D-instantons, mock modular forms and BPS partition functions

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arXiv:1605.05945

(continuation of arXiv:1207.1109)

Plan of the talk

1. Calabi-Yau compactifications and quantum corrected hypermultiplet moduli space

2. Twistorial description of D-instanton corrections

3. D3-instantons: contact potential
   - relation to BPS partition function
   - modularity and mock modularity

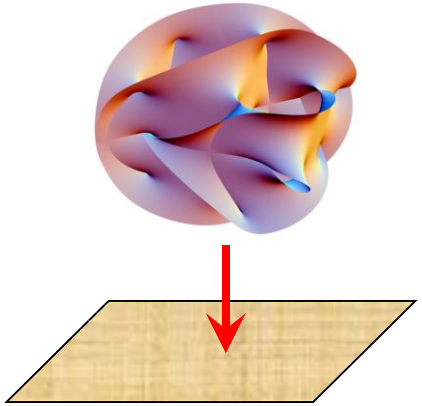
4. D3-instantons: Darboux coordinates on twistor space
   - modularity and indefinite theta functions

Calabi-Yau compactifications

Type II string theory

compactification on a Calabi-Yau

$\mathcal{N}$=2 supergravity in 4d coupled to *vector* and *hypermultiplets*



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$$\mathcal{M}_{\mathrm{VM}} \times \mathcal{M}_{\mathrm{HM}}$$

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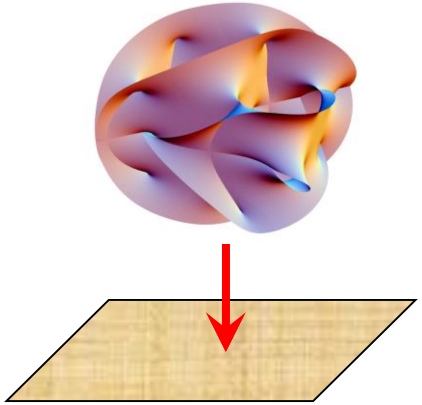

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$\mathcal{N}$=2 supersymmetry:

special Kähler	*quaternion-Kähler*
(determined by holomorphic prepotential)	*twistorial space description*

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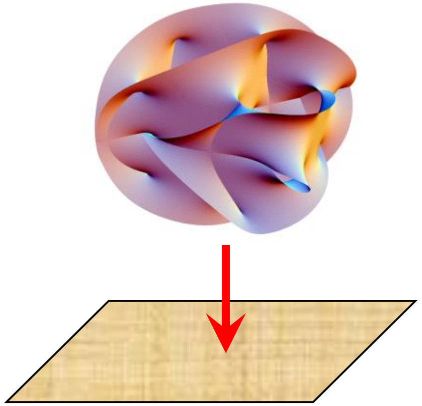

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$\mathcal{N}$=2 supersymmetry:	*special Kähler*	*quaternion-Kähler*
	(determined by holomorphic prepotential)	*twistorial space description*
	no g_s-corrections	various types of g_s-corrections

Quantum corrections and S-duality

Metric on $\mathcal{M}_{\mathrm{HM}}$ = **tree level** + **1-loop** + **D-brane instantons** + **NS5-brane instantons**

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Kaluza-Klein reduction

\+ (in type IIB)

pert. α'-correction

& worldsheet instantons

Ferrara,Sabharwal ’90

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D-branes wrapping cycles of CY

Type IIA	Type IIB
D2	D(-1) D1 D3 D5

S.A.,Pioline,Saueressig, Vandoren '08, S.A. '09

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- pert α'-correct.+1-loop+D(-1)
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(manifest in type IIB)

$$SL(2,\mathbb{R}) \longrightarrow SL(2,\mathbb{Z})$$

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quantum corrections

The problem: to prove (and to derive consequences of)
the modular symmetry of D3-instantons

HM moduli space

Type IIA$/X$ Type IIB$/\tilde{X}$

$$q^{\alpha}_{\mathrm{IIA}}\left\{\begin{array}{ccc} u^a & \text{complex structure/complexified Kähler moduli} & z^a = b^a + \mathrm{i}t^a \\ \zeta^\Lambda, \tilde{\zeta}_\Lambda & \text{periods of RR gauge potentials} & c^0, c^a, \tilde{c}_a, \tilde{c}_0 \\ \sigma & \text{NS-axion (dual to the B-field)} & \psi \\ \phi & \text{dilaton (string coupling } e^\phi \sim g_{(4)}^{-2}\,,\ \tau_2 \sim g_s^{-1}\text{)} & \tau_2 \end{array}\right\} q^{\alpha}_{\mathrm{IIB}}$$

