Black hole degeneracies from worldsheet instantons

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Number Theory and Physics
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Black Holes are sources of

- Energy, angular momentum — No doubt ✓
- Gravity waves — Observed ✓
- Astrophysical power (radiation) — Possible ?

Quantum Black Holes are sources of

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• Heat, entropy	(Bekenstein-Hawking)	✓

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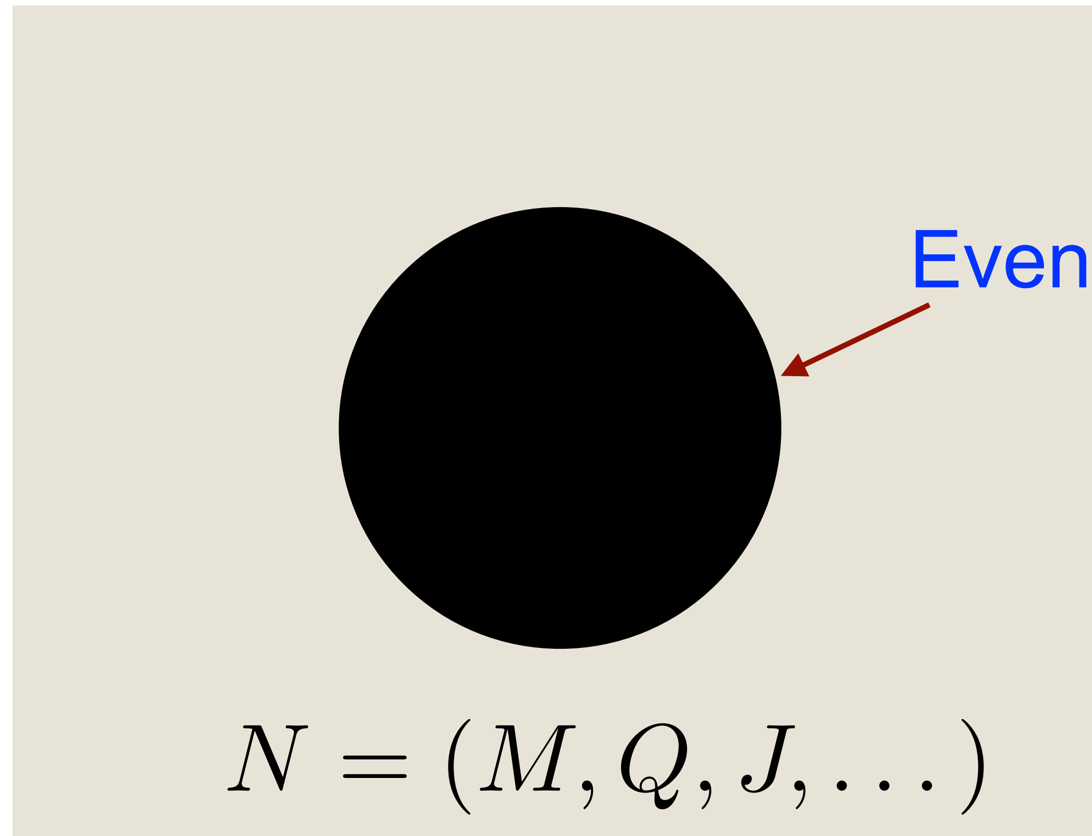
• Modular/automorphic forms	Today’s talk	

A black hole (BH) is a solution to effective theory of graviton, photon, scalars, …

Macroscopic picture
of a black hole



Properties determined
by quantum numbers N

Graviton Photon Scalars

$$\mathcal{L} = e^{-K(X)} R(g) + F_{IJ}(X)\, F^{I}_{\mu\nu} F^{J\,\mu\nu} + F_{IJ}(X)\, D_\mu X^I D^\mu X^J + \cdots$$

Black hole entropy points to an integer *(degeneracy)* associated to a black hole

Universal law in GR

$$S_{\rm BH}^{\rm class}(N) \;=\; \frac{1}{4}\,\frac{A_{\rm H}(N)}{\ell_{\rm Pl}^2}$$

(Bekenstein-Hawking '74)

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$$k_B \log d_{\mathrm{micro}}(N) \;=\; S^{\mathrm{class}}_{\mathrm{BH}}(N) \,+\, \cdots$$

[Boltzmann]

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Deviations from GR!

Universal law in GR

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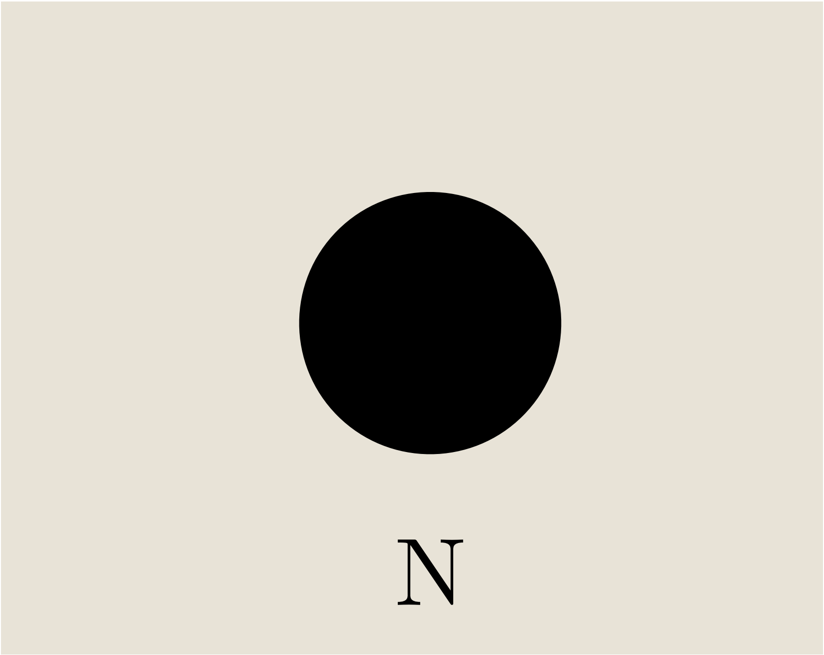
We can access BH degeneracy via dual microscopic picture in string theory models

