witten index, wall-crossing, and threshold bound states

PILJIN YI
Korea Institute for Advanced Study

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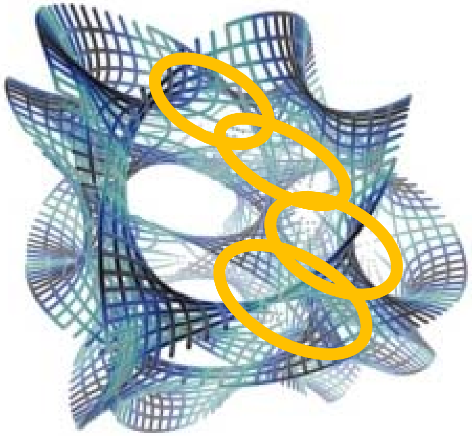
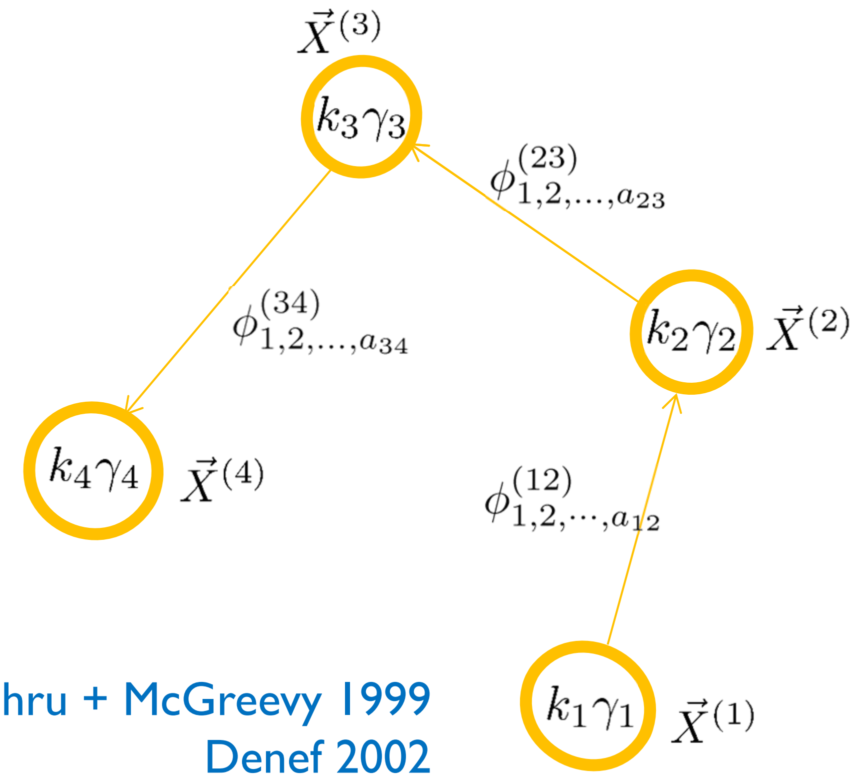
S.J. Lee + P.Y., 2016
H.Kim + S.J. Lee + P.Y. 2015
K.Hori + H.Kim + P. Y. 2014
S.J.Lee + Z.L.Wang + P.Y. 2012/2013

how do we count interesting objects in this picture ?

$$R^{1+3} \quad \times$$



e.g., quiver quantum mechanics on wrapped D-branes

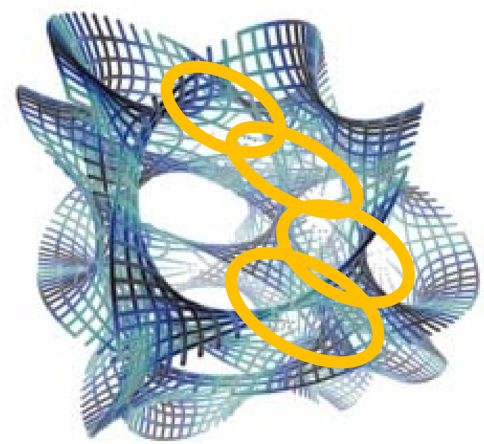


$$\vec{X}^{(1)} \qquad \vec{X}^{(2)} \qquad \vec{X}^{(3)} \qquad \vec{X}^{(4)}$$

$$U(k_1) \times U(k_2) \times U(k_3) \times U(k_4)$$

$$\phi^{(12)}_{1,2,\cdots,a_{12}} \qquad \phi^{(23)}_{1,2,\ldots,a_{23}} \qquad \phi^{(34)}_{1,2,\ldots,a_{34}}$$

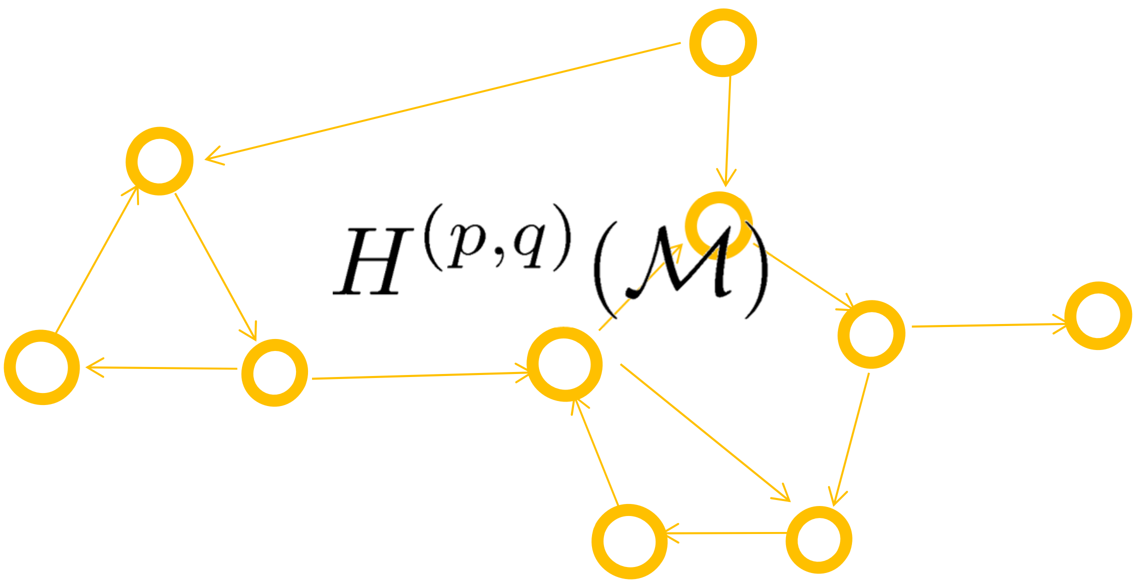
$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$



Kachru + McGreevy 1999
Denef 2002

witten index via path integral

cohomology



$$H^{(p,q)}(\mathcal{M})$$

wall-crossing

quiver invariants

~ J=0 single center black holes

wall-crossing algebra & rational invariants

Kontsevich,Soibelman 2008
cf) Gaiotto,Moore,Neitzke 2008

ingredients

$$\gamma = (g, e) \ \in \ \mathbf{Z}^r \times \mathbf{Z}^r$$

$$V_\gamma V_{\gamma'} - V_{\gamma'} V_\gamma = (-1)^{\langle \gamma, \gamma' \rangle} \langle \gamma, \gamma' \rangle V_{\gamma + \gamma'} \qquad \langle \gamma, \gamma' \rangle = e \cdot g' - g \cdot e'$$

ingredients

$$\gamma = (g, e) \ \in \ \mathbf{Z}^r \times \mathbf{Z}^r$$

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$$\mathcal{I}(\gamma) = 0, \pm 1, \pm 2, \ldots$$

the degeneracy of a given species of charged particle

ingredients

$$\gamma = (g, e) \ \in \ \mathbf{Z}^r \times \mathbf{Z}^r$$

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$$\mathcal{I}(\gamma) = 0, \pm 1, \pm 2, \ldots$$

the degeneracy of a given species of charged particle

Calabi-Yau 3-fold viewpoint

intersection number between the two cycles

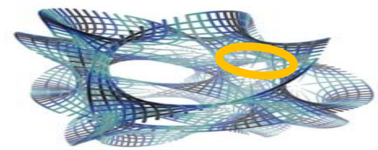
$$\langle \gamma, \gamma' \rangle$$

Euler number of the moduli space of the calibrated cycle

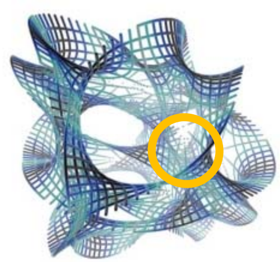
$$\mathcal{I}(\gamma)$$

Kontsevich-Soibelman wall-crossing algebra

$$K_\gamma \equiv \exp\left(\sum_{n=1}^{\infty} \frac{V_{n\gamma}}{n^2}\right)$$

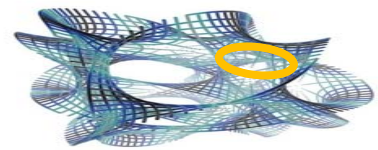


+ side $$\prod_{\gamma} K_\gamma^{\mathcal{I}^+(\gamma)} = {\prod_{\gamma'}}' K_{\gamma'}^{\mathcal{I}^-(\gamma')}$$ - side