$a = 1, \ldots, n_{\mathrm{H}} - 1$

$\Lambda = 0, \ldots, n_{\mathrm{H}} - 1$

$\dim \mathcal{M}_{\mathrm{HM}} = 4n_{\mathrm{H}}$

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The action of SL(2,$\mathbb{Z}$) on type IIB fields:

$$\tau\mapsto\frac{a\tau+b}{c\tau+d}\qquad t^a\mapsto t^a|c\tau+d|\qquad \tilde{c}_a\mapsto\tilde{c}_a$$

$$\begin{pmatrix}c^a\\ b^a\end{pmatrix}\mapsto\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}c^a\\ b^a\end{pmatrix}\qquad \begin{pmatrix}\tilde{c}_0\\ \psi\end{pmatrix}\mapsto\begin{pmatrix}d & -c\\ -b & a\end{pmatrix}\begin{pmatrix}\tilde{c}_0\\ \psi\end{pmatrix}$$

$$\tau=c^0+\mathrm{i}g_s^{-1}$$

HM moduli space

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q^α_{IIA} $\qquad$ q^α_{IIB}

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Classical mirror map

Bohm,Gunther,Hermann,Louis ’99

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RR fields
$$\left\{ \begin{aligned} \zeta^0 &= c^0 \\ \zeta^a &= -(c^a - \tau_1 b^a) \\ \tilde\zeta_a &= \tilde c_a + \tfrac{1}{2}\, \kappa_{abc}\, b^b (c^c - \tau_1 b^c) \\ \tilde\zeta_0 &= \tilde c_0 - \tfrac{1}{6}\, \kappa_{abc}\, b^a b^b (c^c - \tau_1 b^c) \end{aligned} \right.$$

intersection numbers

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Quantum corrections induce corrections to the mirror map

Twistor approach

How to describe instanton corrections to the HM metric? ⟷ How to parametrize QK manifolds?

$\mathcal{M}$ — QK ⟷ Holonomy group $\mathrm{Sp}(n)\times \mathrm{SU}(2)\subset \mathrm{O}(4n)$

$\mathrm{dim}\mathcal{M}=4n$

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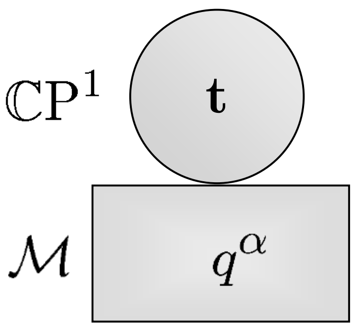
The idea: one should work at the level of the *twistor space*

<u>Quaternionic structure</u>

quaternion algebra of *almost* complex structures

$$J^iJ^j=\varepsilon^{ijk}J^k-\delta^{ij}$$

Twistor space



Connection on the $\mathbb{C}\mathrm{P}^1$ bundle

$$Dt=\mathrm{d}t+p^+-\mathrm{i}tp^3+t^2p^-$$

su(2)-part of the Levi-Civita connection

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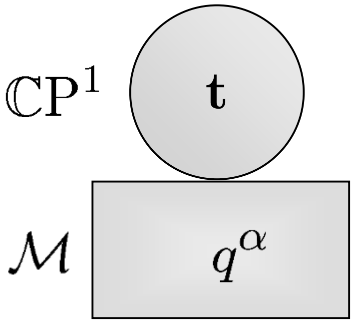

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Twistor space carries:

- *integrable* complex structure
- holomorphic *contact structure*

$$\frac{Dt}{t}\sim\mathcal{X}$$

holomorphic contact 1-form

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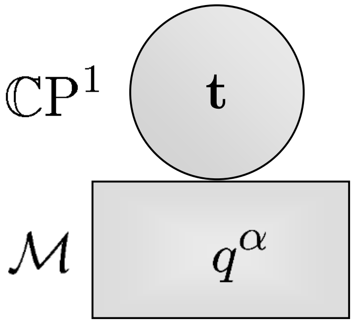

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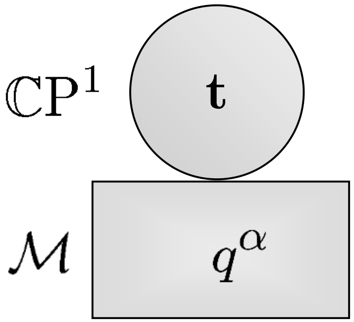