Macroscopic



Bekenstein-Hawking '74

$$S_{\mathrm{BH}}^{\mathrm{class}} = \frac{A_H}{4\ell_{\mathrm{Pl}}^2} = \pi\sqrt{N}$$

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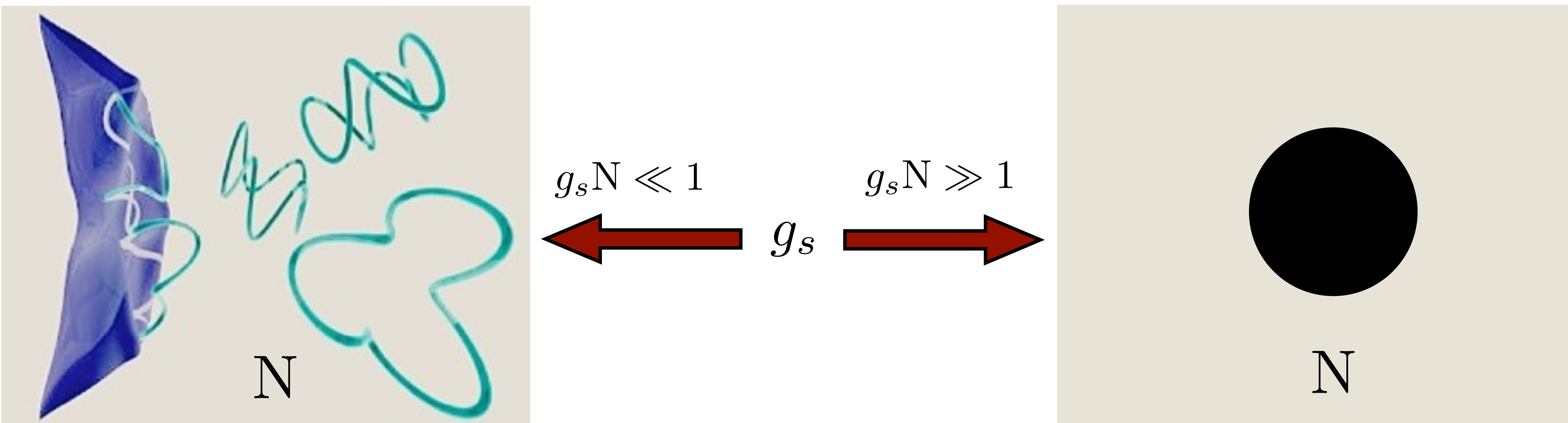
Microscopic | Macroscopic



Strominger-Vafa ’96

$$d_{\rm micro}(\mathrm{N}) = e^{\pi\sqrt{\mathrm{N}}} + \cdots \quad (\mathrm{N} \to \infty)$$

Bekenstein-Hawking ’74

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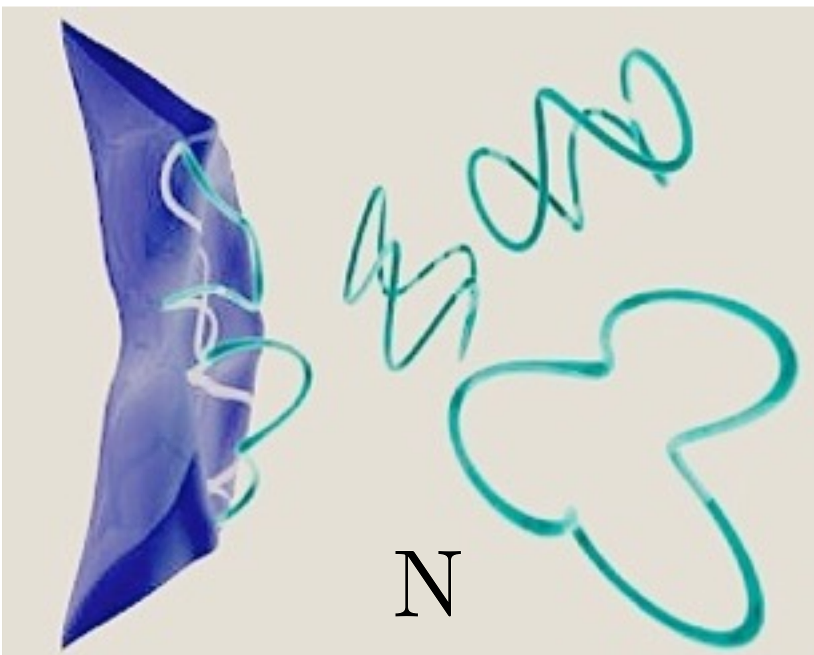

$g_s \mathrm{N} \ll 1$ ⟵ g_s ⟶ $g_s \mathrm{N} \gg 1$

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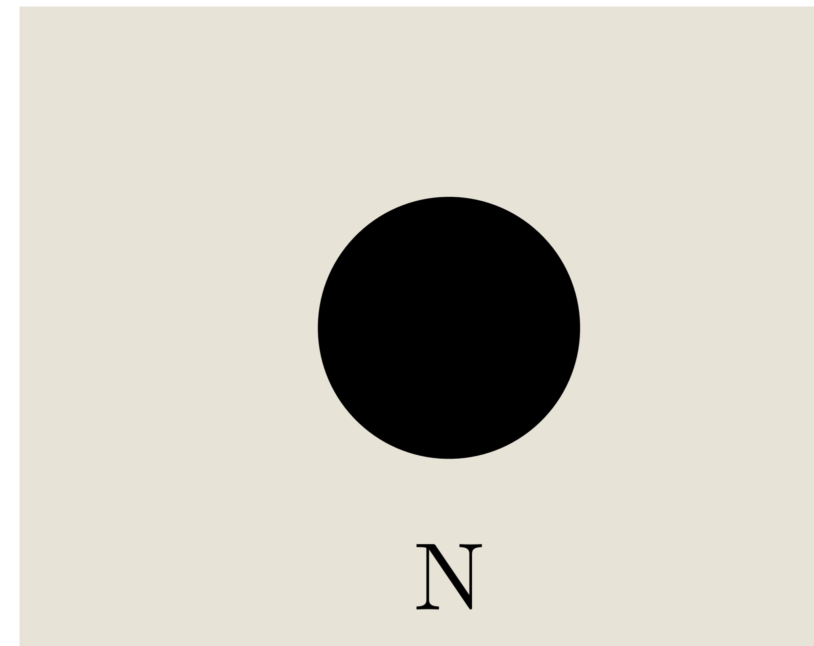