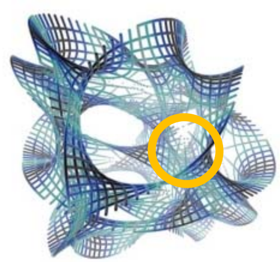


for example, A_1 type Argyres-Douglas theory

$$V_{\gamma_1} V_{\gamma_2} - V_{\gamma_2} V_{\gamma_1} = -V_{\gamma_1+\gamma_2}$$

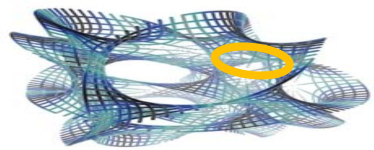


$$K_{\gamma_1} K_{\gamma_2} = K_{\gamma_2} K_{\gamma_1+\gamma_2} K_{\gamma_1}$$

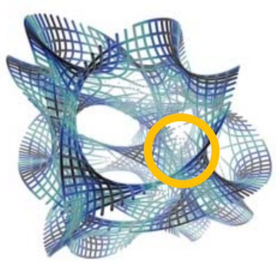


or, pure SU(2) Seiberg-Witten theory

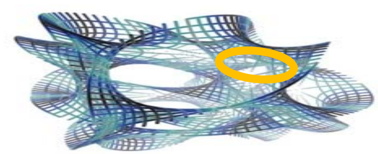
$$V_{\gamma_1} V_{\gamma_2} - V_{\gamma_2} V_{\gamma_1} = 2 V_{\gamma_1 + \gamma_2}$$



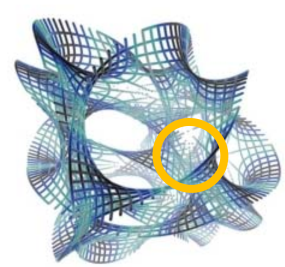
$$K_{\gamma_1} K_{\gamma_2} = K_{\gamma_2} K_{\gamma_1 + 2\gamma_2} K_{2\gamma_1 + 3\gamma_2} \cdots \cdots K^{-2}_{\gamma_1 + \gamma_2} \cdots K_{3\gamma_1 + 2\gamma_2} K_{2\gamma_1 + \gamma_2} K_{\gamma_1}$$



note that, alternatively



+ side $$\prod_{\gamma} \left(e^{V_\gamma}\right)^{\omega^{+}(\gamma)} = \prod_{\gamma}{}' \left(e^{V_\gamma}\right)^{\omega^{-}(\gamma)}$$ - side

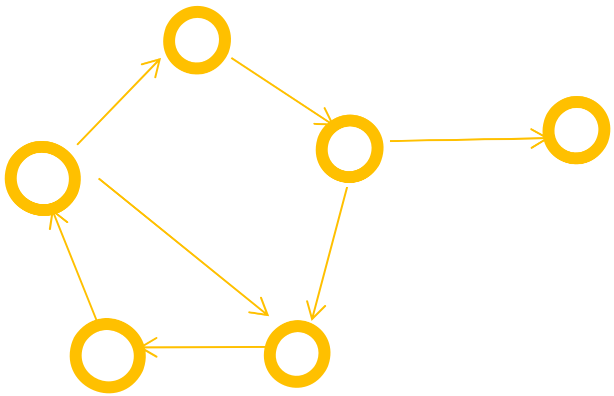


$$\omega(\gamma) \equiv \sum_{p|\gamma} \mathcal{I}(\gamma/p)/p^2$$

via the rational invariants

$$\omega(\gamma) \equiv \sum_{p|\gamma} \frac{\mathcal{I}(\gamma/p)}{p^2}$$

this makes a prominent appearance, in a refined form, again in quiver quantum mechanics



$$\omega(\Gamma) \equiv \sum_{p|\Gamma} \frac{\mathcal{I}(\Gamma/p)}{p^2}$$

$$\omega(\Gamma;\mathbf{y}) \equiv \sum_{p|\Gamma} \mathcal{I}(\Gamma/p;\mathbf{y}^p) \cdot \boxed{\frac{\mathbf{y}-\mathbf{y}^{-1}}{p(\mathbf{y}^p-\mathbf{y}^{-p})}}$$

$$\Gamma = \sum_i n_i \gamma_i$$

one can derive wall-crossing/state counting more directly
by computing index of the relevant susy quantum mechanics

Bak,Lee,Lee,P.Y, 1999
Gauntlett, Kim, Park, P.Y. 2000
Denef 2002

one can derive wall-crossing/state counting more directly
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leading to an explicit & constructive proof of / solution to KS

Manschot, Pioline, Sen /
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Sen 2010-2011

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Kim, Park, Wang, P.Y. /
Sen 2010-2011

also to wall-crossing-safe sector (GLSM/quiver invariant) and, from this, even the entire Hodge diamonds of the moduli space

Lee, Wang, P.Y. /
Bena, Berkooz, deBoer, El Showk, Van den Bleeken /
Manschot, Pioline, Sen
2012-2013
Hori, Kim, P.Y / Lee, Kim, P.Y.
2014-2016

Witten index via localization

$$\{Q, Q^\dagger\} = 2H$$

$$\{Q, (-1)^F\} = 0$$

$$[Q, G_F] = 0$$

refined Witten index of d=1 N≥2 GLSM

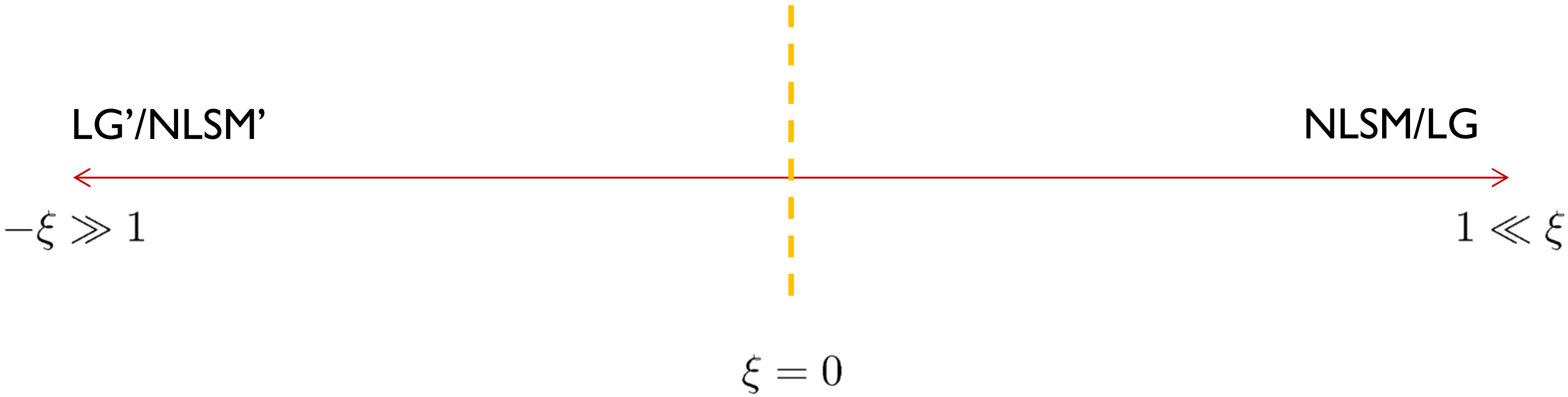
$$\mathcal{I}(x) \equiv \mathrm{Tr}\left[(-1)^F x^{G_F} e^{-\beta H}\right]$$

K.Hori + H.Kim + P.Y. 2014

1d N=2 Gauged Linear Sigma Models

$U(1)_R$

gauge fields	$(A_\mu, \lambda_-, D)^a$	FI constants ξ^i for U(1)'s
chirals	$(X, \psi_+)^I$	
fermi	$(\psi_-, F)^I$	



we will mainly show processes & results for
1d N=4 Gauged Linear Sigma Models

$SU(2)_R \times U(1)_R$ gauge fields $(A_\mu, \lambda_\alpha, \sigma, D)^a$ FI constants ξ^i for U(1)'s

$J_{1,2,3}$ R chirals $(X, \psi_\alpha, F)^I$

LG'/NLSM' NLSM/LG

$-\xi \gg 1$ $1 \ll \xi$

$\xi = 0$

$$\{Q, Q^\dagger\} = 2H$$

$$\{Q, (-1)^{2J_3}\} = 0$$

$$[Q, G_F] = 0$$

$$[Q, R + J_3] = 0$$

refined Witten index of d=1 N≥4 GLSM

$$\mathcal{I}(\mathbf{y}; x) \equiv \mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2(R+J_3)}x^{G_F}e^{-\beta H}\right]$$