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Darboux coordinates

The metric can be encoded in Darboux coordinates as functions of q^α and t

D-instantons

S.A.,Pioline,Saueressig,
Vandoren ’08, S.A. ‘09

It is convenient to work with

$$\mathcal{X}_\gamma = e^{-2\pi \mathrm{i}\left(q_\Lambda \xi^\Lambda - p^\Lambda \tilde\xi_\Lambda\right)}$$

Classical
result

$$\mathcal{X}^{\rm sf}_\gamma(t) = e^{-\frac{\pi \mathrm{i}\tau_2}{2}\left(t^{-1}Z_\gamma - t\bar Z_\gamma\right) - 2\pi \mathrm{i}\langle\gamma, C\rangle}$$

$$\alpha^{\rm sf}(t) = -\frac{1}{2}\,\sigma + \text{polynomial in } t \text{ and } t^{-1}$$

Notations:

$\gamma = (p^\Lambda, q_\Lambda)$ D-brane charge

$\Xi = (\xi^\Lambda, \tilde\xi_\Lambda)$ vector of D.c.

$C = (\zeta^\Lambda, \tilde\zeta_\Lambda)$ vector of RR fields

$Z_\gamma = q_\Lambda u^\Lambda - p^\Lambda F_\Lambda(u)$ central charge

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Integral equations on Darboux coordinates

$$\mathcal{X}_\gamma = \mathcal{X}^{\rm sf}_\gamma \exp\left[\frac{1}{4\pi \mathrm{i}}\sum_{\gamma'} \Omega(\gamma')\langle\gamma,\gamma'\rangle \int_{\ell_{\gamma'}} \frac{\mathrm{d}t'}{t'}\frac{t+t'}{t-t'}\log\left(1-\mathcal{X}_{\gamma'}(t')\right)\right]$$

Donaldson-Thomas invariants

joins $t=0$ and $t=\infty$ along direction determined by $\arg Z_\gamma$

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rational Donaldson-Thomas invariants

joins $t=0$ and $t=\infty$ along direction determined by $\arg Z_\gamma$

$$\bar{\Omega}(\gamma) = \sum_{d|\gamma} \frac{1}{d^2} \Omega(\gamma/d)$$

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Notations:

$\gamma=(p^\Lambda,q_\Lambda)$ D-brane charge

$\Xi=(\xi^\Lambda,\tilde\xi_\Lambda)$ vector of D.c.

$C=(\zeta^\Lambda,\tilde\zeta_\Lambda)$ vector of RR fields

$Z_\gamma=q_\Lambda u^\Lambda-p^\Lambda F_\Lambda(u)$ central charge

Integral equations on Darboux coordinates

$$\mathcal{X}_\gamma=\mathcal{X}^{\rm sf}_\gamma\exp\left[\frac{-1}{4\pi\mathrm{i}}\sum_{\gamma'}\bar\Omega(\gamma')\langle\gamma,\gamma'\rangle\int_{\ell_{\gamma'}}\frac{\mathrm{d}t'}{t'}\,\frac{t+t'}{t-t'}\,\mathcal{X}_{\gamma'}(t')\right]$$

rational Donaldson-Thomas invariants

joins $t=0$ and $t=\infty$ along direction determined by $\arg Z_\gamma$

$$\bar\Omega(\gamma)=\sum_{d|\gamma}\frac{1}{d^2}\Omega(\gamma/d)$$

$$\alpha=\alpha^{\rm sf}+\frac{1}{16\pi^3\mathrm{i}}\sum_\gamma\bar\Omega(\gamma)\int_{\ell_\gamma}\frac{\mathrm{d}t'}{t'}\,\frac{t+t'}{t-t'}\left(1+q_\Lambda\xi^\Lambda(t')+\cdots\right)\mathcal{X}_\gamma(t')$$

S-duality in twistor space

All isometries of $\mathcal{M}$ can be lifted to *holomorphic* isometries on twistor space which are realized as *contact* transformations.

Classical twistor space

On $\mathbb{CP}^1$ fiber: $t \mapsto \dfrac{c\tau_2 + \left((c\tau_1+d)+|c\tau+d|\right)t}{(c\tau_1+d)+|c\tau+d| - c\tau_2 t}$

One can use instead:

$$z = \frac{t+\mathrm{i}}{t-\mathrm{i}} \qquad z \mapsto \frac{c\bar{\tau}+d}{|c\tau+d|}\, z$$