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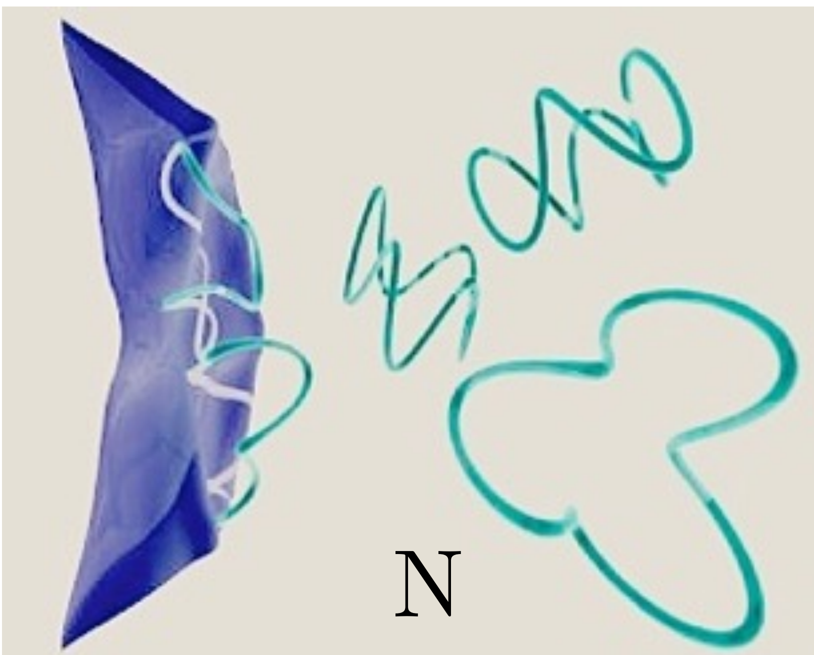

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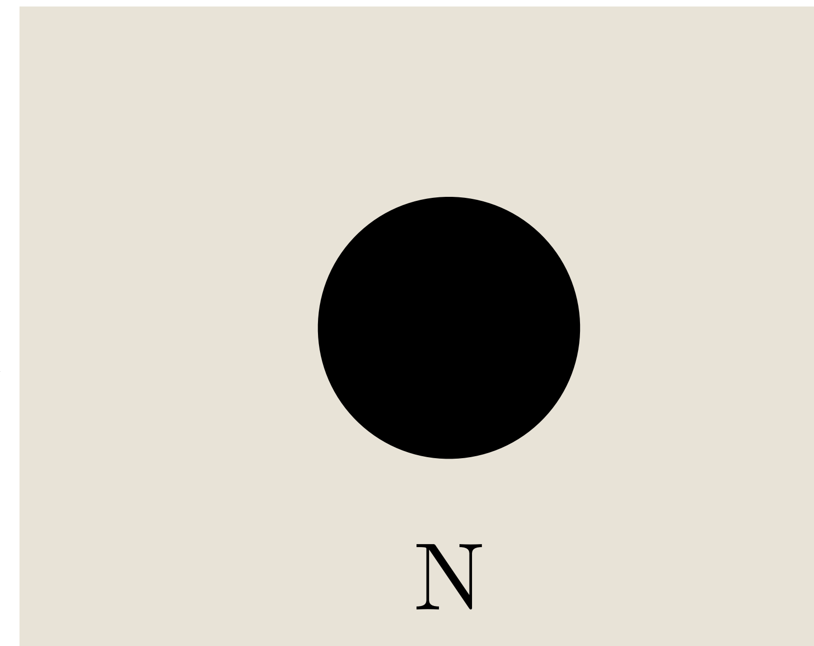

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$$\log d_{\mathrm{micro}} = S_{\mathrm{BH}}^{\mathrm{class}} + \cdots$$

→ $S_{\mathrm{BH}}^{\mathrm{quant}}$ (finite N)

Macroscopic physics encodes the integer degeneracy through asymptotic expansion

$$S_{\rm BH}^{\rm quant} = \frac{1}{4}A + a_0 \log(A) + a_1 \frac{1}{A} + a_2 \frac{1}{A^2} + \cdots$$
$$+ b_1(A) e^{-A} + \cdots$$

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$$\log d_{\rm micro}(N) = \pi\sqrt{N} + a_0' \log N + a_1' \frac{1}{N} + a_2' \frac{1}{N^2} + \cdots$$
$$+ b_1'(N) e^{-N} + \cdots$$

Prototype: $\frac{1}{8}$-BPS BHs in string compactification with 32 supercharges

U-Duality group $E_{7,7}(\mathbb{Z})$

$\frac{1}{8}$-BPS states labelled by quartic invariant N.

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$\frac{1}{8}$ -BPS states labelled by quartic invariant N.

Microscopic degeneracies are given by the Fourier coefficients of: [J. Maldacena, G. Moore, A. Strominger ('99)]

$$\varphi_{-2,1}(\tau, z) = \frac{\vartheta_1(\tau, z)^2}{\eta(\tau)^6}$$

Jacobi form of weight -2 and index 1.

$$\vartheta_1(\tau, z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} (-1)^n q^{n^2/2} \zeta^n$$

$$\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n)$$

$$q = e^{2\pi i \tau} \qquad \zeta = e^{2\pi i z}$$

Jacobi forms Review: definitions

Jacobi form of weight k, index m

$\varphi(\tau, z)$ Holomorphic function $\mathbb{H} \times \mathbb{C} \to \mathbb{C}$

Modular property:

$$\varphi\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) = (c\tau+d)^k \, e^{\frac{2\pi i m c z^2}{c\tau+d}} \, \varphi(\tau, z)$$

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Elliptic property:

$$\varphi(\tau, z + \lambda\tau + \mu) \;=\; e^{-2\pi i m(\lambda^2\tau + 2\lambda z)} \varphi(\tau, z) \quad \forall\, \lambda,\, \mu \in \mathbb{Z}$$

