible.

Method 2: Redefined fluctuation variables

Redefinition of the perturbation variable

$$\delta g_{\mu\nu} = \left(\delta^{\beta}_{\mu} S^{\lambda}_{\nu} + \delta^{\beta}_{\nu} S^{\lambda}_{\mu}\right) \delta g'_{\beta\lambda}$$

- The other variables (δS, δS^{-1}) can then be expressed as a function of $\delta g'_{\beta\lambda}$.

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$\delta g'_{\beta\lambda}$ is also a solution of the Sylvester equation:

$$g^{-1}\delta g = S\, g^{-1}\delta g' + g^{-1}\delta g'\, S$$

- There is a unique solution for $g^{-1}\delta g'$ iff S **and** $-S$ **do not have common eigenvalues.**
- It decreases the number of terms we have to deal with: the variation of δS is hidden in $\delta g'_{\beta\lambda}$.

Search for a scalar constraint

Counting the degrees of freedom

- 4 vector constraints: $\nabla^{\nu}\delta E_{\mu\nu} = 0$
- Scalar constraint: unlike in the F-P theory, it cannot be obtained from a linear combination of $g^{\mu\nu}\delta E_{\mu\nu}$ and $\nabla^{\mu}\nabla^{\nu}\delta E_{\mu\nu}$.

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Generalised traces and divergences of the field equations

1. We define all possible ways of tracing $\delta E_{\mu\nu}$ with $S^\mu{}_\nu$:

$$\begin{aligned}\Phi_i &\equiv [S^i]^{\mu\nu}\,\delta E_{\mu\nu}\,, \qquad 0 \le i \le 3\\ \Psi_i &\equiv [S^i]^{\mu\nu}\nabla_\nu\nabla^\lambda\,\delta E_{\lambda\mu} \qquad 0 \le i \le 3\,.\end{aligned}$$

2. Find a linear combination of these 8 scalars for which the 2nd derivative terms vanish :

$$\sum_{i=0}^{3}\left(u_i\,\Phi_i + v_i\,\Psi_i\right) \sim 0,$$

More details on the search for a scalar constraint

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$$\sum_{i=0}^{3}(u_i\,\Phi_i + v_i\,\Psi_i) \sim \sum_{i=1}^{26}\alpha_i \aleph_i = 0,$$

$$\aleph_i = \{\nabla_\rho\nabla_\sigma\,h^{\rho\sigma}, ..., [S^3]^{\rho\sigma}\,[S^3]^{\mu\nu}\,\nabla_\rho\nabla_\sigma\,h_{\mu\nu}\}$$

- $\alpha_i = 0$: 26 equations for 7 unknowns $\{u_i, v_i\}$, only the trivial solution.
- All the $\aleph_i$ are not all independent from each other : non trivial identities (**syzygies**) linking them $\Longrightarrow$ Reduces the number of equations to be solved.

A particular case: the beta 1 model

We assume $\boldsymbol{\beta_2} = \boldsymbol{\beta_3} = 0$ and keep $\boldsymbol{\beta_0}$, $\boldsymbol{\beta_1} \neq 0$ and $\boldsymbol{f_{\mu\nu}}$ arbitrary.

Field equations

$$\mathcal{G}_{\mu\nu} + m^2 \Big[\beta_0 \, g_{\mu\nu} + \beta_1 \, g_{\mu\rho} \big(e_1(S) \delta^\rho_\nu - S^\rho{}_\nu \big) \Big] = 0 \, ,$$

It can be solved for $S^\mu{}_\nu$:

$$S^\rho{}_\nu = \frac{1}{\beta_1 m^2} \left[R^\rho{}_\nu - \frac{1}{6} \delta^\rho_\nu R - \frac{m^2 \beta_0}{3} \, \delta^\rho_\nu \right] .$$

- **It is only possible in the $\boldsymbol{\beta_1}$ model.**
- It can be used to eliminate any occurrences of S in the linearized field equations.

A particular case: the beta 1 model

- In the $\boldsymbol{\beta_1}$ model, we can express the linearized field equations as a function of $\boldsymbol{g_{\mu\nu}}$ and its curvature.
- **We now take these equations as our starting point, no more assuming that $g_{\mu\nu}$ is a background solution.**

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The fifth scalar constraint

$$\boxed{-\frac{m^2\,\beta_1\,e_4}{2}\,\Phi_0 - e_3\,\Psi_0 + e_2\,\Psi_1 - e_1\,\Psi_2 + \Psi_3 = 0\,.}$$

- **Massive graviton (with 5 dof) propagating in a single arbitrary background.**

Beyond the beta 1 model: the general case

$$\overline{\Psi} = [S^{-1}]^{\mu\nu} \nabla_\nu \nabla^\lambda \, \delta E_{\lambda\mu} = \frac{1}{e_4} \left(e_3 \, \Psi_0 - e_2 \, \Psi_1 + e_1 \, \Psi_2 - \Psi_3 \right).$$

1. $\boldsymbol{\beta_3 = 0}$

$$\boxed{\frac{m^2 \, \beta_1}{2} \, \Phi_0 + m^2 \, \beta_2 \, \Phi_1 + \overline{\Psi} = 0 \, .}$$

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1. $\boldsymbol{\beta_3 = 0}$

$$\boxed{\frac{m^2 \, \beta_1}{2} \, \Phi_0 + m^2 \, \beta_2 \, \Phi_1 + \overline{\Psi} = 0 \,.}$$

2. $\boldsymbol{\beta_3 \neq 0}$

$$\frac{m^2 \, \beta_1}{2} \, \Phi_0 + m^2 \, \beta_2 \, \Phi_1 - m^2 \, \beta_3 \left(\Phi_2 - e_1 \Phi_1 + \tfrac{1}{2} e_2 \, \Phi_0 \right) + \overline{\Psi}$$
$$\sim m^2 \beta_3 \left(S^{\mu\lambda} [S^2]^{\nu\beta} - S^{\mu\nu} [S^2]^{\beta\lambda} \right) \nabla_\mu \nabla_\nu \delta g'_{\beta\lambda} \,.$$

- **It is not a covariant constraint but all the second time derivatives acting on the lapse and the shift vanish.**

Applications

- For flat and Einstein space-times the constraint reduces to the expected one $h = 0$.
- Application to cosmology (β_1 model): The equations of motion are those of a massive graviton propagating in an arbitrary FLRW space-time.
- We used this formalism to obtain the linearized field equations in bimetric theories and study the covariant constraints.

Conclusion

- Linearized equations of massive gravity (and bi-gravity) in the general case.
- Consistent theory for a massive graviton propagating in a single arbitrary background metric (β_1 model).
- Five covariant constraints in a metric formulation, when $\beta_3 = 0$.
- Non-covariant scalar constraint when $\beta_3 \neq 0$ and in bimetric theories.