Holomorphic representation of SL(2,$\mathbb{Z}$)

$$\xi^0 \mapsto \frac{a\xi^0+b}{c\xi^0+d} \qquad \xi^a \mapsto \frac{\xi^a}{c\xi^0+d}$$

$$\tilde\xi_a \mapsto \tilde\xi_a + \frac{c}{2(c\xi^0+d)}\kappa_{abc}\xi^b\xi^c$$

$$\begin{pmatrix}\tilde\xi_0\\ \alpha\end{pmatrix} \mapsto \begin{pmatrix} d & -c\\ -b & a\end{pmatrix}\begin{pmatrix}\tilde\xi_0\\ \alpha\end{pmatrix} + \begin{matrix}\text{non-linear}\\ \text{terms}\end{matrix}$$

$$\mathcal{X} \mapsto \frac{\mathcal{X}}{c\xi^0+d}$$

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contact transformation

Quantum corrections should be consistent with this action up to a *local contact transformation*

proven for α'-corrections and D1-D(-1) instantons

S.A.,Saueressig ‘09

Contact potential

Complications with D3-instantons:

- One cannot solve integral equations for Darboux coordinates
- *Wall-crossing:* DT invariants $\bar\Omega(\gamma)$ jump at walls of marginal stability

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contact 1-form canonical (1,0)-form

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- Geometrically: gives the Kähler pot. on $\mathcal{Z}$

$$K_{\mathcal{Z}} = \log \frac{1+t\bar{t}}{|t|} + \mathrm{Re}\,\Phi(t)$$

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$$e^{\Phi} = \frac{\mathrm{i}\tau_2^2}{16}\left(\bar u^\Lambda F_\Lambda - u^\Lambda \bar F_\Lambda\right) - \frac{\chi_{\mathrm{CY}}}{192\pi} + \frac{\mathrm{i}\tau_2}{64\pi^2}\sum_\gamma \bar\Omega(\gamma)\int_{\ell_\gamma}\frac{\mathrm{d}t}{t}\left(t^{-1}Z_\gamma - t\bar Z_\gamma\right)\mathcal{X}_\gamma(t)$$

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S-duality transformation

$$\left.\begin{array}{l} \mathcal{X} \mapsto \dfrac{\mathcal{X}}{c\xi^0+d} \\[2ex] \dfrac{\mathrm{d}t}{t} \mapsto \dfrac{|c\tau+d|}{c\xi^0+d}\,\dfrac{\mathrm{d}t}{t} \end{array}\right\} \Longrightarrow \boxed{e^{\Phi} \mapsto \frac{e^{\Phi}}{|c\tau+d|}}$$

Quantum corrected mirror map

Way to proceed:

- Evaluate e^{Φ} in terms of type IIB fields →
- Apply S-duality

Requires:

- an approximation
- quantum mirror map

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- D3-instantons: D-brane charges of the form $\gamma = (0, p^a, q_a, q_0)$
- 2-instanton approx.: second order in DT invariants $\bar{\Omega}(\gamma)$
- large volume limit $t^a \to \infty$

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Quantum corrections to the mirror map:

(follows from the general formalism of S.A.,Pioline '12, S.A.,Banerjee '13)

$$\delta u^a = -\frac{\mathrm{i}}{8\pi^2\tau_2}\sum_{\gamma\in\Gamma_+}\sigma_\gamma\bar{\Omega}(\gamma)p^a\left[\int_{\ell_\gamma}\mathrm{d}z\,(1-z)\,\mathcal{X}_\gamma+\int_{\ell_{-\gamma}}\frac{\mathrm{d}z}{z^3}(1-z)\,\mathcal{X}_{-\gamma}\right]$$

$$\delta\zeta^a = 0$$

$$\delta\tilde{\zeta}_a = \frac{1}{4\pi^2}\,\kappa_{abc}t^b\sum_{\gamma\in\Gamma_+}\sigma_\gamma\bar{\Omega}(\gamma)p^c\,\mathrm{Im}\left(\int_{\ell_\gamma}\mathrm{d}z\,\mathcal{X}_\gamma\right)$$

$$\delta\tilde{\zeta}_0 = -\frac{1}{4\pi^2}\,\kappa_{abc}t^b\sum_{\gamma\in\Gamma_+}\sigma_\gamma\bar{\Omega}(\gamma)p^c\,\mathrm{Im}\int_{\ell_\gamma}\mathrm{d}z\left(b^a-\frac{\mathrm{i}}{2}\,t^a z\right)\mathcal{X}_\gamma$$

$$\delta\sigma = -\frac{1}{4\pi^2}\,\kappa_{abc}t^b\sum_{\gamma\in\Gamma_+}\sigma_\gamma\bar{\Omega}(\gamma)p^c\,\mathrm{Im}\int_{\ell_\gamma}\mathrm{d}z\left(c^a-4\mathrm{i}\tau_2 b^a-\left(\frac{\mathrm{i}}{2}\,\tau_1+3\tau_2\right)t^a z\right)\mathcal{X}_\gamma$$

