Aspects of the dimer model

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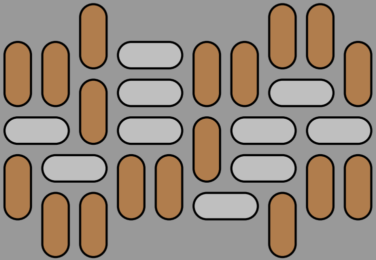


Based on joint work with Alexi Morin-Duchesne & Philippe Ruelle

Outline

Dimer model

- Bits and pieces
- Lieb's transfer matrix formulation
- Exact solvability

Temperley-Lieb algebra

- Dimer representation of TL at $\beta = 0$
- Spanning webs and critical dense polymers

Integrability

- Transfer tangles with inhomogeneities
- Commuting transfer matrices

Conformal data

- Integrals of motion
- Conformal $c = -2$ description

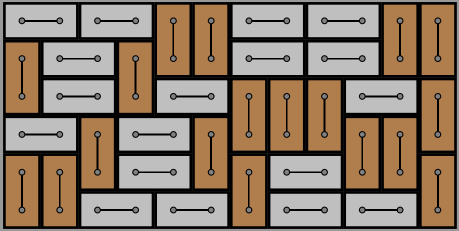
Domino tilings

Dimer model on a graph

- **Graph G**: edges connecting vertices pairwise
- **Perfect matching**: set of edges covering every vertex exactly once
- **Dimer model on G**: the set of perfect matchings on G

Domino tilings

- Planar **domino tiling**: perfect matching on (a subgraph of) $\mathbb{Z}^2$
- 106,912,793 possible domino tilings of the 6×12 rectangle
- Example:



Counting problems

Number of domino tilings of an $M \times N$ rectangle

$$D_{M,N} = \prod_{j=1}^{\lceil \frac{M}{2} \rceil} \prod_{k=1}^{\lceil \frac{N}{2} \rceil} \left(4\cos^2 \frac{j\pi}{M+1} + 4\cos^2 \frac{k\pi}{N+1} \right)$$

- Solved by Kasteleyn (1961) and Fisher & Temperley (1961), and analysed by Stephenson, Lieb, Ferdinand, Wu, Hartwig, ...

A source of geeky dinner-table quizzes

- In how many ways can you cover a 3×7 rectangle?
- In how many ways can you cover a chess board with two diagonally opposite corners removed?
- Can you always cover a chess board if one black and one white square have been removed?

Scenarios

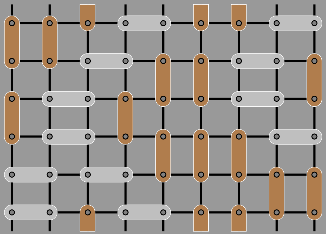
Many possibilities

- **Tiles**: monomers, dimers, trimers, tetrominoes, ...
- **Geometry**: rectangle, cylinder, torus, ...
- **Boundaries**: without protrusion, with protrusion, ...

Our focus today

Dimer model on the square lattice
wrapped around a horizontal cylinder, without protrusion

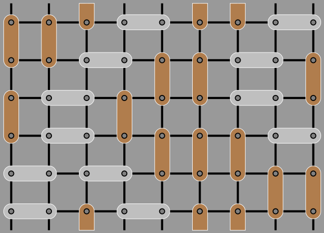
- Example (6×9):



Stat-mech description

Boltzmann weights

- Domino tilings are weighted according to their tile content
- A vertical domino has weight 1; a horizontal domino has weight α
- Example:



$\longrightarrow \quad \alpha^{12}$

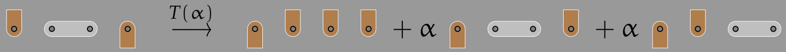
Partition function

$$Z_{M,N} = \sum_{\text{configs}} \alpha^h, \qquad h = \#[\text{horizontal dimers}]$$

Transfer matrix approach

In terms of dimer configurations

- The transfer matrix $T(\alpha)$ acts on a row of vertices
- It outputs all possible dimer configurations in the row just above
- It assigns the appropriate weights to the various configurations
- Example ($N = 4$):