K.Hori + H.Kim + P.Y. 2014

N≥4 compact and geometric $\rightarrow$

$$\mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2(R+J_3)}x^{G_F}e^{-\beta H}\right]$$

$$=\sum_{p,q}(-1)^{p+q-d}\mathbf{y}^{2p-d}\dim H^{(p,q)}(\mathcal{M})$$

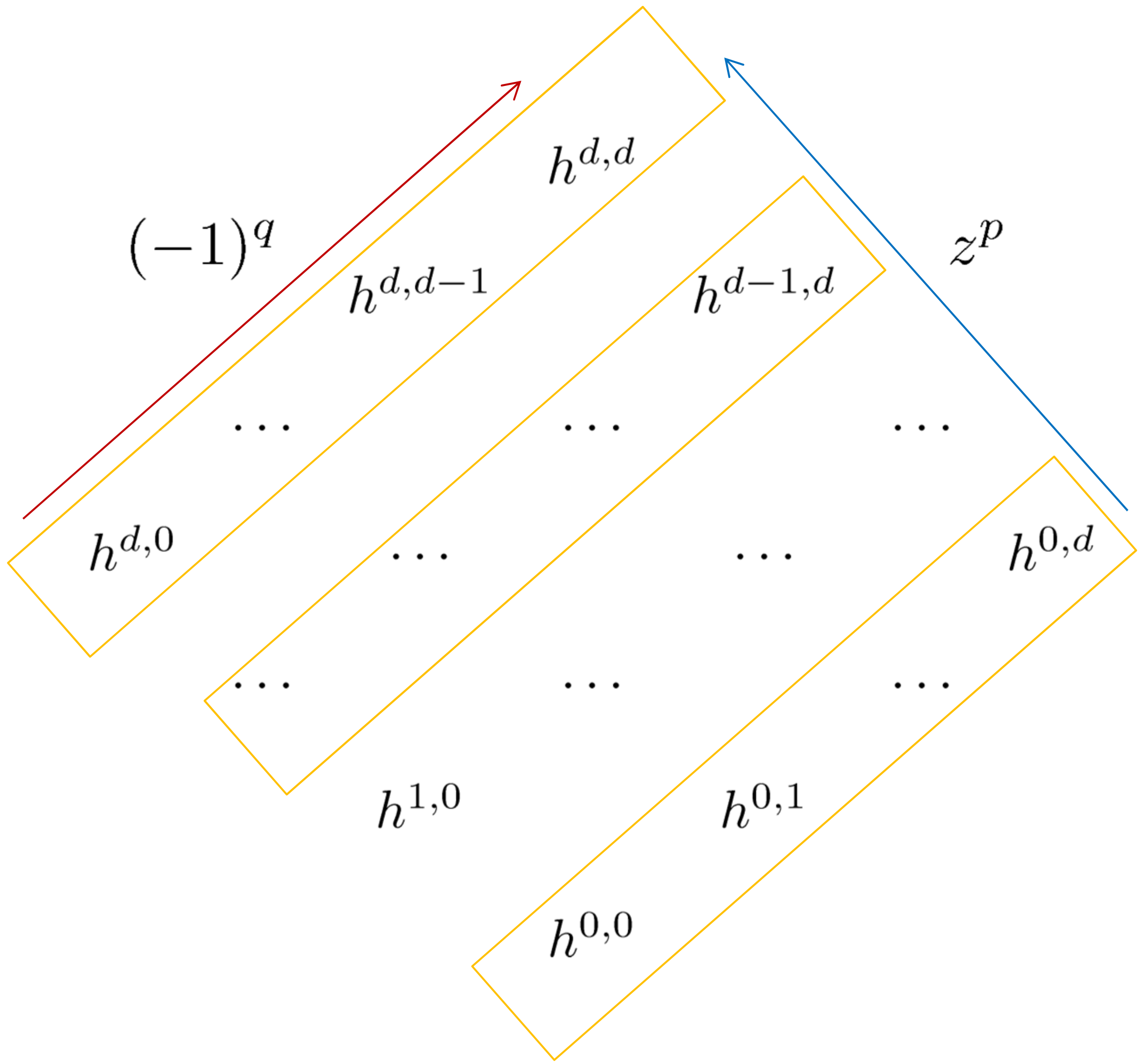
$$=(-\mathbf{y})^{-d}\mathcal{I}_{\text{Hirzebruch}}(z=-\mathbf{y}^2)$$

$\boxed{x\text{-independent}}$

$$\mathcal{I}_{\text{Hirzebruch}}(z) = \sum_p z^p \sum_q (-1)^q h^{p,q}(\mathcal{M})$$

$$\begin{array}{ccccccc}
 & & & h^{d,d} & & & \\
 & & h^{d,d-1} & & h^{d-1,d} & & \\
 & \cdots & & \cdots & & \cdots & \\
h^{d,0} & & \cdots & & \cdots & & h^{0,d} \\
 & \cdots & & \cdots & & \cdots & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array}$$

$$\mathcal{I}_{\text{Hirzebruch}}(z) = \sum_p z^p \sum_q (-1)^q h^{p,q}(\mathcal{M})$$



the most complete and general method known so far is via localization of the path integral

$$Z_t(\mathcal{O}) = \int e^{-S-tQ\Psi}\mathcal{O}$$

$$\partial_t Z_t(\mathcal{O}) = 0$$

$$[Q, \mathcal{O}] = 0$$

$$Q^2 = G$$

$$[G, \mathcal{O}] = 0$$

the localization we perform is a deformation

$$e^2 \to 0$$

$$\mathcal{L}_{\mathrm{vector}} = \frac{1}{e^2}\,\mathrm{Re}\left(\int d\theta^2\ \mathrm{tr}\, W_\alpha W^\alpha\right)$$

$$\mathcal{L}_{\mathrm{chiral}} = \frac{1}{g^2}\ \int d\theta^2 d\bar{\theta}^2\ \mathrm{tr}\,\bar{\Phi} e^V \Phi$$

$$\mathcal{L}_{\mathrm{usperpotential}} = \ \int d\theta^2 W(\Phi) + c.c.$$

$$\mathcal{L}_{\mathrm{FI}} = \xi\ \int d\theta^2 d\bar{\theta}^2 \mathrm{tr}\, V$$

Benini + Eager + Hori + Tachikawa 2013
Hori + Kim + P.Y. 2014

the localization we perform is a deformation

$$e^2 \to 0$$

$$\Omega \equiv \lim_{e^2\to 0} \mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta Q^2}\right] \qquad [Q, J_3+R]=0$$

$$= \lim_{e^2\to 0}\int_{\text{periodic}} [dX\cdots d\phi\cdots]\; e^{-\int_0^\beta d\tau \mathcal{L}_E}\Big|_{\partial_\tau\to\partial_\tau+(2J_3+2R)\log(\mathbf{y})/\beta+\cdots}$$

cf) $$\mathcal{I} \equiv \lim_{\beta\to\infty} \mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta Q^2}\right]$$

localization

$$e^2 \to 0$$

$$\Omega \equiv \lim_{e^2 \to 0} \mathrm{Tr}\left[(-1)^{2J_3} \mathbf{y}^{2J_3+2R} x^{G_F} e^{-\beta Q^2}\right] \qquad u = A_3 + iA_\tau \Big|_{\text{zeromode}}^{\text{Cartan}}$$

$$= \int_{M_u} du\, d\bar{u} \int_{\mathbf{R}+i\delta} dD \left[h(u,\bar{u};D) \cdot g(u,\bar{u};D) \cdot e^{-\frac{D^2}{e^2}+i\xi D}\right]$$

zero mode
of gauge multiplets

from integral over
gaugino zero mode

one-loop determinants
of everything else

localization

$$e^2 \to 0$$

$$\Omega \equiv \lim_{e^2 \to 0} \mathrm{Tr}\left[(-1)^{2J_3} \mathbf{y}^{2J_3+2R} x^{G_F} e^{-\beta Q^2}\right] \qquad u = A_3 + iA_\tau \Big|_{\text{zeromode}}^{\text{Cartan}}$$

$$= \int_{M_u} du\, d\bar{u} \int_{\mathbf{R}+i\delta} dD \left[h(u,\bar{u};D) \cdot g(u,\bar{u};D) \cdot e^{-\frac{D^2}{e^2}+i\xi D}\right]$$

$$= \int_{\partial M_u} du \int_{\mathbf{R}+i\delta} \frac{dD}{D}\, g(u,\bar{u};D) \cdot e^{-\frac{D^2}{e^2}+i\xi D}$$

$$g(u,\bar{u};D) \sim \prod_Q \prod_n \frac{(2\pi n i + Qu - (R-2)\log(\mathbf{y}+\cdots)) \cdot (-2\pi n i + Qu - R\log(\mathbf{y}) + \cdots)}{|2\pi n i + Qu - R\log(\mathbf{y}) + \cdots|^2 - iQD}$$

localization

$$e^2 \to 0$$

$$\Omega \equiv \lim_{e^2\to 0} \mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta Q^2}\right]$$

$$= \int_{\partial M_u} du \int_{\mathbf{R}+i\delta} \frac{idD}{D}\, g(u,\bar{u};D)\cdot e^{-\frac{D^2}{e^2}+i\xi D}$$

$$u = A_3 + iA_\tau\Big|_{\text{zeromode}}^{\text{Cartan}} \qquad M_u = (C^*)^{\text{rank}} \setminus \cup H_*^Q$$