Jacobi forms Review: Fourier coefficients

Weak Jacobi forms

$$\varphi(\tau, z) \;=\; \sum_{n\geq 0, \ell} c(n,\ell)\, q^n\, \zeta^\ell$$

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Elliptic property ⇒

$$c(n,\ell) \,=\, C_\mu(\Delta)$$

$$\Delta = 4nm - \ell^2$$

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Special coefficients: the *polar coefficients*

$C_\mu(\Delta)$ with $\Delta < 0$

Jacobi forms Review: Polar coefficients completely determine the Jacobi form

$$\varphi(\tau, z) \;=\; \sum_{n\geq 0,\ell} c(n,\ell)\, q^n\, \zeta^\ell$$

weight $k = w + 1/2$

index m

Hardy-Ramanujan-Rademacher expansion

$$C_\ell(\Delta) = (2\pi)^{2-w} \sum_{c=1}^{\infty} c^{w-2} \sum_{\widetilde{\ell}\,(\mathrm{mod} 2m)} \sum_{\widetilde{\Delta}<0} C_{\widetilde{\ell}}(\widetilde{\Delta})\, Kl(\Delta,\ell,\widetilde{\Delta},\widetilde{\ell};c) \times$$

$$\times \left|\frac{\widetilde{\Delta}}{4m}\right|^{1-w} \widetilde{I}_{1-w}\Big(\frac{\pi}{mc}\sqrt{|\widetilde{\Delta}|\Delta}\Big)$$

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Only input: Polar coefficients

Prototype: $\frac{1}{8}$-BPS BHs in string compactification with 32 supercharges

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$$c(n,\ell) = C(4n-\ell^2) \quad , \qquad d_{\text{micro}}(N) = (-1)^N\, C(N)$$

Exact asymptotic expansion of degeneracy: a good guide for quantum gravity

(Hardy-Ramanujan-Rademacher)

$$d_{\rm micro}(N) = \sum_{c=1}^{\infty} c^{-9/2} K_c(N)\, \widetilde{I}_{7/2}\Big(\frac{\pi\sqrt{N}}{c}\Big)$$

$$= \widetilde{I}_{7/2}(\pi\sqrt{N})\Big(1 + O(e^{-\pi\sqrt{N}/2})\Big)$$

$$= e^{\pi\sqrt{N}}\left(1 - \frac{15}{4}\log N + O\Big(\frac{1}{N}\Big)\right).$$

$K_c(N)$ Kloosterman sum

$\widetilde{I}_\rho(z) = 2\pi\Big(\frac{z}{4\pi}\Big)^{-\rho} I_\rho(z)$ I-Bessel function

We can recover integer degeneracies from macroscopic (continuum) BH physics

(A.Dabholkar, J.Gomes, S.M. '10, '11, '14)

$$\begin{aligned} \exp(S_{\mathrm{BH}}^{\mathrm{quant}}(N)) &= \sum_{c=1}^{\infty} c^{-9/2} K_c(N)\, \widetilde{I}_{7/2}\Big(\frac{\pi\sqrt{N}}{c}\Big) \\ &= \widetilde{I}_{7/2}(\pi\sqrt{N})\Big(1 + O(e^{-\pi\sqrt{N}/2})\Big) \\ &= e^{\pi\sqrt{N}}\Big(1 - \frac{15}{4}\log N + O(\frac{1}{N})\Big)\,. \end{aligned}$$

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Bekenstein-Hawking

One-loop corrections

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All-order perturbation using localization

Bekenstein-Hawking

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Orbifolds of AdS_2

All-order perturbation using localization

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A macroscopic source of modular forms

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Broad Questions

- How generic are these ideas?
- Can we use the BH macroscopics to reconstruct the microscopic degeneracy in new cases?
- What kind of generating functions do we get in general?

How generic are these ideas?

In theories with lower supersymmetry:

a. Modular symmetry is broken due to wall-crossing.

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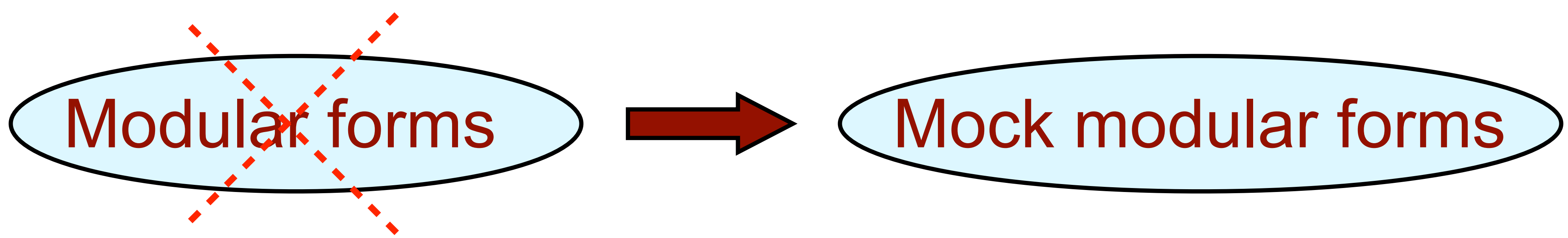
In N=4 string theory:

A. Dabholkar, S.M., D. Zagier ’12



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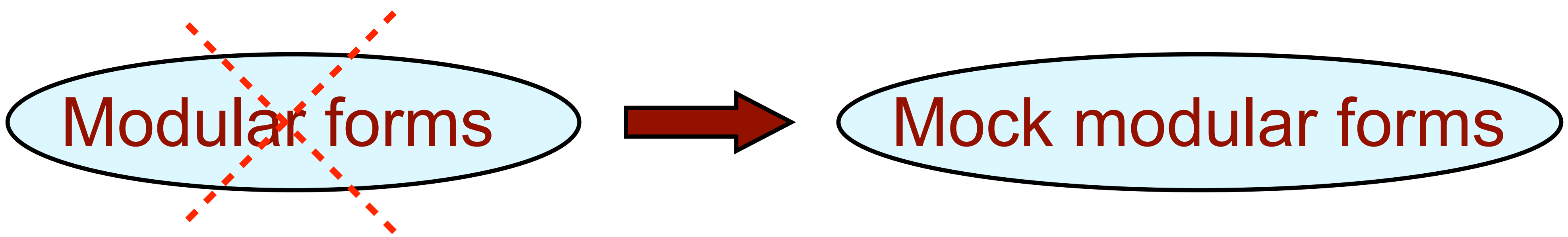