The charge lattice

$$\Gamma_+ = \{\gamma = (0, p^a, q_a, q_0):\ (pt^2) \equiv \kappa_{abc}p^a t^b t^c > 0\}$$

Quantum corrected contact potential

Define the function:

$$\mathcal{F} = \frac{1}{4\pi^2} \sum_{\gamma\in\Gamma_+} \bar{\Omega}(\gamma) \int_{\ell_\gamma} \mathrm{d}z\, \mathcal{X}_\gamma$$

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$$= \sum_{p}\mathcal{F}^{(1)}_{p} + \sum_{p_1,p_2}\mathcal{F}^{(2)}_{p_1 p_2} + \cdots$$

Expansion of Darboux coordinates

$$\mathcal{X}_\gamma = \mathcal{X}^{\mathrm{cl}}_\gamma\left[1+\frac{1}{2\pi}\sum_{\gamma'\in\Gamma_+}\Omega(\gamma')\int_{\ell_{\gamma'}}\mathrm{d}z'\left((tpp') - \frac{\mathrm{i}\langle\gamma,\gamma'\rangle}{z'-z}\right)\mathcal{X}^{\mathrm{cl}}_{\gamma'}(z') + \cdots\right]$$

from mirror map

from integral equation

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⬇

$$\tilde{\mathcal{F}}_p = \mathcal{F}_p^{(1)} + \frac{1}{2} \sum_{p_1+p_2=p} \mathcal{F}_{p_1p_2}^{(2)}$$

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$$\Downarrow$$

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Modular covariant derivative

$$\mathcal{D}_{\mathfrak{h}}\ :\ f_{\mathfrak{h},\bar{\mathfrak{h}}} \mapsto f'_{\mathfrak{h}+2,\bar{\mathfrak{h}}}$$

D3-instanton contribution to the contact potential

$$\delta e^{\Phi} = \frac{\tau_2}{2}\,\mathrm{Re}\sum_p \mathcal{D}_{-\frac{3}{2}}\tilde{\mathcal{F}}_p - \frac{1}{8}\sum_{p_1,p_2}(p_1p_2t)\tilde{\mathcal{F}}_{p_1}\overline{\tilde{\mathcal{F}}_{p_2}}$$

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δe^{Φ} transforms as modular form of weight $\left(-\frac{1}{2},-\frac{1}{2}\right)$

provided $\tilde{\mathcal{F}}_p$ transforms as modular form of weight $\left(-\frac{3}{2},\frac{1}{2}\right)$

DT and MSW

DT invariants depend on Kähler moduli via wall crossing.

Define $\Omega^{\mathrm{MSW}}_{\gamma} = \Omega(\gamma; z^a_\infty(\gamma))$

attractor point

counts states in SCFT
constructed in Maldacena,Strominger,Witten '97

$$z^a_\infty(\gamma) = \lim_{\lambda \to +\infty} (-\kappa^{ab} q_b + \mathrm{i}\lambda p^a)$$

$\kappa_{ab} = \kappa_{abc} p^c$
quadratic form
of signature
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Manschot '09

Decomposition of DT in terms of MSW

$$\bar\Omega(\gamma; z^a) = \bar\Omega^{\mathrm{MSW}}_{\gamma} + \frac{1}{2} \sum_{\substack{\gamma_{1,2}\in\Gamma_+ \\ \gamma_1+\gamma_2=\gamma}} (-1)^{\langle\gamma_1,\gamma_2\rangle} \langle\gamma_1, \gamma_2\rangle \Delta^t_{\gamma_1\gamma_2} \bar\Omega^{\mathrm{MSW}}_{\gamma_1} \bar\Omega^{\mathrm{MSW}}_{\gamma_2} + \cdots$$

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sign factor

Crucial property

MSW invariants are invariant under spectral flow symmetry

spectral flow

$q_a \mapsto q_a - \kappa_{ab}\epsilon^b$

$q_0 \mapsto q_0 - \epsilon^a q_a + \frac{1}{2}\,\kappa_{ab}\epsilon^a\epsilon^b$

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charge decomposition $(q_0,q_a)\to(\hat q_0,\mu_a,\epsilon^a)$