Hori + Kim + P.Y. 2014

scale up FI to send $e\xi$ to infinite,

then, after a long, long, long song and dance, ……

reduces to a contour integral of JK type,
which, in the presence of FI constant, looks like

$$\Omega \equiv \lim_{e^2\to 0} \mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta Q^2}\right] = \sum \text{JK-Res}_{\eta:\{Q_i\}}\, g(u;0)$$

$$\partial M_u = \partial M_\infty + \cup_Q \partial\Delta^Q \qquad M_u = (C^*)^{\mathrm{rank}} \setminus \cup H_*^Q$$

$$\{Q_i\} = \{Q^{\mathrm{chiral}}\} \cup \{Q^{\mathrm{vector}}\} \cup \{Q_\infty = -\xi\}$$

$$\text{JK-Res}_{\eta:\{Q_i\}} \frac{d^r u}{(Q_1\cdot u)(Q_2\cdot u)\cdots(Q_r\cdot u)} = \left\{ \begin{array}{cc} \frac{1}{|\mathrm{Det}Q|} & \eta = \sum a_i^{>0} Q_i \\ & \\ 0 & \text{otherwise} \end{array} \right\}$$

Hori + Kim + P.Y. 2014

can be simplified further if the FI constant is generic

$$\Omega \equiv \lim_{e^2\to 0} \mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta Q^2}\right] = \sum \text{JK-Res}_{\xi:\{Q_i\}}g(u;0)$$

$$\partial M_u = \cancel{\partial M_\infty} + \cup_Q \partial\Delta^Q$$

$$\{Q_i'\} = \{Q^{\text{chiral}}\} \cup \{Q^{\text{vector}}\} \cup \{\cancel{Q_\infty = -\xi}\}$$

Hori + Kim + P.Y. 2014
Szenes + Vergne 2004
Brion + M.Vergne 1999
Jeffrey + Kirwan 1993

the derivation is closely related to that for 2d elliptic genus when the 2d version of GLSM is free of axial anomaly

$$\mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_A+2J_V}\cdots\right] \quad = \quad \int_{\mathrm{periodic}}[dX\cdots d\phi\cdots]\, e^{-S_E^{\mathbf{y}}+\cdots}$$

but with very different behavior in the end

$$M_u=(C^*)^{\mathrm{rank}}\setminus\cup H_*^Q \qquad \text{vs.} \qquad M_u=\left(T^2\right)^{\mathrm{rank}}\setminus\cup H_*^Q$$

2d GLSM Elliptic Genera

Benini + Eager + Hori + Tachikawa / Gadde + Gukov 2013

$\xi < 0$ $\xi = 0$ $0 < \xi$

1d GLSM Equivariant Index

Hori + Kim + P.Y. 2014

N=4 CP(N -1)

chirals	$U(1)$	$U(N)_F$
X	1	N

null

$\xi < 0$

$0 < \xi$

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 1 & & 0 & & 0 & \\
0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
 & 0 & & 0 & & 1 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}$$

$N = 6$

quintic CY3 hypersurface in CP4

	P	$X_{1,2,3,4,5}$
$U(1)$	-5	1

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array} \qquad \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

$\xi < 0$ $\qquad$ $0 < \xi$

N=4 rank 2 GLSM Q.M. for CY3 in WCP$_{(11222)}$

	P	$X_{1,2}$	$Y_{1,2,3}$	Z
$U(1)_1$	-4	0	1	1
$U(1)_2$	0	1	0	-2

hybrid

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 1 & & 86 & & 86 & & 1 \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array}$$

geometric

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 2 & & 0 & \\ 1 & & 86 & & 86 & & 1 \\ & 0 & & 2 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Landau-Ginsburg

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 1 & & 83 & & 83 & & 1 \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array}$$

orbifold

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 83 & & 83 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

$\mathcal{I}$ **vs.** Ω

but all four pieces are individually Q-exact for some supercharge Q

$$\mathcal{L}_{\text{vector}} = \frac{1}{e^2}\,\mathrm{Re}\left(\int d\theta^2\ \mathrm{tr}\, W_\alpha W^\alpha\right)$$

$$\mathcal{L}_{\text{chiral}} = \frac{1}{g^2}\ \int d\theta^2 d\bar{\theta}^2\ \mathrm{tr}\,\bar{\Phi} e^V \Phi$$

$$\mathcal{L}_{\text{usperpotential}} = \ \int d\theta^2 W(\Phi) + c.c.$$

$$\mathcal{L}_{\text{FI}} = \xi\ \int d\theta^2 d\bar{\theta}^2 \mathrm{tr}\, V$$

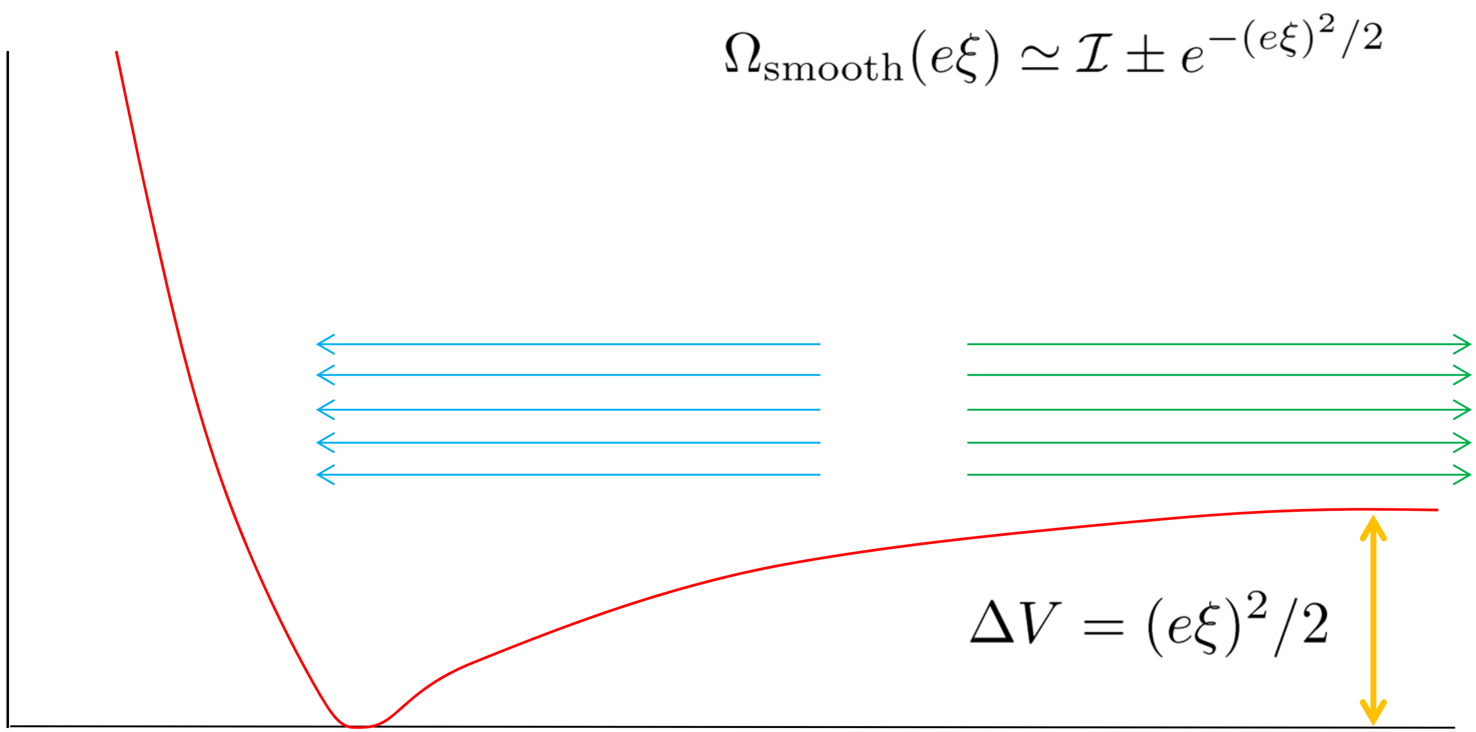
Hori + Kim + P.Y. 2014

so, whatever happened to the subsequent ξ-independence?

such a naïve invariance argument always assumes
“small” deformation of the parameters,
meaning, nothing drastic should happen

however, vanishing FI constants always implies new
asymptotic runaway direction along vector multiplets,
invalidating Q-exactness across $\xi = 0$

nonintegral contributions from the continuum, interpolating across $\xi = 0$ which is why we had to scale up $e\xi$



this reminds us of many subtleties that can appear
if such an asymptotic direction is unavoidable

for example, the entire classes of ADHM or of D-brane probe theories for noncompact Calabi-Yau's fall under this category

can we still count the relevant Witten index,
say, under some physical boundary condition such as L2,
reliably via this type of localization computation ?