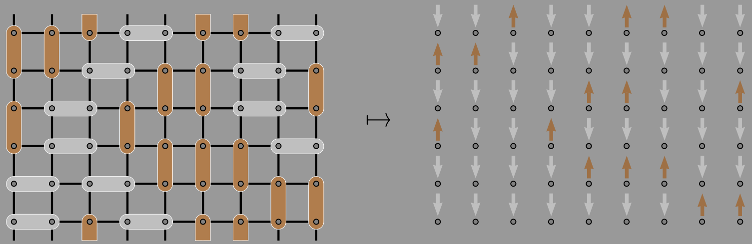
Partition function

$$Z = \mathrm{Tr}\big(T^M(\alpha)\big)$$

Arrow configurations

Vertical edges are replaced by arrows

- A **vertical domino** becomes an **up arrow**
- **Absence** of a vertical domino is indicated by a **down arrow**
- Example:



On the horizontal cylinder, the map is one-to-one:
domino tilings $\leftrightarrow$ arrow configurations

Lieb's transfer matrix I

In terms of arrow configurations

- The transfer matrix $T(\alpha)$ acts on a row of vertical edges
- It outputs all possible arrow configurations in the row just above
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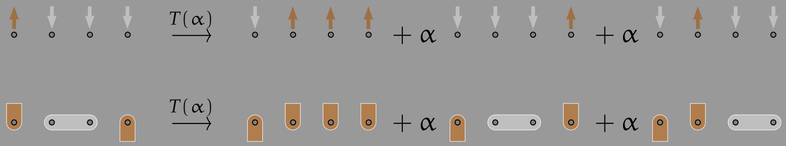
- Example ($N = 4$):



Lieb's transfer matrix II

- Each vertical edge carries a copy of $\mathbb{C}^2$ spanned by

$$\begin{pmatrix}1\\0\end{pmatrix} = \uparrow, \qquad \begin{pmatrix}0\\1\end{pmatrix} = \downarrow$$

- In a given row, at edge i, the relevant Pauli matrices act as

$$\sigma_i^x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} : \uparrow \leftrightarrow \downarrow, \qquad \sigma_i^- = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} : \uparrow \mapsto \downarrow, \ \downarrow \mapsto 0$$

Transfer matrix (Lieb, 1967)

$$T(\alpha) = V_3 V_1 \qquad V_1 = \prod_{i=1}^{N} \sigma_i^x \qquad V_3 = \prod_{i=1}^{N-1} (\mathbb{I} + \alpha\, \sigma_i^- \sigma_{i+1}^-)$$

$$\uparrow \downarrow \downarrow \downarrow \xrightarrow{V_1} \downarrow \uparrow \uparrow \uparrow$$

$$\downarrow \uparrow \uparrow \uparrow \xrightarrow{V_3} \downarrow \uparrow \uparrow \uparrow + \alpha \downarrow \downarrow \downarrow \uparrow + \alpha \downarrow \uparrow \downarrow \downarrow$$

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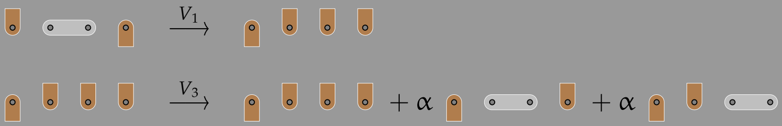
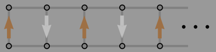


Variation index

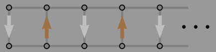
- Convenient to work with

$$T^2(\alpha) = V_3 V_3^T = \prod_{i=1}^{N-1} (\mathbb{I} + \alpha\, \sigma_i^- \sigma_{i+1}^-) \prod_{i=1}^{N-1} (\mathbb{I} + \alpha\, \sigma_i^+ \sigma_{i+1}^+)$$

- One-dimensional subspaces invariant under $T^2(\alpha)$:



and



Variation index

$$[T^2(\alpha), \mathcal{V}] = 0, \qquad \mathcal{V} = \frac{1}{2} \sum_{i=1}^{N} (-1)^i \sigma_i^z, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- $\mathcal{V}$ is diagonal in the arrow basis, with eigenvalues

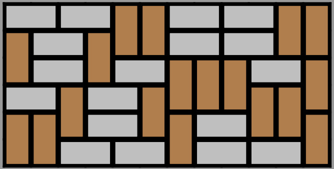
$$v \in \{-\tfrac{N}{2}, -\tfrac{N}{2}+1, \ldots, \tfrac{N}{2}\}$$

- Irreducible $T^2(\alpha)$-invariant subspaces: E_N^v with $\dim E_N^v = \binom{N}{\frac{N}{2} - v}$