$\hat q_0\equiv q_0-\frac{1}{2}\,\kappa^{ab}q_aq_b$ – invariant charge

$q_a=\mu_a+\frac{1}{2}\,\kappa_{ab}p^b+\kappa_{ab}\epsilon^b$

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$$\bar\Omega(\gamma; z^a) = \bar\Omega^{\rm MSW}_{\gamma} + \frac12 \sum_{\substack{\gamma_{1,2}\in\Gamma_+\\ \gamma_1+\gamma_2=\gamma}} (-1)^{\langle\gamma_1,\gamma_2\rangle}\langle\gamma_1,\gamma_2\rangle \Delta^t_{\gamma_1\gamma_2}\bar\Omega^{\rm MSW}_{\gamma_1}\bar\Omega^{\rm MSW}_{\gamma_2} + \cdots$$

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$q_a \mapsto q_a - \kappa_{ab}\epsilon^b$

$q_0 \mapsto q_0 - \epsilon^a q_a + \frac12\,\kappa_{ab}\epsilon^a\epsilon^b$

$$\Omega^{\rm MSW}_{\gamma} = \Omega^{\rm MSW}_{p,\mu}(\hat q_0)$$

charge decomposition $(q_0, q_a) \to (\hat q_0, \mu_a, \epsilon^a)$

$\hat q_0 \equiv q_0 - \frac12\,\kappa^{ab}q_a q_b$ – invariant charge

$q_a = \mu_a + \frac12\,\kappa_{ab}p^b + \kappa_{ab}\epsilon^b$

BPS partition function

Define $\mathcal{Z}_p(\tau, z^a, c^a) = \sum_{q_\Lambda} \bar{\Omega}(\gamma)\, e^{-2\pi\tau_2|Z_\gamma| - 2\pi i\tau_1\left(q_0 + b^a q_a + \frac{1}{2}b^2\right) + 2\pi i c\cdot\left(q + \frac{1}{2}b\right)}$

Boltzmann factor

couplings to axions

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$$\mathcal{F}_p^{(1)} = \frac{1}{4\pi^2}\sum_{\gamma\in\Gamma_+}\bar{\Omega}(\gamma)\int_{\ell_\gamma}\mathrm{d}z\,\mathcal{X}_\gamma^{\mathrm{cl}} = \frac{e^{2\pi\mathrm{i}p^a\tilde{c}_a}}{4\pi^2\sqrt{2\tau_2(pt^2)}}\,\mathcal{Z}_p$$

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Expand in terms of MSW invariants $\mathcal{Z}_p = \sum_{n\geq 1} \mathcal{Z}_p^{(n)}$

$\mathcal{Z}_p^{(1)}$

Gaiotto,Strominger,Yin ’06
de Boer,Cheng,Dijkgraaf,
Manschot,Verlinde ‘07

elliptic genus of MSW SCFT

Jacobi form of weight $\left(-\frac{3}{2}, \frac{1}{2}\right)$

the right modular behavior of the contact potential in the 1-instanton (MSW) approx.

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Boltzmann factor

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Expand in terms of MSW invariants $\mathcal{Z}_p = \sum_{n\geq 1}\mathcal{Z}^{(n)}_p$

$$\mathcal{Z}^{(1)}_p = \sum_\mu h_{p,\mu}(\tau)\,\theta_{p,\mu}$$

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Jacobi form of weight $\left(-\frac{3}{2},\frac{1}{2}\right)$

the right modular behavior of the contact potential in the 1-instanton (MSW) approx.

$$\theta_{p,\mu} = \sum_{k\in\mathbb{Z}^{b_2}+\mu+\frac{1}{2}p} q^{-\frac{1}{2}(k+b)^2_-}\,\bar{q}^{\frac{1}{2}(k+b)^2_+}\,e^{-2\pi\mathrm{i}c\cdot(k+\frac{1}{2}b)}$$

$(q = e^{2\pi\mathrm{i}\tau})$

Seigel-Narain theta series ― vector valued modular form of weight $\left(\frac{b_2-1}{2},\frac{1}{2}\right)$

$$h_{p,\mu}(\tau) = \sum_{\hat{q}_0\leq\hat{q}_0^{\max}}\bar{\Omega}_{p,\mu}(\hat{q}_0)\,e^{-2\pi\mathrm{i}\hat{q}_0\tau}$$

generating func. of MSW invariants ― vector valued modular form of weight $\left(-\frac{b_2}{2}-1, 0\right)$

2-instanton contribution

The full 2-instanton result

double theta series

$$\tilde{\mathcal{F}}_p = \frac{e^{2\pi \mathrm{i} p^a \tilde{c}_a}}{4\pi^2\sqrt{2\tau_2(pt^2)}} \left[\mathcal{Z}_p + \frac{1}{2} \sum_{p_1+p_2=p} \sum_{\mu_1,\mu_2} h_{p_1,\mu_1}\, h_{p_2,\mu_2}\, \Psi^{(+)}_{p_1,p_2,\mu_1,\mu_2} \right]$$