the only generic answer to the last question has to be "NO"

yet, this never stopped people from computing Ω for problems with noncompact dynamics, such as ADHM, where one is forced to introduce flavor chemical potentials examples displayed above, where the spectrum is discrete, flavor chemical potentials were merely innocuous tools

$$\begin{aligned} &\mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta H}\right] \\ &= \mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}e^{-\beta H}\right] \end{aligned}$$

chemical potentials translated to extra mass terms,
so represents huge deformation for theories with continuum
as seen easily here for a single free chiral theory

$$\mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta H}\right]$$

e.g., NLSM onto C

$$\Omega = \frac{x^{1/2}\mathbf{y}^{-1} - \mathbf{y}x^{-1/2}}{x^{1/2} - x^{-1/2}}$$

chemical potentials translated to extra mass terms,
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as seen easily here for a single free chiral theory

the result of the computation is clearly nonsense:

e.g., NLSM onto C

$$\Omega = \frac{x^{1/2}\mathbf{y}^{-1} - \mathbf{y}x^{-1/2}}{x^{1/2} - x^{-1/2}} \longrightarrow \begin{array}{c} ? \\ \mathbf{y} + (\mathbf{y} - \mathbf{y}^{-1})(x + x^2 + \cdots) \\ \mathbf{y}^{-1} + (\mathbf{y}^{-1} - \mathbf{y})(x^{-1} + x^{-2} + \cdots) \end{array}$$

which cautions us against trusting flavor chemical potentials
to infrared-regulate continuum of states correctly
if one is interested in counting physical ground states

another simple example: line bundles over projective spheres

chirals	$U(1)$	$[U(N)\times U(K)]_{\rm F}$
X	$+1$	$(N,1)$
Y	-1	$(1,K)$

$N > K$

$$\Omega^{\xi>0}(\mathbf{y})\Big|_{x\to 0} = (-1)^{N-K-1}\left(\mathbf{y}^{1+K-N}+\mathbf{y}^{3+K-N}+\cdots+\mathbf{y}^{N+K-1}\right)$$

$$\Omega^{\xi>0}(\mathbf{y})\Big|_{1/x\to 0} = (-1)^{N-K-1}\left(\mathbf{y}^{1-K-N}+\mathbf{y}^{3+K-N}+\cdots+\mathbf{y}^{N-K-1}\right)$$

vs.

$$\mathcal{I}^{\xi>0}(\mathbf{y}) = (-1)^{N-K-1}\left(\mathbf{y}^{1+K-N}+\mathbf{y}^{3+K-N}+\cdots+\mathbf{y}^{N-K-1}\right)$$

S.J. Lee + P.Y., 2016

do things get better with higher supersymmetry?

not really

a single instanton ADHM for U(N)

$N=8$ $U(1)$ GLSM with N Fundamental Hypers and a Singe Adjoint Hyper

$\mathbf{y}$ $\mathbf{z}$ x

$$\Omega(\mathbf{y},\mathbf{z},x)\Big|_{\mathbf{z}\to\mathbf{0}}^{x\text{ flavor singlet}} = 1+\mathbf{y}^2+\cdots+\mathbf{y}^{2N-2}$$

$$\Omega(\mathbf{y},\mathbf{z},x)\Big|_{1/\mathbf{z}\to\mathbf{0}}^{x\text{ flavor singlet}} = \mathbf{y}^{2-2N}+\cdots+\mathbf{y}^{-2}+1$$

vs.

$$\mathcal{I}(\mathbf{y})=1$$

S.J. Lee + P.Y., 2016

$\mathcal{I}$ from Ω

a cohomology fact for asymptotically conical geometry

Hausel, Hunsicker, Mazzeo 2002

$$H^n_{L^2}(M) = \begin{cases} H^n(M, \partial M) & n < d = (\dim_R M)/2 \\ \mathrm{Im}\,(H^n(M, \partial M) \to H^n(M)) & n = d \\ H^n(M) & n > d \end{cases}$$

which suggests that perhaps Ω encodes $\mathcal{I}$ in a simple manner ?

line bundles over projective spheres

$$H^*(M, \partial M) \qquad \Omega^{\xi>0}(\mathbf{y})\Big|_{x\to 0} = (-1)^{N-K-1}\left(\mathbf{y}^{1+K-N} + \mathbf{y}^{3+K-N} + \cdots + \mathbf{y}^{N+K-1}\right)$$

$$H^*(M) \qquad \Omega^{\xi>0}(\mathbf{y})\Big|_{1/x\to 0} = (-1)^{N-K-1}\left(\mathbf{y}^{1-K-N} + \mathbf{y}^{3+K-N} + \cdots + \mathbf{y}^{N-K-1}\right)$$

vs.

$$H^*_{L^2}(M) \qquad \mathcal{I}^{\xi>0}(\mathbf{y}) = (-1)^{N-K-1}\left(\mathbf{y}^{1+K-N} + \mathbf{y}^{3+K-N} + \cdots + \mathbf{y}^{N-K-1}\right)$$

S.J. Lee + P.Y., 2016

which suggests that perhaps Ω encodes $\mathcal{I}$ in a simple manner ?

a single instanton ADHM for U(N)

$$H^*(M, \partial M) \qquad \Omega(\mathbf{y}, \mathbf{z}, x)\Big|_{\mathbf{z}\to\mathbf{0}}^{x \text{ flavor singlet}} = 1 + \mathbf{y}^2 + \cdots + \mathbf{y}^{2N-2}$$

$$H^*(M) \qquad \Omega(\mathbf{y}, \mathbf{z}, x)\Big|_{1/\mathbf{z}\to\mathbf{0}}^{x \text{ flavor singlet}} = \mathbf{y}^{2-2N} + \cdots + \mathbf{y}^{-2} + 1$$

vs.

$$H^*_{L^2}(M) \qquad \mathcal{I}(\mathbf{y}) = 1$$

S.J. Lee + P.Y., 2016

unfortunately, the general story is more involved than these simple examples suggest:

no single comprehensive relation appears possible with multiparticle states contributing to Ω fractional or even integral pieces beyond $\mathcal{I}$

one of more interesting class of phenomena occurs when the gapless asymptotic directions comes from the vector multiplets

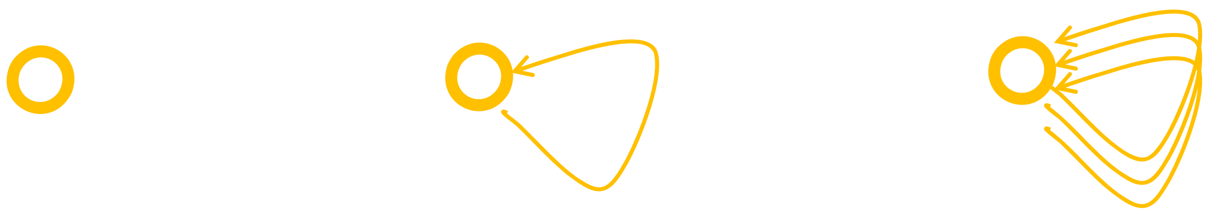
Ω produces rational functions of $\mathbf{y}$
which organize themselves in a particularly useful manner
that allows one to extract integral refined index $\mathcal{I}$ effortlessly

S.J. Lee + P.Y., 2016

back to the basic:

$$\mathcal{N} = 4, 8, 16$$

supersymmetric Yang-Mills quantum mechanics



S.J. Lee + P.Y., 2016

pure $\mathcal{N} = 4$ Yang-Mills quantum mechanics

$$\Omega^{G}_{\mathcal{N}=4}(\mathbf{y}) \quad = \frac{1}{|W|} \sum_{w}{}^{\prime} \frac{1}{\mathrm{Det}\,(\mathbf{y}^{-1} - \mathbf{y} \cdot w)}$$

Weyl group

elliptic Weyl elements only
$0 \neq \mathrm{Det}\,(1 - w)$

P.Y. / Green+Gutperle 1997

elliptic Weyl elements for some classical groups

G	W	Elliptic Weyl Elements
$SU(N)$	S_N	$(123\cdots N)$
$SO(4)$	$Z_2 \times S_2$	$(\dot{1})(\dot{2})$
$SO(5)/Sp(2)$	$(Z_2)^2 \times S_2$	$(1\dot{2}), \quad (\dot{1})(\dot{2})$
$SO(6)$	$(Z_2)^2 \times S_3$	$(1\dot{2})(\dot{3})$
$SO(7)/Sp(3)$	$(Z_2)^3 \times S_3$	$(\dot{1}\dot{2}\dot{3}), \quad (12\dot{3}), \quad (1\dot{2})(\dot{3}), \quad (\dot{1})(\dot{2})(\dot{3})$
$SO(8)$	$(Z_2)^3 \times S_4$	$(\dot{1}\dot{2}\dot{3})(\dot{4}), \quad (12\dot{3})(\dot{4}), \quad (1\dot{2})(3\dot{4}), \quad (\dot{1})(\dot{2})(\dot{3})(\dot{4})$