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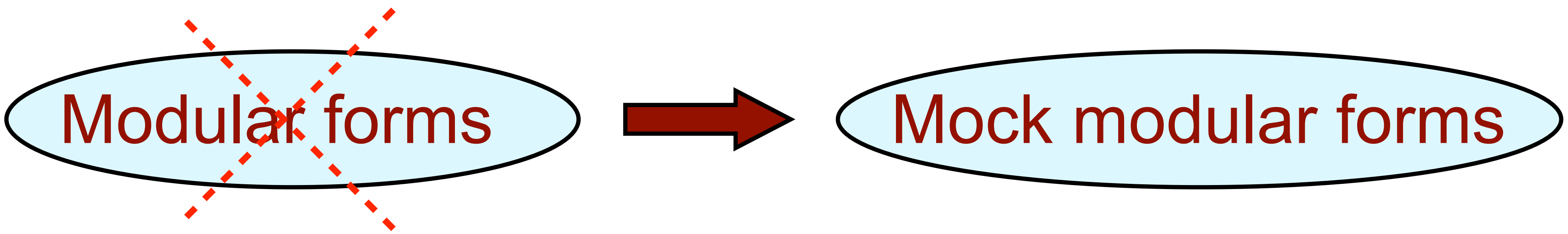

b. Instantons contribute to the gravitational theory.

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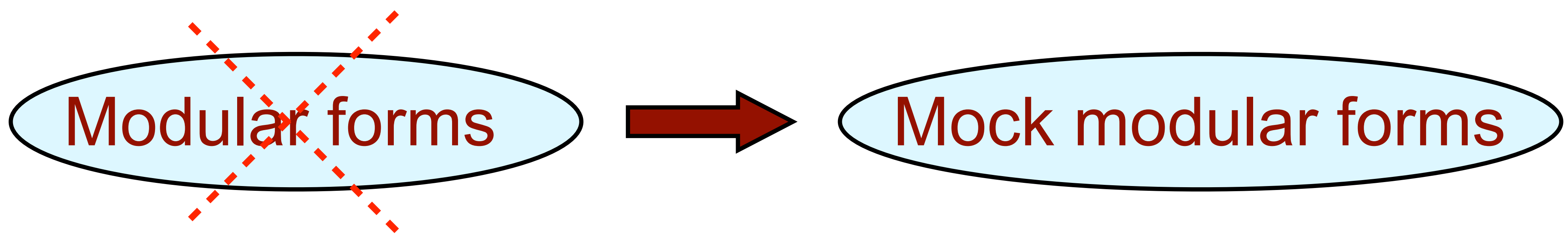

b. Instantons contribute to the gravitational theory.

a. Can we see the mock modular symmetry from gravity?

b. How do the instanton degeneracies encode the BH degeneracies?

Where we are headed

Using these ideas, I will present the beginnings of a formula in purely mathematical terms.

In the context of compactification of Type II string theory on $M_6 = K3 \times T^2$, this formula gives a simple relation between the degeneracies of worldsheet instantons on M_6 — the Gromov-Witten invariants — and the degeneracies of BHs.

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Still in progress: help from number theorists would be appreciated.

Brief summary of macroscopic quantum entropy computation

Origin of corrections in physics:

- Higher derivative corrections to the Wilsonian GR+matter Lagrangian (integrating out massive fields).
- Quantum effects: loops of massless fields (non-local).

The quantity $Z_{AdS_2}(N) = \exp(S_{\rm BH}^{\rm quant}(N))$ is the result of a gravitational functional integral with AdS_2 boundary conditions.

(A. Sen '08, '09)

Brief summary of macroscopic quantum entropy computation

- Functional integral has been computed in concrete models with varying degrees of supersymmetry (8-16-32).

- Computations use input from string scattering, supergravity and the technique of supersymmetric localization.

- Localization reduces the perturbative functional integral a finite dimensional integral.

Simple formula for exact entropy of $\frac{1}{2}$-BPS BH in theories with 8 supercharges

(A.Dabholkar, J.Gomes, S.M. '10) (c.f. Ooguri-Strominger-Vafa '04)

4d N=2 supergravity coupled to n_{v} vector multiplets,

BH carrying charges (p^I, q_I) $I = 0, 1, \cdots, n_{\mathrm{v}}$

$$Z_{AdS_2}(q,p) = \int \prod_{I=0}^{n_{\mathrm{v}}} [d\phi^I] \exp(\mathcal{S}_{\mathrm{ren}}(\phi, p, q))$$

$$\mathcal{S}_{\mathrm{ren}}(\phi, p, q) = -\pi q_I \phi^I + \mathrm{Im} F(\phi^I + i p^I)$$

Here the function $F(X^I)$ is the holomorphic prepotential of N=2 supergravity.

Prototype: 1/8 BPS black holes in N=8 string theory

- Truncation of N=8 to N=2 theory with 8 vectors.
- F-term action (prepotential) exact at tree-level.

$$F(X) = -\frac{1}{2}\frac{X^1 C_{ab} X^a X^b}{X^0}\,, \qquad a,b = 2,\ldots,7\,.$$

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Exact quantum gravitational entropy

$$e^{S_{BH}^{\rm qu}}(N) = \int \frac{d\sigma}{\sigma^{9/2}} \exp\left(\sigma + \pi^2 N/4\sigma\right) = \widetilde{I}_{7/2}(\pi\sqrt{N})$$

Type II string theory on $K3 \times T^2$ $\equiv$ Heterotic string theory on T^6

- U-Duality group $SO(22,6) \times SL(2,\mathbb{Z})$

- $\frac{1}{4}$-BPS dyonic states labelled by $(Q^2,\, Q.P,\, P^2) \equiv (n, \ell, m)$

- We work in a regime with fixed P and varying Q.

- The Fourier coefficients $d_m^{\rm micro}(n,\ell)$ have the formal structure of a Jacobi-like form.