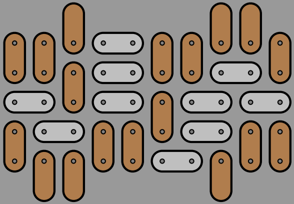
Examples

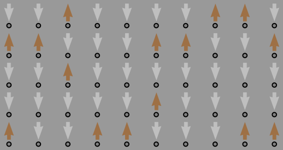
$v = \frac{N}{2}$

$v = 0$



$v = -1,\ 1,\ -1,\ 1,\ -1$





Exact solvability

$T(\alpha)$: symmetric $\rightarrow$ diagonalisable with real spectrum

Jordan-Wigner transformation $\rightarrow$ fermions $\rightarrow$ closed expressions for eigenvalues

$$\Lambda_{\mu,\nu} = \prod_{\substack{k=1 \\ k=N-1 \bmod 2}}^{N-1} \left(\alpha \sin p_k + \sqrt{1+\alpha^2 \sin^2 p_k}\right)^{2(1-\mu_k-\nu_k)}$$

$$\mu_k, \nu_k \in \{0,1\} \qquad p_k = \frac{\pi k}{2(N+1)}$$

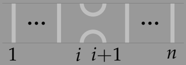
In general, no commutativity: $[T(\alpha), T(\alpha')] \neq 0$

Exact solvability without (Yang-Baxter) integrability?!

Temperley-Lieb algebra

Generators **A connectivity**

$I =$ 

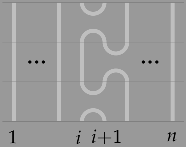
$e_i =$ 

$c =$ 

- Multiplication is by vertical concatenation

$(c = e_1 e_4 e_2 e_3)$

Example

$e_i \, e_{i+1} \, e_i =$ 

$=$ 

$= e_i$

Algebraic definition

$$TL_n(\beta) = \langle I, e_i \, ; i = 1, \ldots, n-1 \rangle$$

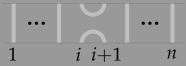
$$I\,A = A\,I = A \qquad (A \in TL_n(\beta))$$

$$e_i^2 = \beta e_i \qquad e_i e_{i\pm1} e_i = e_i \qquad e_i e_j = e_j e_i \qquad (|i-j| > 1)$$

Temperley-Lieb algebra

Generators

$I =$ 

$e_i =$ 

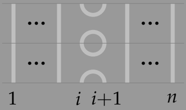
A connectivity

$c =$ 

- Multiplication is by vertical concatenation

$(c = e_1 e_4 e_2 e_3)$

Example

$(e_i)^2 =$ 

$= \beta$ 

$= \beta\, e_i$

Algebraic definition

$$TL_n(\beta) = \langle I,\, e_i\,;\, i = 1, \ldots, n-1 \rangle$$

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$$e_i^2 = \beta e_i \qquad e_i e_{i\pm 1} e_i = e_i \qquad e_i e_j = e_j e_i \qquad (|i-j| > 1)$$

Dimer representation

Representation of $TL_n(\beta = 0)$ on $(\mathbb{C}^2)^{\otimes n-1}$

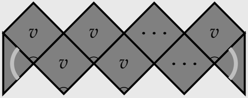
$$\tau(I) = \mathbb{I} \qquad \tau(e_i) = \begin{cases} \sigma^-_{i-1}\sigma^-_i + \sigma^-_i\sigma^-_{i+1} & i \text{ even} \\ \sigma^+_{i-1}\sigma^+_i + \sigma^+_i\sigma^+_{i+1} & i \text{ odd} \end{cases}$$

Tilted transfer tangle

$$T^2(\alpha) = \prod_{i=1}^{\lfloor \frac{N-2}{2} \rfloor}\Big(\mathbb{I}+\alpha\,(\sigma^-_{2i-1}\,\sigma^-_{2i}+\sigma^-_{2i}\,\sigma^-_{2i+1})\Big)\prod_{i=1}^{\lfloor \frac{N-1}{2} \rfloor}\Big(\mathbb{I}+\alpha\,(\sigma^+_{2i-2}\,\sigma^+_{2i-1}+\sigma^+_{2i-1}\,\sigma^+_{2i})\Big)$$

is a matrix representative of an element of $TL_n(0)$ for $\underline{n = N+1}$:

$$T^2(\alpha) = (1+\alpha^2)^{N/2}\,\tau(\boldsymbol{\mathcal{D}}(v)) \qquad \alpha = \tan v$$