S.J. Lee + P.Y., 2016

pure $\mathcal{N}=4$ Yang-Mills quantum mechanics

$$\Omega^{SU(p)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{p!}\sum_{p-cyclic\ w}{}^{'} \frac{1}{\mathrm{Det}\,(\mathbf{y}^{-1}-\mathbf{y}\cdot w)}$$

$$= \frac{(p-1)!}{p!}\frac{\mathbf{y}-\mathbf{y}^{-1}}{\mathbf{y}^{p}-\mathbf{y}^{-p}} = \frac{\mathbf{y}-\mathbf{y}^{-1}}{p(\mathbf{y}^{p}-\mathbf{y}^{-p})}$$

$$\mathbf{y}\to 1 \;\Downarrow$$

$$\frac{1}{p^2}$$

P.Y. / Green+Gutperle 1997

which has a common origin as the universal function that enters the rational invariant $\omega(\Gamma;\mathbf{y})$ of wall-crossing

Kim, Park, Wang, P.Y. 2011

$$\Omega^{SU(p)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{p!}\sum_{p-cyclic\ w}{}' \frac{1}{\mathrm{Det}\left(\mathbf{y}^{-1}-\mathbf{y}\cdot w\right)}$$

$$= \frac{(p-1)!}{p!}\,\frac{\mathbf{y}-\mathbf{y}^{-1}}{\mathbf{y}^{p}-\mathbf{y}^{-p}} = \boxed{\frac{\mathbf{y}-\mathbf{y}^{-1}}{p(\mathbf{y}^{p}-\mathbf{y}^{-p})}}$$

$$\omega(\Gamma;\mathbf{y}) \equiv \sum_{p|\Gamma}\mathcal{I}(\Gamma/p;\mathbf{y}^{p})\cdot \boxed{\frac{\mathbf{y}-\mathbf{y}^{-1}}{p(\mathbf{y}^{p}-\mathbf{y}^{-p})}}$$

S.J. Lee + P.Y., 2016

for general gauge groups : rank 2 examples

-

$$\Omega^{SU(3)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{3}\frac{1}{(\mathbf{y}^{-2}+1+\mathbf{y}^2)}$$

$$\Omega^{SO(4)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{4}\frac{1}{(\mathbf{y}^{-1}+\mathbf{y})^2}$$

$$\Omega^{SO(5)/Sp(2)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{8}\left[\frac{2}{\mathbf{y}^{-2}+\mathbf{y}^2}+\frac{1}{(\mathbf{y}^{-1}+\mathbf{y})^2}\right]$$

$$\Omega^{G_2}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{12}\left[\frac{2}{\mathbf{y}^{-2}-1+\mathbf{y}^2}+\frac{2}{\mathbf{y}^{-2}+1+\mathbf{y}^2}+\frac{1}{(\mathbf{y}^{-1}+\mathbf{y})^2}\right]$$

S.J. Lee + P.Y., 2016

for general gauge groups : more examples

$$\Omega^{SU(4)/SO(6)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{4}\frac{1}{(\mathbf{y}^{-3}+\mathbf{y}^{-1}+\mathbf{y}+\mathbf{y}^{3})}$$

$$\Omega^{SO(7)/Sp(3)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{48}\left[\frac{8}{\mathbf{y}^{-3}+\mathbf{y}^{3}}+\frac{6}{(\mathbf{y}^{-2}+\mathbf{y}^{2})(\mathbf{y}^{-1}+\mathbf{y})}+\frac{1}{(\mathbf{y}^{-1}+\mathbf{y})^{3}}\right]$$

$$\Omega^{SO(8)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{192}\left[\frac{32}{(\mathbf{y}^{-3}+\mathbf{y}^{3})(\mathbf{y}^{-1}+\mathbf{y})}+\frac{12}{(\mathbf{y}^{-2}+\mathbf{y}^{2})^{2}}+\frac{1}{(\mathbf{y}^{-1}+\mathbf{y})^{4}}\right]$$

$$\mathcal{I} \quad = \quad \mathcal{I}_{\text{bulk}} \quad + \quad \delta\mathcal{I}$$

$$\lim_{\beta\to 0} \mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta Q^2}\right]$$

$$\Omega \equiv \lim_{e^2\to 0} \mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta Q^2}\right]$$

S.J. Lee + P.Y., 2016

pure $\mathcal{N} = 4$ Yang-Mills quantum mechanics

-

$$\Omega^{G}_{\mathcal{N}=4} \quad = \mathcal{I}^{G}_{\mathcal{N}=4;\mathrm{bulk}}$$

pure $\mathcal{N}=4$ Yang-Mills quantum mechanics

$$\Omega^{G}_{\mathcal{N}=4} \quad = \mathcal{I}^{G}_{\mathcal{N}=4;\mathrm{bulk}} \quad = -\delta\mathcal{I}^{G}_{\mathcal{N}=4}$$

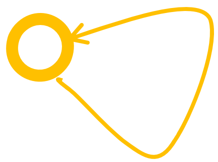
$$= -\delta\mathcal{I}^{U(1)^r/W}_{\mathcal{N}=4}$$

$$= \mathcal{I}^{U(1)^r/W}_{\mathcal{N}=4;\mathrm{bulk}}$$

P.Y. 1997

S.J. Lee + P.Y., 2016

pure $\mathcal{N}=8$ Yang-Mills quantum mechanics



$$\Omega^G_{\mathcal{N}=8}(\mathbf{y},x) = \frac{1}{|W|}{\sum_w}' \frac{1}{\mathrm{Det}\,(\mathbf{y}^{-1}-\mathbf{y}\cdot w)}\cdot \frac{\mathrm{Det}\,\left(\mathbf{y}^{-1}x^{1/2}-\mathbf{y}x^{-1/2}\cdot w\right)}{\mathrm{Det}\,\left(x^{1/2}-x^{-1/2}\cdot w\right)}$$

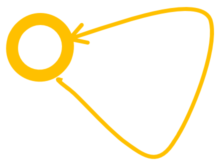
Weyl group

elliptic Weyl elements only

$0\neq \mathrm{Det}\,(1-w)$

S.J. Lee + P.Y., 2016

pure $\mathcal{N}=8$ Yang-Mills quantum mechanics



$$\Omega^{G}_{\mathcal{N}=8} = \mathcal{I}^{G}_{\mathcal{N}=8;\text{bulk}} = -\delta\mathcal{I}^{G}_{\mathcal{N}=8}$$

$$= -\delta\mathcal{I}^{U(1)^r/W}_{\mathcal{N}=8}$$

$$= \mathcal{I}^{U(1)^r/W}_{\mathcal{N}=8;\text{bulk}}$$

P.Y. 1997

S.J. Lee + P.Y., 2016

$\mathcal{N}=16$ SU(N) theories, a.k.a. D0-brane bound state problem

$$\Omega^{SU(N)}_{\mathcal{N}=16}(\mathbf{y},x) \quad = \mathcal{I}^{SU(N)}_{\mathcal{N}=16} + \sum_{p|N;p<N} \mathcal{I}^{SU(N/p)}_{\mathcal{N}=16} \cdot \Delta^{SU(p)}_{\mathcal{N}=16}$$

$$\Delta^{G}_{\mathcal{N}=16}(\mathbf{y},x) = \frac{1}{|W|}\sum_{w}{}' \frac{1}{\mathrm{Det}\left(\mathbf{y}^{-1}-\mathbf{y}\cdot w\right)} \cdot \prod_{a=1,2,3} \frac{\mathrm{Det}\left(\mathbf{y}^{R_a-1}x^{F_a/2}-\mathbf{y}^{1-R_a}x^{-F_a/2}\cdot w\right)}{\mathrm{Det}\left(\mathbf{y}^{R_a}x^{F_a/2}-\mathbf{y}^{-R_a}x^{-F_a/2}\cdot w\right)}$$

S.J. Lee + P.Y., 2016

$\mathcal{N}=16$ SU(N) theories, a.k.a. D0-brane bound state problem

$$\Omega^{SU(N)}_{\mathcal{N}=16}(\mathbf{y},x) \quad = \mathcal{I}^{SU(N)}_{\mathcal{N}=16} + \sum_{p|N;p<N} \mathcal{I}^{SU(N/p)}_{\mathcal{N}=16} \cdot \Delta^{SU(p)}_{\mathcal{N}=16}$$