Macroscopic side of the story

- F-term action (prepotential) receives contributions from worldsheet instantons.

$$F(X) = -\frac{X^1 X^a C_{ab} X^b}{X^0} + \frac{1}{2\pi i}\mathcal{F}^{(1)}(X^1/X^0)$$

$$\mathcal{F}^{(1)}(\tau) = \log \eta(\tau)^{24}$$

Instanton contributions

Macroscopic side of the story

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$$\mathcal{F}^{(1)}(\tau) \;=\; \log \eta(\tau)^{24}$$

Instanton contributions

Fourier expansion gives the instanton degeneracies
$(q = e^{2\pi i \tau})$

$$e^{-\mathcal{F}^{(1)}(\tau)} = \sum_{n=-1}^{\infty} d(n)\, q^n \quad = q^{-1} + 24 + 324q + 3200q^2 + \cdots$$

Macroscopic side of the story

S.M., V.Reys arXiv:1512.01553

- Using this expansion in our supergravity formula

$$Z_{AdS_2}(q,p) = \int \prod_{I=0}^{n_{\rm v}} [d\phi^I] \exp(-\pi q_I \phi^I + {\rm Im} F(\phi^I + ip^I))$$

  we get a series of Bessel functions.

- This step needs a certain contour of integration for which we use the one proposed in J.Gomes arXiv:1511.07061

- Assume a certain measure factor (Full first-principles derivation of measure remains to be done).

Macroscopic quantum entropy formula

We obtain a sum over Bessel functions with numerical coefficients depending on the instanton degeneracies

$$Z_{AdS_2}(n,\ell,m) \approx \sum_{0\le \ell' \le m} \sum_{4n'-\frac{\ell'^2}{m}<0} (\ell'-2n')\, d(m+n'-\ell')\, d(n')\, \cos\left(\pi(m-\ell')\frac{\ell}{m}\right) \times$$
$$\times \frac{2\pi}{\sqrt{m}} \left(\frac{\left|4n'-\frac{\ell'^2}{m}\right|}{n-\frac{\ell^2}{4m}}\right)^{n_{\rm v}/4} I_{n_{\rm v}/2}\left(2\pi\sqrt{\left|4n'-\frac{\ell'^2}{m}\right|\left(n-\frac{\ell^2}{4m}\right)}\right)$$

(This formula receives corrections from subleading saddle points, and from certain “edge terms”.)

Microscopic side of the story

Partition function is the inverse of the Igusa cusp form

$$\frac{1}{\Phi_{10}(\tau, z, \sigma)} = \sum_{n,\ell,m} d(n,\ell,m)\, e^{2\pi i(n\tau+\ell z+m\sigma)}$$

(R. Dijkgraaf, E.+H. Verlinde, 1994)

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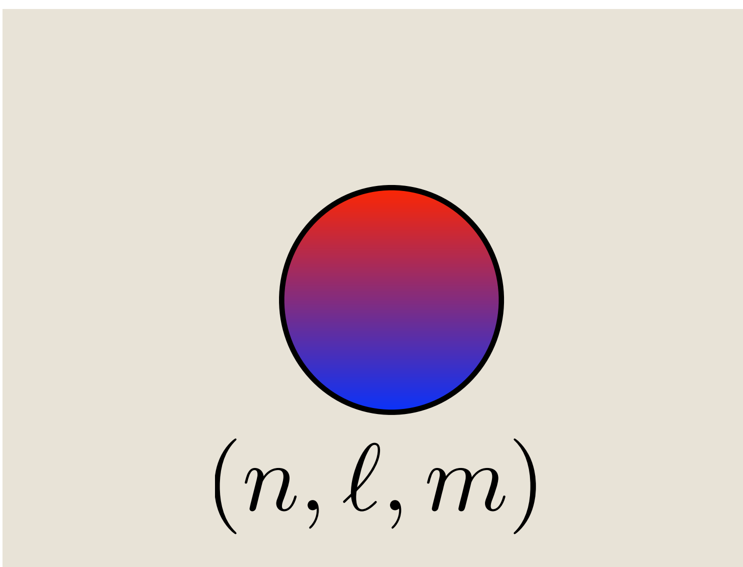
Fourier Expansion is ill-defined due to meromorphicity!

$$\Phi_{10}(\tau, z, \sigma) = 4\pi z^2 \eta(\tau)^{24}\, \eta(\sigma)^{24} + O(z^4)$$

"Phenomenology" of the N=4 theory (Meaning of ambiguity in physics)

$\frac{1}{4}$ -BPS dyonic BH

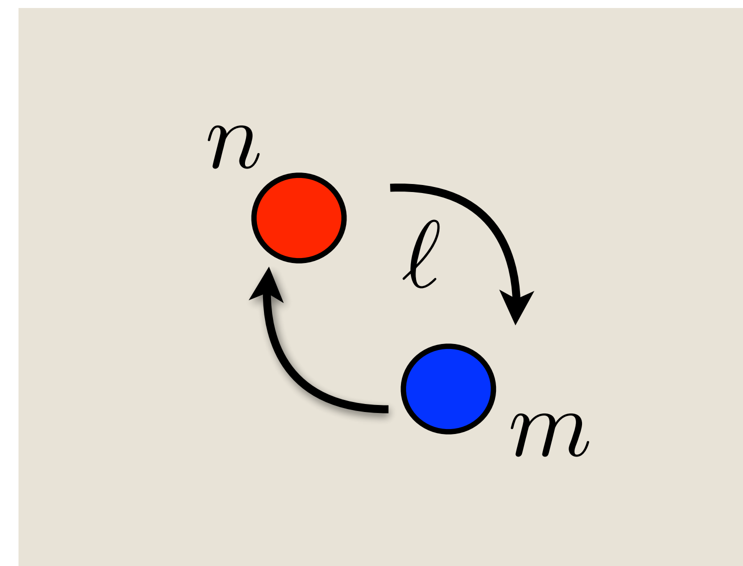


$$d_{\mathrm{BH}}(m, \ell, n) \approx e^{\pi\sqrt{4mn-\ell^2}}$$

Exists everywhere in moduli space

2-centered BH bound state

(Each $\frac{1}{2}$ -BPS)



$$d^{(2)}(m, \ell, n) = p_{24}(m+1)\, p_{24}(n+1)\, \ell$$
$$\approx e^{4\pi(\sqrt{n}+\sqrt{m})}$$

(Dis)appears on crossing a co-dimension one surface (wall) in moduli space

(Denef '00)

One can isolate the BH degeneracies

$$\frac{1}{\Phi_{10}(\tau, z, \sigma)} = \sum_{m=-1}^{\infty} \psi_m(\tau, z)\, e^{2\pi i m \sigma}$$

Expansion in
M-theory limit

$\psi_m(\tau, z)$ Jacobi form of index m *meromorphic in z!*

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Canonical decomposition (A.Dabholkar, S.M., D. Zagier '12)

$$\psi_m(\tau, z) = \psi_m^{\mathrm{BH}}(\tau, z) + \psi_m^{\mathrm{multi}}(\tau, z)\,.$$

Partition function of the isolated BH is a *mock Jacobi form*.