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$$\mathcal{Z}_p^{(1)} \qquad \mathcal{Z}_p^{(2)} \qquad = \widehat{\mathcal{Z}}_p =$$

$$\sum_\mu h_{p,\mu}\,\theta_{p,\mu} + \frac{1}{2}\sum_{p_1+p_2=p}\;\sum_{\mu_1,\mu_2} h_{p_1,\mu_1}\, h_{p_2,\mu_2}\left(\Psi_{p_1,p_2,\mu_1,\mu_2} + \Psi^{(+)}_{p_1,p_2,\mu_1,\mu_2}\right)$$

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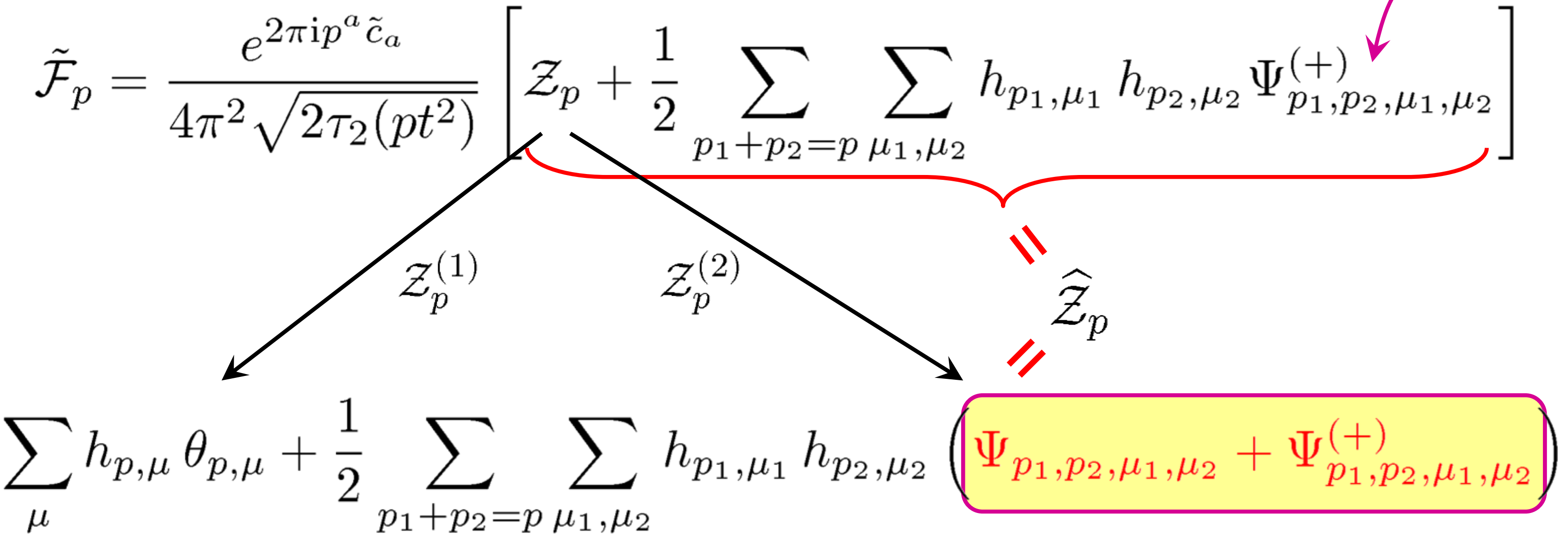

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Results of Manschot ‘09

- Ψ is a *mock* Jacobi form
- There exists a modular completion of weight $\left(b_2+\frac{1}{2},\frac{1}{2}\right)$

$$\widehat{\Psi} = \Psi + \Psi^{(+)} + \Psi^{(-)}$$

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- Ψ is a *mock* Jacobi form
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There seems to be a clash between modular symmetry of $\widehat{\mathcal{Z}}_p$ and $\widehat{\Psi}$

Mock modularity

$$\widehat{\mathcal{Z}}_p = \sum_{\mu} h_{p,\mu}\,\theta_{p,\mu} + \frac{1}{2}\sum_{p_1+p_2=p}\sum_{\mu_1,\mu_2} h_{p_1,\mu_1}\,h_{p_2,\mu_2}\left(\widehat{\Psi}_{p_1,p_2,\mu_1,\mu_2} - \Psi^{(-)}_{p_1,p_2,\mu_1,\mu_2}\right)$$