$\mathbf{y} \to 1$

$$\to \sum_{p|N} 1 \times \frac{1}{p^2}$$

$$\mathcal{I}^{SU}_{\mathcal{N}=16} = 1$$

P.Y. / Sethi,Stern 1997

Nekrasov, Moore, Shatashibli 1998

$$\Delta^{G}_{\mathcal{N}=16}(\mathbf{y},x) = \frac{1}{|W|}\sum_{w}{}' \frac{1}{\mathrm{Det}\,(\mathbf{y}^{-1}-\mathbf{y}\cdot w)} \cdot \prod_{a=1,2,3} \frac{\mathrm{Det}\,\left(\mathbf{y}^{R_a-1}x^{F_a/2}-\mathbf{y}^{1-R_a}x^{-F_a/2}\cdot w\right)}{\mathrm{Det}\,\left(\mathbf{y}^{R_a}x^{F_a/2}-\mathbf{y}^{-R_a}x^{-F_a/2}\cdot w\right)}$$

S.J. Lee + P.Y., 2016

$\mathcal{N}=16$ with general simple Lie groups

$$\Omega^{G}_{\mathcal{N}=16}(\mathbf{y},x) \quad = \mathcal{I}^{G}_{\mathcal{N}=16} + \sum_{G'\subset G;G'\neq G} \# \cdot \Delta^{G'}_{\mathcal{N}=16}$$

$$\Delta^{G}_{\mathcal{N}=16}(\mathbf{y},x) = \frac{1}{|W|}\sum_{w}{}' \frac{1}{\mathrm{Det}\,(\mathbf{y}^{-1}-\mathbf{y}\cdot w)} \cdot \prod_{a=1,2,3} \frac{\mathrm{Det}\,\left(\mathbf{y}^{R_a-1}x^{F_a/2}-\mathbf{y}^{1-R_a}x^{-F_a/2}\cdot w\right)}{\mathrm{Det}\,\left(\mathbf{y}^{R_a}x^{F_a/2}-\mathbf{y}^{-R_a}x^{-F_a/2}\cdot w\right)}$$

$$\Omega^{G}_{\mathcal{N}=4,8,16}(\mathbf{y},x)\Big|_{\mathbf{y}\to 1}$$

	$\mathcal{N}=4,8$	$\mathcal{N}=16$
$SU(N)$	$\frac{1}{N^2}$	$\sum_{p\mid N}\frac{1}{p^2}$
$SO(4)$	$\frac{1}{16}$	$\frac{25}{16}$
$SO(6)=SU(4)$	$\frac{1}{16}$	$\frac{21}{16}$
$SO(8)$	$\frac{59}{1024}$	$\frac{3755}{1024}$
$SO(5)$	$\frac{5}{32}$	$\frac{53}{32}$
$SO(7)$	$\frac{15}{128}$	$\frac{267}{128}$
$SO(9)$	$\frac{195}{2048}$	$\frac{7555}{2048}$
$Sp(2)$	$\frac{5}{32}$	$\frac{53}{32}$
$Sp(3)$	$\frac{15}{128}$	$\frac{395}{128}$
$Sp(4)$	$\frac{195}{2048}$	$\frac{8067}{2048}$
G_2	$\frac{35}{144}$	$\frac{395}{144}$

cf) Moore, Nekrasov, Shatashibili 1998
Kac, Smilga 1999
Staudacher 2000
Pestun 2002

S.J. Lee + P.Y., 2016

or, more informatively

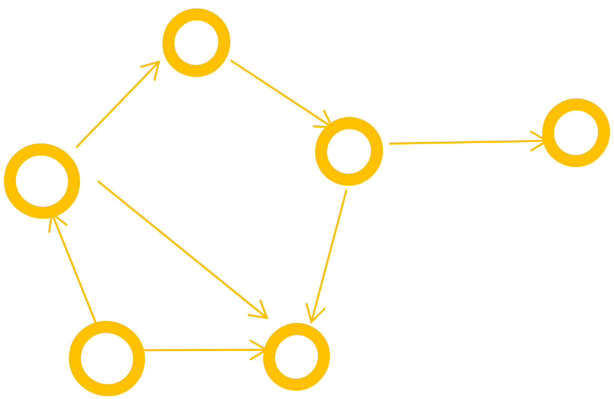
$$
\begin{aligned}
\Omega^{SO(5)/Sp(2)}_{\mathcal{N}=16} &= 1 + 2\Delta^{SO(3)/Sp(1)}_{\mathcal{N}=16} + \Delta^{SO(5)/Sp(2)}_{\mathcal{N}=16} \\
\Omega^{G_2}_{\mathcal{N}=16} &= 2 + 2\Delta^{SU(2)}_{\mathcal{N}=16} + \Delta^{G_2}_{\mathcal{N}=16} \\
\Omega^{SO(7)}_{\mathcal{N}=16} &= 1 + 3\Delta^{SO(3)}_{\mathcal{N}=16} + \left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^2 + \Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(7)}_{\mathcal{N}=16} \\
\Omega^{Sp(3)}_{\mathcal{N}=16} &= 2 + 3\Delta^{Sp(1)}_{\mathcal{N}=16} + \left(\Delta^{Sp(1)}_{\mathcal{N}=16}\right)^2 + \Delta^{Sp(2)}_{\mathcal{N}=16} + \Delta^{Sp(3)}_{\mathcal{N}=16} \\
\Omega^{SO(8)}_{\mathcal{N}=16} &= 2 + 4\Delta^{SO(3)}_{\mathcal{N}=16} + 2\left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^2 + \left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^3 + 3\Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(8)}_{\mathcal{N}=16} \\
\Omega^{SO(9)}_{\mathcal{N}=16} &= 2 + 4\Delta^{SO(3)}_{\mathcal{N}=16} + 2\left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^2 + 2\Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(3)}_{\mathcal{N}=16}\cdot\Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(7)}_{\mathcal{N}=16} + \Delta^{SO(9)}_{\mathcal{N}=16} \\
\Omega^{Sp(4)}_{\mathcal{N}=16} &= 2 + 5\Delta^{Sp(1)}_{\mathcal{N}=16} + 2\left(\Delta^{Sp(1)}_{\mathcal{N}=16}\right)^2 + 2\Delta^{Sp(2)}_{\mathcal{N}=16} + \Delta^{Sp(1)}_{\mathcal{N}=16}\cdot\Delta^{Sp(2)}_{\mathcal{N}=16} + \Delta^{Sp(3)}_{\mathcal{N}=16} + \Delta^{Sp(4)}_{\mathcal{N}=16}
\end{aligned}
$$

$$
\Delta^{G}_{\mathcal{N}=16}(\mathbf{y},x) = \frac{1}{|W|}\sum_{w}{}' \frac{1}{\mathrm{Det}\left(\mathbf{y}^{-1} - \mathbf{y}\cdot w\right)}\cdot\prod_{a=1,2,3}\frac{\mathrm{Det}\left(\mathbf{y}^{R_a-1}x^{F_a/2} - \mathbf{y}^{1-R_a}x^{-F_a/2}\cdot w\right)}{\mathrm{Det}\left(\mathbf{y}^{R_a}x^{F_a/2} - \mathbf{y}^{-R_a}x^{-F_a/2}\cdot w\right)}
$$

even without the full understanding of the recursive structure for the continuum contributions, the results suffice for reading off the Witten index $\mathcal{I}^{G}_{\mathcal{N}=16}$ from the unique integral part

$$
\begin{aligned}
\mathcal{I}^{SO(5)=Sp(2)}_{\mathcal{N}=16} &= 1 \\
\mathcal{I}^{G_2}_{\mathcal{N}=16} &= 2 \\
\mathcal{I}^{SO(7)}_{\mathcal{N}=16} &= 1 \\
\mathcal{I}^{Sp(3)}_{\mathcal{N}=16} &= 2 \\
\mathcal{I}^{Sp(8)}_{\mathcal{N}=16} &= 2 \\
\mathcal{I}^{SO(9)}_{\mathcal{N}=16} &= 2 \\
\mathcal{I}^{Sp(4)}_{\mathcal{N}=16} &= 2 \\
&\vdots
\end{aligned}
$$

the fact that these features are not limited to adjoint-only Yang-Mills quantum mechanics can be inferred from the appearance of the rational invariant in the general wall-crossing story proposal : the twisted partition functions of quivers compute these rational invariants rather than Witten indices