Multi-centered BH contribution.

Practical implication of *mock* nature

Mock means that the function itself is not quite modular, but one can add a specific non-holomorphic function (called the shadow function) to it so that the sum is modular (but not holomorphic).

So the power of modularity is resurrected!

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In particular, there is a Rademacher-type formula for mock Jacobi forms, but with some modifications from the modular case.

(Bringmann+Ono ’07; Bringmann+Manschot; …)

Microscopic entropy formula

$$c_m(n,\ell) \approx \sum_{0\le \ell' \le m} \sum_{4n' - \frac{\ell'^2}{m} < 0} c(n',\ell') \cos\left(\pi(m-\ell')\frac{\ell}{m}\right) \times$$

$$\times \frac{2\pi}{\sqrt{m}} \left(\frac{\left|4n' - \frac{\ell'^2}{m}\right|}{n - \frac{\ell^2}{4m}} \right)^{n_{\rm v}/4} I_{n_{\rm v}/2}\left(2\pi \sqrt{\left|4n' - \frac{\ell'^2}{m}\right| \left(n - \frac{\ell^2}{4m}\right)} \right)$$

Polar coefficients of mock Jacobi form

(This is the c=1 term of the Rademacher expansion for true Jacobi forms, one can estimate the nature of corrections due to the mock nature.)

The mock Jacobi forms encoding the N=4 BH degeneracies are explicitly known

(A.Dabholkar, S.M. D. Zagier '12) (K. Bringmann, S.M.'12)

m=1

$$\psi_1^{\mathrm{F}}(\tau,z) = \frac{1}{\eta(\tau)^{24}}\left(3E_4(\tau)A(\tau,z) - 648\mathcal{H}_1(\tau,z)\right)$$

$$= (3\zeta + 48 + 3\zeta^{-1})q^{-1} + (48\zeta^2 + 600\zeta - 648 + 600\zeta^{-1} + 48\zeta^{-2}) + \cdots$$

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m=2

$$\psi_2^{\mathrm{F}}(\tau,z) = \frac{1}{3\eta(\tau)^{24}}\left(22E_4AB - 10E_6A^2 - 9600\mathcal{H}_2\right)$$

and so on …

Microscopic vs macroscopic formula

In each case a sum over Bessel functions with some numerical coefficients (the polar coefficients)

$$c_m(n,\ell) \approx \sum_{0 \le \ell' \le m} \sum_{4n' - \frac{\ell'^2}{m} < 0} c_m(n',\ell') \cos\left(\pi(m-\ell')\frac{\ell}{m}\right) \times$$
$$\times \frac{2\pi}{\sqrt{m}} \left(\frac{\left|4n' - \frac{\ell'^2}{m}\right|}{n - \frac{\ell^2}{4m}} \right)^{n_{\rm v}/4} I_{n_{\rm v}/2}\left(2\pi \sqrt{\left|4n' - \frac{\ell'^2}{m}\right| \left(n - \frac{\ell^2}{4m}\right)} \right)$$

$$Z_{AdS_2}(n,\ell,m) \approx \sum_{0 \le \ell' \le m} \sum_{4n' - \frac{\ell'^2}{m} < 0} (\ell' - 2n')\, d(m+n'-\ell')\, d(n') \cos\left(\pi(m-\ell')\frac{\ell}{m}\right) \times$$
$$\times \frac{2\pi}{\sqrt{m}} \left(\frac{\left|4n' - \frac{\ell'^2}{m}\right|}{n - \frac{\ell^2}{4m}} \right)^{n_{\rm v}/4} I_{n_{\rm v}/2}\left(2\pi \sqrt{\left|4n' - \frac{\ell'^2}{m}\right| \left(n - \frac{\ell^2}{4m}\right)} \right)$$

Predicted relation between instanton degeneracies and BH degeneracies

S.M., V.Reys arXiv:1512.01553

Single-centered BH (polar degeneracies)

Instanton degeneracies

$$c_m(n,\ell) = (\ell - 2n)\, d(m+n-\ell)\, d(n)$$

$$4mn - \ell^2 < 0, \quad n \geq -1,\ 0 \leq \ell \leq m$$

This formula can still get corrections from lower order terms on both sides, that we have not calculated yet.

Checks of prediction

$\mathbf{m = 1}$:

Δ	(n,ℓ)	$c_1(n,\ell)$	$(\ell - 2n)\, d(1+n-\ell)\, d(n)$
-5	$(-1,1)$	3	3
-4	$(-1,0)$	$\mathbf{48}$	48
-1	$(0,1)$	600	576

$\mathbf{m = 2}$:

Δ	(n,ℓ)	$c_2(n,\ell)$	$(\ell - 2n)\, d(2+n-\ell)\, d(n)$
-12	$(-1,2)$	4	4
-9	$(-1,1)$	72	72
-8	$(-1,0)$	$\mathbf{648}$	648
-4	$(0,2)$	1152	1152
-1	$(0,1)$	8376	7776

Checks of prediction

$\mathbf{m = 3}$:

Δ	(n, ℓ)	$c_3(n, \ell)$	$(\ell - 2n)\, d(3 + n - \ell)\, d(n)$
-21	$(-1, 3)$	5	5
-16	$(-1, 2)$	96	96
-13	$(-1, 1)$	972	972
-12	$(-1, 0)$	$\mathbf{6404}$	6400
-9	$(0, 3)$	1728	1728
-4	$(0, 2)$	15600	15552
-1	$(0, 1)$	85176	76800

We checked this up to m=7 (in principle we can continue).