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Proposal: **when the divisor given by p^a is reducible, $h_{p,\mu}$ transforms as a mock modular form with the modular completion given by**

$$\boxed{\widehat{h}_{p,\mu} = h_{p,\mu}(\tau) - \frac{1}{2} R_{p,\mu}(\tau,\bar{\tau}) + \cdots}$$

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- consistent with some results for non-compact CYs
- the elliptic genus $\mathcal{Z}^{(1)}_p = \sum h_{p,\mu}\theta_{p,\mu}$ is only mock modular
- can be thought as a result of the continuum of states in the spectrum

Darboux coordinates

Darboux coordinates are analyzed in the limit $z \to 0$ with zt^a kept constant

All of them can be expressed in terms of two functions: $\tilde{\mathcal{F}}_p$ and $\tilde{\mathcal{J}}_p(z)$

$$\mathcal{J}(z) = \frac{1}{4\pi^2} \sum_{\gamma \in \Gamma_+} \bar{\Omega}(\gamma) \int_{\ell_\gamma} \frac{\mathrm{d}z'}{z-z'} \mathcal{X}_\gamma(z') = \sum_p \mathcal{J}_p^{(1)}(z) + \sum_{p_1,p_2} \mathcal{J}_{p_1 p_2}^{(2)}(z) + \cdots$$

$$\Downarrow$$

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But this is not true even at one instanton level!

$$\underset{\substack{\text{represented as}\\ \text{a period integral}}}{\mathcal{J}_p^{(1)}(z)} \mapsto \frac{1}{c\tau+d}\left(\mathcal{J}_p^{(1)}(z) + \underbrace{\sum_\mu h_{p,\mu} \int_{-d/c}^{-\mathrm{i}\infty} \frac{\mathrm{d}w}{(w-\tau)^{1/2}} \overset{\text{shadow}}{\Theta_{p,\mu}}(w,\tau,z)}_{\text{modular anomaly}}\right)$$

Removing anomaly

The idea: the modular completion can be generated by a local contact transformation

local coordinate transformation preserving the contact 1-form up to a factor

All such transformations can be generated by holomorphic functions – "*Hamiltonians*" $\mathcal{G}(\xi, \tilde{\xi}, \alpha)$

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In our case:

$$\mathcal{G}=\sum_p e^{2\pi \mathrm{i} p^a\tilde\xi_a} f_p(\xi)$$

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holomorphic

$$\begin{cases}\mathcal{G}=\sum\limits_p \mathcal{G}^{(1)}_p+\sum\limits_{p_1,p_2}\mathcal{G}^{(2)}_{p_1p_2}+\cdots \\ \tilde{\mathcal{G}}_p=\mathcal{G}^{(1)}_p+\frac{1}{2}\sum\limits_{p_1+p_2=p}\mathcal{G}^{(2)}_{p_1p_2}\end{cases}$$

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At 1-instanton level, the anomaly is canceled by the *indefinite theta series*

$$f_p(\xi) = \sum_{q_\Lambda} \Delta^{tt'}_q e^{-2\pi \mathrm{i} q_\Lambda \xi^\Lambda}$$

$$\sim e^{-2\pi \mathrm{i} \hat{q}_0 \xi^0} e^{-\pi \mathrm{i} \xi^0 \kappa^{ab} q_a q_b}$$

quadratic form of indefinite signature

$$\Delta^{tt'}_q = \frac{1}{2}\left[\mathrm{sgn}\left(\mathrm{Im}\left(z + \frac{\mathrm{i}(q\cdot t)}{(pt^2)}\right)\right) - \mathrm{sgn}\left(\mathrm{Im}\left(z + \frac{\mathrm{i}(q\cdot t')}{2(ptt')}\right)\right)\right]$$

t'^a – vector belonging to the boundary of the Kähler cone: $(pt'^2) = 0$

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What is this vector?

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What is f_p for 2-instantons?

Conclusions

Main results:

- The elliptic genus for a *reducible* divisor of CY is *mock modular* with the modular completion resulting from $\widehat{h}_{p,\mu} = h_{p,\mu}(\tau) - \frac{1}{2} R_{p,\mu}(\tau, \bar{\tau})$
- Relation between the *contact potential* of the twistor formalism and (the modular completion of) *the BPS partition function*
- Modularity of Darboux coordinates
  - Anomaly cancellation by indefinite theta functions
  - New modular forms from (double) integrals on the twistor space
- Instanton corrected mirror map

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Some open questions:

- Modular completion for Darboux coordinates at 2-instanton level
- The nature of the light-like vector t'^a
- Derivation of $R_{p,\mu}$ from CFT
- Consequences for the counting of states of BPS black holes
- Extension beyond our approximation (large volume & 2-instanton)