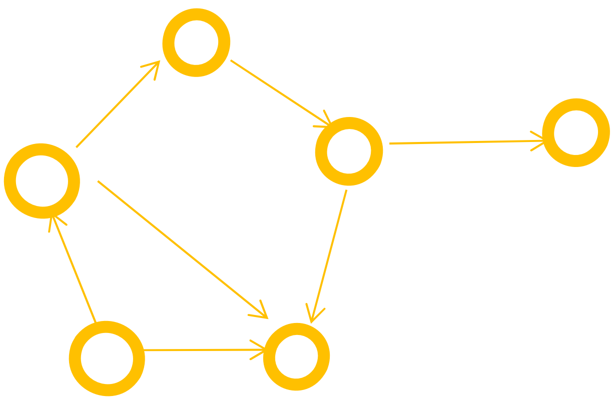


$$\Omega(\Gamma;\mathbf{y}) = \mathcal{I}_{\rm bulk} = \omega(\Gamma;\mathbf{y})$$

for quivers, with compact chiral sector

S.J. Lee + P.Y., 2016

proposal : the twisted partition functions of quivers compute these rational invariants rather than Witten indices



$$\Omega(\Gamma;\mathbf{y}) = \mathcal{I}_{\rm bulk} = \omega(\Gamma;\mathbf{y})$$

for quivers, with compact chiral sector

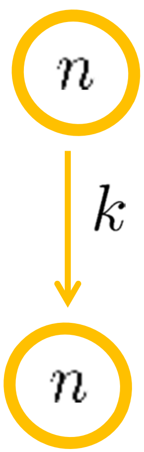
S.J. Lee + P.Y., 2016

$$\mathcal{I}(\Gamma;\mathbf{y}) = \sum_{p|\Gamma} \mu(p)\cdot\Omega(\Gamma/p;\mathbf{y}^p)\cdot\frac{\mathbf{y}-\mathbf{y}^{-1}}{p(\mathbf{y}^p-\mathbf{y}^{-p})}$$

this allows a simple extraction of the Witten index $\mathcal{I}$ from the localization computation of Ω, even when the quiver is non-primitive and thus, relevant bound states are at threshold

example : nonprimitive Kronecker quiver

$$\mathcal{I}(\mathcal{Q}^k_{n,n};\mathbf{y}) = \sum_{p|n} \mu(p) \cdot \Omega(\mathcal{Q}^k_{n/p,n/p};\mathbf{y}^p) \cdot \frac{\mathbf{y}-\mathbf{y}^{-1}}{p(\mathbf{y}^p-\mathbf{y}^{-p})}$$



$$\begin{aligned}
\mathcal{I}(\mathcal{Q}^1_{2,2};\mathbf{y}) &= 0 \\
\mathcal{I}(\mathcal{Q}^2_{2,2};\mathbf{y}) &= 0 \\
\mathcal{I}(\mathcal{Q}^3_{2,2};\mathbf{y}) &= -\chi_{5/2}(\mathbf{y}^2) \\
\mathcal{I}(\mathcal{Q}^4_{2,2};\mathbf{y}) &= -\chi_{9/2}(\mathbf{y}^2) - \chi_{5/2}(\mathbf{y}^2) \\
\mathcal{I}(\mathcal{Q}^5_{2,2};\mathbf{y}) &= -\chi_{13/2}(\mathbf{y}^2) - 2\chi_{9/2}(\mathbf{y}^2) - \chi_{5/2}(\mathbf{y}^2)
\end{aligned}$$

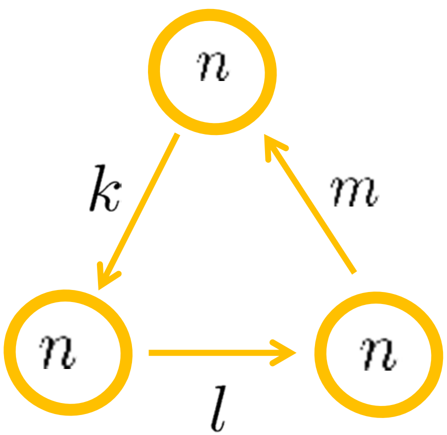
example : nonprimitive 3-node quiver

$$\mathcal{I}(\mathcal{Q}^{k,l}_{n,n,n};\mathbf{y}) = \sum_{p|n} \mu(p)\cdot \Omega(\mathcal{Q}^{k,l}_{n/p,n/p,n/p};\mathbf{y}^p)\cdot \frac{\mathbf{y}-\mathbf{y}^{-1}}{p(\mathbf{y}^p-\mathbf{y}^{-p})}$$

$$\begin{aligned}
\mathcal{I}(\mathcal{Q}^{1,1}_{2,2,2};\mathbf{y}) &= 0\\
\mathcal{I}(\mathcal{Q}^{1,2}_{2,2,2};\mathbf{y}) &= 0\\
\mathcal{I}(\mathcal{Q}^{1,3}_{2,2,2};\mathbf{y}) &= -\chi_{5/2}(\mathbf{y}^2)\\
\mathcal{I}(\mathcal{Q}^{1,4}_{2,2,2};\mathbf{y}) &= -\chi_{9/2}(\mathbf{y}^2)-\chi_{5/2}(\mathbf{y}^2)\\
\mathcal{I}(\mathcal{Q}^{1,5}_{2,2,2};\mathbf{y}) &= -\chi_{13/2}(\mathbf{y}^2)-2\chi_{9/2}(\mathbf{y}^2)-\chi_{5/2}(\mathbf{y}^2)\\
\mathcal{I}(\mathcal{Q}^{2,2}_{2,2,2};\mathbf{y}) &= -\chi_{5/2}(\mathbf{y}^2)-\chi_{3/2}(\mathbf{y}^2)-\chi_{1/2}(\mathbf{y}^2)\\
\mathcal{I}(\mathcal{Q}^{2,3}_{2,2,2};\mathbf{y}) &= -\chi_{9/2}(\mathbf{y}^2)-\chi_{7/2}(\mathbf{y}^2)-3\chi_{5/2}(\mathbf{y}^2)-\chi_{3/2}(\mathbf{y}^2)-\chi_{1/2}(\mathbf{y}^2)
\end{aligned}$$

example : nonprimitive triangle quiver

$$\mathcal{I}(\mathcal{Q}^{k,l,m}_{n,n,n};\mathbf{y}) = \sum_{p|n} \mu(p)\cdot\Omega(\mathcal{Q}^{k,l,m}_{n/p,n/p,n/p};\mathbf{y}^p)\cdot\frac{\mathbf{y}-\mathbf{y}^{-1}}{p(\mathbf{y}^p-\mathbf{y}^{-p})}$$



$$\begin{aligned}
\mathcal{I}(\mathcal{Q}^{1,1,-1}_{2,2,2};\mathbf{y}) &= 0\\
\mathcal{I}(\mathcal{Q}^{2,1,-1}_{2,2,2},;\mathbf{y}) &= -\chi_{5/2}(\mathbf{y}^2)-\chi_{3/2}(\mathbf{y}^2)-\chi_{1/2}(\mathbf{y}^2)\\
\mathcal{I}(\mathcal{Q}^{1,1,-2,}_{2,2,2};\mathbf{y}) &= -\chi_{5/2}(\mathbf{y}^2)\\
\mathcal{I}(\mathcal{Q}^{2,2,-1}_{2,2,2};\mathbf{y}) &= -\chi_{9/2}(\mathbf{y}^2)-\chi_{7/2}(\mathbf{y}^2)-3\chi_{5/2}(\mathbf{y}^2)-\chi_{3/2}(\mathbf{y}^2)-\chi_{1/2}(\mathbf{y}^2)\\
\mathcal{I}(\mathcal{Q}^{2,1,-2}_{2,2,2};\mathbf{y}) &= -\chi_{9/2}(\mathbf{y}^2)-\chi_{7/2}(\mathbf{y}^2)-3\chi_{5/2}(\mathbf{y}^2)-2\chi_{3/2}(\mathbf{y}^2)-\chi_{1/2}(\mathbf{y}^2)\\
\mathcal{I}(\mathcal{Q}^{3,1,-1}_{2,2,2};\mathbf{y}) &= -\chi_{9/2}(\mathbf{y}^2)-\chi_{7/2}(\mathbf{y}^2)-3\chi_{5/2}(\mathbf{y}^2)-2\chi_{3/2}(\mathbf{y}^2)-2\chi_{1/2}(\mathbf{y}^2)\\
\mathcal{I}(\mathcal{Q}^{1,1,-3}_{2,2,2};\mathbf{y}) &= -\chi_{9/2}(\mathbf{y}^2)-\chi_{5/2}(\mathbf{y}^2)\\
\mathcal{I}(\mathcal{Q}^{2,2,-2}_{2,2,2};\mathbf{y}) &= -\chi_{13/2}(\mathbf{y}^2)-\chi_{11/2}(\mathbf{y}^2)-4\chi_{9/2}(\mathbf{y}^2)\\
&\quad -3\chi_{7/2}(\mathbf{y}^2)-4\chi_{5/2}(\mathbf{y}^2)-\chi_{3/2}(\mathbf{y}^2)-\chi_{1/2}(\mathbf{y}^2)
\end{aligned}$$

twisted partition function Ω

$\neq$ equivariant witten index $\mathcal{I}$

the former is computationally more accessible

but it is the latter that carries physical/mathematical importance twisted partition function Ω

$\neq$ equivariant witten index $\mathcal{I}$

relationships btw them are not universal, but

we identified several that allowed us to extract

$\mathcal{I}$ from Ω

despite the bound states being at threshold two immediate, unanswered questions:

systematic understanding of the rational contributions
to Ω for noncompact GLSM involving SO/Sp gauge groups?

how to compute $\mathcal{I}$ for a GLSM/quiver at $\xi = 0$
as wall-crossing-safe GLSM/quiver invariant,
a.k.a., single center black hole degeneracy ?