$\boldsymbol{\mathcal{D}}(v) =$ 

$= \cos v$  $+ \sin v$ 

Module structure

Variation index decomposition

$$(\mathbb{C}^2)^{\otimes n-1} = \bigoplus_v E^v_{n-1}$$

$$\Big(\mathcal{I}^d_n : \text{ irred module over } TL_n(0)\Big)$$

Structure for n odd:

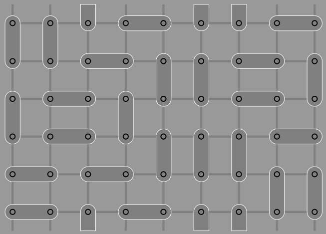
$$E^v_{n-1} = \bigoplus_{i=0}^{\left\lfloor \frac{n-1-2|v|}{4} \right\rfloor} \mathcal{I}_n^{2|v|+4i+1}$$

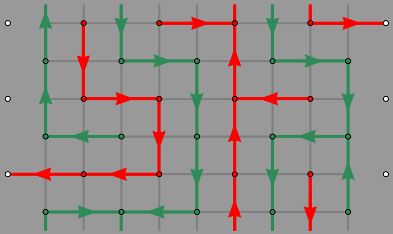
Structure for n even, $v > 0$:

$$E^v_{n-1} = \begin{cases} \begin{matrix} \mathcal{I}_n^{2v+1} & & \mathcal{I}_n^{2v+5} & & \mathcal{I}_n^{n-2} & & \\ & \searrow \quad \swarrow & & \searrow \quad \swarrow & & \searrow & \\ & \mathcal{I}_n^{2v+3} & & \dots & & \mathcal{I}_n^{n} & \end{matrix} & \frac{n-1-2v}{2} \text{ odd} \\ \begin{matrix} \mathcal{I}_n^{2v+1} & & \mathcal{I}_n^{2v+5} & & \mathcal{I}_n^{n-4} & & \mathcal{I}_n^{n} \\ & \searrow \quad \swarrow & & \searrow \quad \swarrow & & \searrow \quad \swarrow & \\ & \mathcal{I}_n^{2v+3} & & \dots & & \mathcal{I}_n^{n-2} & \end{matrix} & \frac{n-1-2v}{2} \text{ even} \end{cases}$$

E^{-v}_{n-1} is the module contragredient to E^v_{n-1}

Spanning webs and loop models

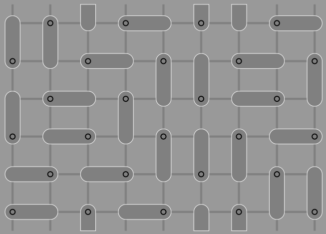


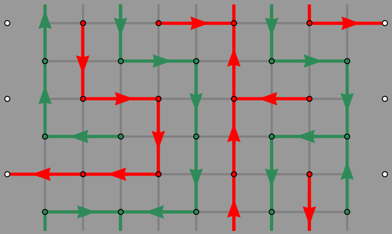


Series of maps (including Temperley's correspondence)

dimer coverings $\longrightarrow$ **oriented spanning webs** $\longrightarrow$ critical dense polymer configurations $\longrightarrow$ *TL* algebra

Spanning webs and loop models





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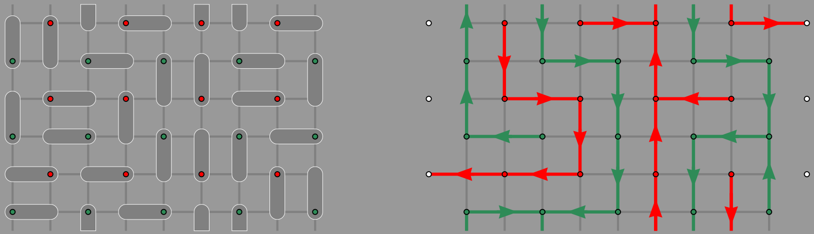

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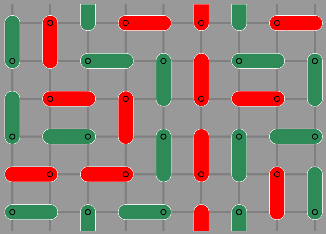