In each case, the formula agrees in its regime of validity

Checks of prediction

$\mathbf{m = 7}$:

Δ	(n,ℓ)	$c_7(n,\ell)$	$(\ell-2n)\,d(7+n-\ell)\,d(n)$
-77	$(-1,7)$	9	9
-64	$(-1,6)$	192	192
-53	$(-1,5)$	2268	2268
-49	$(0,7)$	4032	4032
-44	$(-1,4)$	19200	19200
-37	$(-1,3)$	128250	128250
-36	$(0,6)$	46656	46656
-32	$(-1,2)$	705030	$\boxed{705024}$
-29	$(-1,1)$	3222780	$\boxed{3221160}$
-28	$(-1,0)$	$\mathbf{11963592}$	11860992
-25	$(0,5)$	384000	384000
-21	$(1,7)$	524880	524880
-16	$(0,4)$	2462496	2462400
-9	$(0,3)$	12713760	12690432
-8	$(1,6)$	4147848	4147200
-4	$(0,2)$	52785360	51538560
-1	$(0,1)$	173032104	142331904

Modification due to mock nature

Expect corrections to sugra formula

Lessons and outlook

- Degrees of freedom of a BH are encoded in an intricate manner in gravity. Modular symmetry is useful to decode.

- More generally gravity path integral seems to know about mock nature.

- Instantons in supergravity encode the BH degeneracies (note: single-centered BHs) via an explicit relation (and an intricate interplay with modular invariance).

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- Many interesting things to do.

Thank you for your attention!

Some more details

"Phenomenology" of the N=4 theory (Meaning of ambiguity in physics)

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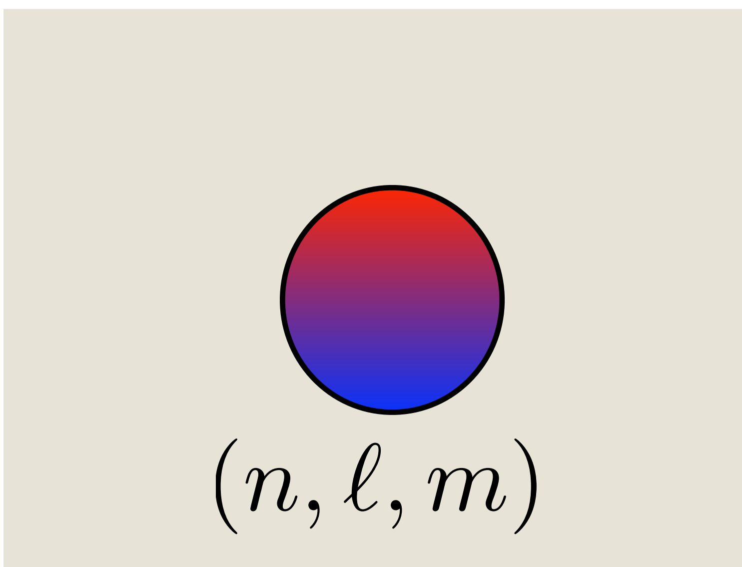

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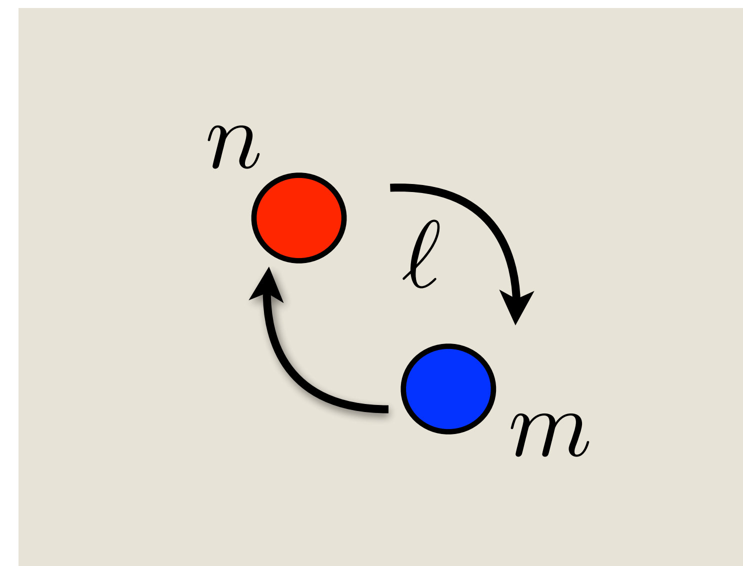

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(Denef '00)

We can recover integer degeneracies from macroscopic (continuum) BH physics

- The quantity $S_{\mathrm{BH}}^{\mathrm{quant}}$ is the result of a functional integral with AdS_2 boundary conditions.

- Computation uses input from string scattering, supergravity, and the technique of supersymmetric localization.

- Localization reduces the perturbative functional integral a one-dimensional integral = leading Bessel function.

- Can identify sub-leading orbifold saddle points: fluctuation integral over them make up sub-leading Bessels.

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where $g^*(\tau) = \left(\frac{i}{2\pi}\right)^{k-1} \int_{-\bar{\tau}}^{\infty} (z+\tau)^{-k}\, \overline{g(-\bar{z})}\, dz$

$$g(\tau) = \sum_{n>0} b_n\, q^n \Rightarrow \quad g^*(\tau) = \sum_{n>0} n^{k-1}\, \bar{b}_n\, \Gamma(1-k, 4\pi n \tau_2)\, q^{-n}$$

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⇒

Holomorphic anomaly equation

$$(4\pi\tau_2)^k\, \frac{\partial \widehat{f}(\tau,\bar\tau)}{\partial\bar\tau} = -2\pi i\, \overline{g(\tau)}\,.$$

Prototype: $\frac{1}{8}$-BPS BHs in string compactification with 32 supercharges

Microscopic degeneracies are given by the Fourier coefficients of a Jacobi form: [J. Maldacena, G. Moore, A. Strominger ('99)]

$$\varphi_{-2,1}(\tau, z) = \frac{\vartheta_1(\tau, z)^2}{\eta(\tau)^6}$$

$$\vartheta_1(\tau, z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} (-1)^n q^{n^2/2} \zeta^n , \qquad \eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n)$$

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$$c(n,\ell) = C(4n - \ell^2) \quad , \qquad d_{\text{micro}}(N) = (-1)^N\, C(N)$$

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