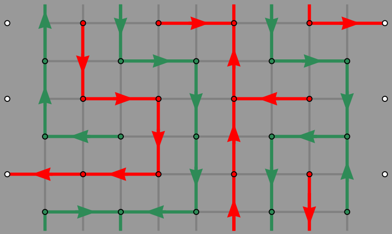

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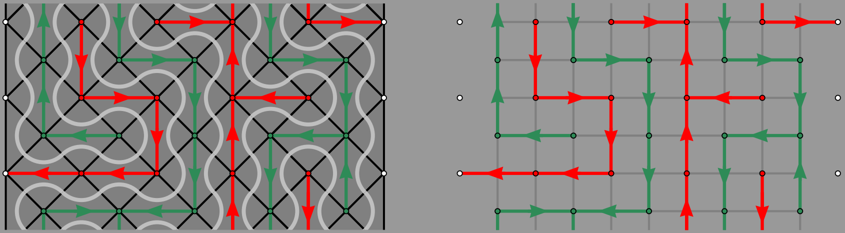

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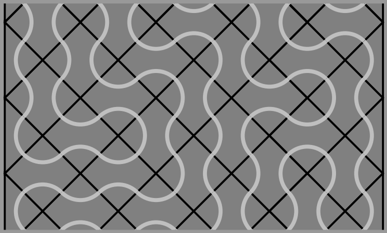

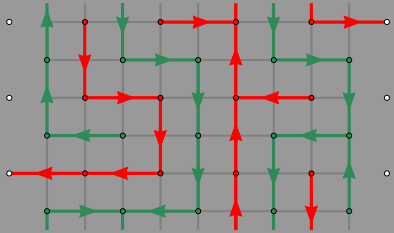

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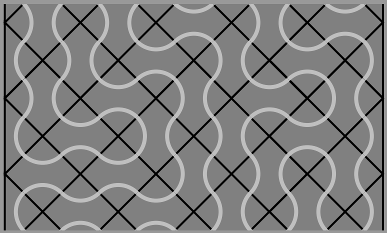

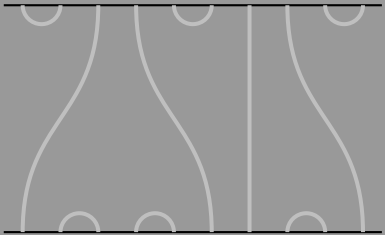

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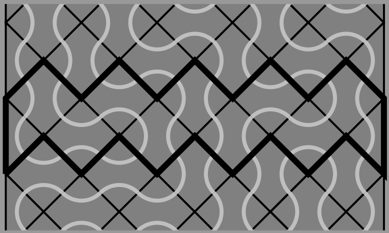
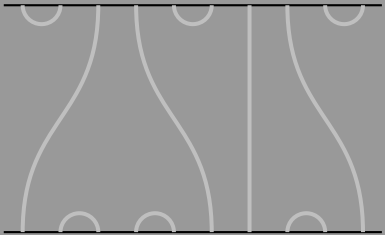




Series of maps (including Temperley's correspondence)

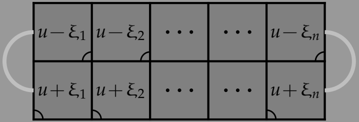
dimer coverings $\rightarrow$ oriented spanning webs $\rightarrow$ **critical dense polymer configurations** $\rightarrow$ **TL algebra**

Critical dense polymers

Lattice loop model where contractible loops are disallowed ($\beta = 0$)

Double-row transfer tangle

$$\boldsymbol{D}(u, \xi) = \frac{1}{\sin 2u}$$



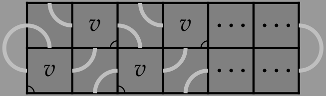
$\in TL_n(0)$

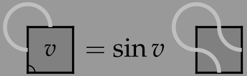
- $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ are inhomogeneities
- Commuting transfer tangles: $[\boldsymbol{D}(u, \xi), \boldsymbol{D}(v, \xi)] = 0$

Tilted transfer tangle

Special choice of parameters: $u=\frac{v}{2}$, $\xi=\xi_v=(\frac{v}{2},-\frac{v}{2},\frac{v}{2},-\frac{v}{2},\dots)$

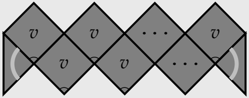
$$\boldsymbol{D}(\tfrac{v}{2},\xi_v)=\frac{1}{\sin v}$$





$\Rightarrow$

$$\boldsymbol{D}(\tfrac{v}{2},\xi_v)=$$



$$=\boldsymbol{\mathcal{D}}(v)$$

Tangles commuting with $\boldsymbol{\mathcal{D}}(v)$

For each $u\in\mathbb{C}$:

$$\Big[\boldsymbol{D}(u,\xi_v),\underbrace{\boldsymbol{D}(\tfrac{v}{2},\xi_v)}_{=\boldsymbol{\mathcal{D}}(v)}\Big]=0$$

Integrability

Dimer representation

$$T^2(\alpha) \simeq \tau\big(\boldsymbol{\mathcal{D}}(v)\big)$$

General absence of commutativity

$$\xi_v \neq \xi_{v'} \quad \to \quad \big[\underbrace{\boldsymbol{D}(v, \xi_v)}_{=\boldsymbol{\mathcal{D}}(v)}, \underbrace{\boldsymbol{D}(v', \xi_{v'})}_{=\boldsymbol{\mathcal{D}}(v')}\big] \neq 0 \quad \to \quad [T^2(\alpha), T^2(\alpha')] \neq 0$$

Integrability from commutativity

$$[\boldsymbol{D}(u, \xi_v), \boldsymbol{\mathcal{D}}(v)] = 0 \;\to\; \boxed{[\tau\big(\boldsymbol{D}(u, \xi_v)\big), T^2(\alpha)] = 0} \quad (u \in \mathbb{C},\; \alpha = \tan v)$$

Different values of α label different integrable families

Continuum scaling limit

Asymptotic expansion of spectra

$$\mathrm{Eig}^{(\mu,\nu)}\left(-\tfrac{1}{2}\log T^2(\alpha)\right) = n f_{\mathrm{bulk}} + f_{\mathrm{bdy}} + \sum_{p=1}^{\infty} \frac{a_p^{(\mu,\nu)}}{n^{2p-1}}$$

- The coefficients decompose as

$$a_p^{(\mu,\nu)} = \lambda_p(\alpha)\, I_{2p-1}^{(\mu,\nu)}$$

Integrals of motion

- Abelian subalg of the enveloping algebra of the Virasoro algebra:

$$\mathbf{I}_1 = L_0 - \frac{c}{24}, \quad \mathbf{I}_3 = 2\sum_{n=1}^{\infty} L_{-n}L_n + L_0^2 - \frac{c+2}{12}L_0 + \frac{c(5c+22)}{2880}, \quad \mathbf{I}_5 = \ldots$$

- In each given v-sector (as verified for the lowest-lying vectors),

$I_{2p-1}^{(\mu,\nu)}$ match the spectra of $\mathbf{I}_{2p-1}$ on certain Virasoro modules for $c = -2$

Feigin-Fuchs modules

Limiting Virasoro module

$$E^v_{n-1} \quad \longrightarrow \quad E^v$$

Structure for n odd

$$E^v \simeq E^{-v} \;:\quad \overset{\Delta_v}{\bullet} \oplus \overset{\Delta_{v+2}}{\bullet} \oplus \overset{\Delta_{v+4}}{\bullet} \oplus \overset{\Delta_{v+6}}{\bullet} \oplus \cdots \qquad (v \geq 0)$$

Structure for n even

$$E^v \;:\; \begin{cases} \overset{\Delta_v}{\bullet} \longleftarrow \overset{\Delta_{v+1}}{\circ} \longrightarrow \overset{\Delta_{v+2}}{\bullet} \longleftarrow \overset{\Delta_{v+3}}{\circ} \longrightarrow \overset{\Delta_{v+4}}{\bullet} \longleftarrow \overset{\Delta_{v+5}}{\circ} \longrightarrow \cdots & (v \geq \frac{1}{2}) \\[2ex] \overset{\Delta_{-v}}{\circ} \longrightarrow \overset{\Delta_{-v+1}}{\bullet} \longleftarrow \overset{\Delta_{-v+2}}{\circ} \longrightarrow \overset{\Delta_{-v+3}}{\bullet} \longleftarrow \overset{\Delta_{-v+4}}{\circ} \longrightarrow \overset{\Delta_{-v+5}}{\bullet} \longleftarrow \cdots & (v \leq -\frac{1}{2}) \end{cases}$$

Conformal weights

$$\Delta_v = \frac{4v^2 - 1}{8}$$

Summary and outlook

Summary

- Dimer model → TL loop model with $\beta = 0$
- Integrability and solvability reconciled
- Indecomposable (zigzag) modules over $TL_n(0)$
- Feigin-Fuchs modules over the Virasoro algebra at $c = -2$
- Results compatible with conformal integrals of motion

Outlook

- Torus scenario and its conformal properties
- Inclusion of monomers and more general polyominoes
- Dimer models on different lattices
- Dimer-like representations of TL for $\beta \neq 0$