Quantum Modular Forms from Knots & 3-manifolds

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Outline

Introduction to Quantum Modular Form (Zagier 2010)

- colored Jones polynomial for knots
- Witten–Reshetikhin–Turaev Invariant for 3-manifolds

Quantum Modular Form (Zagier 2010)

A quantum modular form is $f : \mathbb{Q} \to \mathbb{C}$ s.t. for ${}^{\forall}\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \subset SL(2;\mathbb{Z})$ the function

$$h_\gamma(x) := f(x) - \chi(\gamma)\,(cx+d)^{-k}\, f\left(\frac{ax+b}{cx+d}\right)$$

has "some properties" of continuity or analyticity.

“some properties”: Eichler integral

The Eichler integral of wt $k \in \mathbb{Z}_{\geq 2}$ cusp form $f(\tau) = \sum_{n=1}^{\infty} a_n q^n$ is

$$\widetilde{f}(\tau) := \sum_{n=1}^{\infty} \frac{a_n}{n^{k-1}} q^n \qquad \tau \in \mathbb{H}$$

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It satisfies for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$

$$\widetilde{f}(\tau) - (c\tau + d)^{k-2} \widetilde{f}(\gamma(\tau)) = -\frac{(2\pi \mathrm{i})^{k-1}}{(k-2)!} \int_{\gamma^{-1}(\mathrm{i}\infty)}^{\mathrm{i}\infty} f(z) (\tau - z)^{k-2} dz$$

period polynomial

“some properties”: mock theta functions

mock theta functions (Ramanujan 1920, Zwegers 2003):
the Appell–Lerch sum

$$\mu(z;\tau) := \frac{\mathrm{i}\mathrm{e}^{\pi \mathrm{i} z}}{\theta_{11}(z;\tau)} \sum_{n\in\mathbb{Z}} \frac{(-\mathrm{e}^{2\pi \mathrm{i} z})^n q^{\frac{1}{2}n(n+1)}}{1-\mathrm{e}^{2\pi \mathrm{i} z} q^n}$$

satisfies

$$\mu(z;\tau) + \sqrt{\frac{\mathrm{i}}{\tau}}\mu(z/\tau; -1/\tau) = \frac{1}{2}\int_0^{\mathrm{i}\infty} \frac{[\eta(z)]^3}{\sqrt{-\mathrm{i}(z+\tau)}} dz$$

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Mathieu moonshine

$$-8 \sum_{w\in\{\frac{1}{2},\frac{1+\tau}{2},\frac{\tau}{2}\}} \mu(w;\tau) = q^{-\frac{1}{8}}\left(-2 + 90q + 462q^2 + 1540q^3 + 4554q^4 + \cdots\right)$$

Examples of Quantum Modular Forms

1. Kontsevich (1997), Zagier (2001), Kashaev (1996): $q^N = 1$

$$F(q) := \sum_{n=0}^{\infty} (q)_n = 1 + (1-q) + (1-q)(1-q^2) + (1-q)(1-q^2)(1-q^3) + \cdots$$

2. Kashaev (1996)

$$J(q) := \sum_{n=0}^{\infty} \left| (q)_n \right|^2$$

3. Lawrence–Zagier (1999):

$$W(q) := \frac{1}{2G} \sum_{\substack{\beta \mod 60N \\ \beta \neq 0 \mod N}} \frac{(1-q^{\beta/5})(1-q^{\beta/3})}{1+q^{\beta/2}} q^{-\frac{\beta^2}{120}}$$

Kontsevich–Zagier Series

Kontsevich (1997) studied $F(q)$ at $\zeta_N := \mathrm{e}^{2\pi\mathrm{i}/N}$

$$F(q) := \sum_{n=0}^{\infty} (q)_n = 1 + (1-q) + (1-q)(1-q^2) + \cdots$$

By numerical computations, he observed

$$F(\zeta_N) \sim N^{3/2} \mathrm{e}^{-\frac{\pi\mathrm{i}}{12}\left(N-3+\frac{1}{N}\right)} + \sum_{n=0}^{\infty} \frac{b_n}{n!} \left(-\frac{2\pi\mathrm{i}}{N}\right)^n$$

with $b_0 = 1$, $b_1 = 1$, $b_2 = 3$, $b_3 = 19$

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Zagier (2001) proved that $\varphi(\alpha) := \mathrm{e}^{\pi\mathrm{i}\alpha/12} F(\mathrm{e}^{2\pi\mathrm{i}\alpha})$ for $\alpha \in \mathbb{Q}$ fulfills

$$\varphi(\alpha) + (\mathrm{i}\alpha)^{-3/2} \varphi(-1/\alpha) = g(\alpha)$$

Kontsevich–Zagier Series and Eichler Integral

$$\varphi(\alpha) := \mathrm{e}^{\pi \mathrm{i} \alpha/12} F(\mathrm{e}^{2\pi \mathrm{i} \alpha})$$

Let $\widetilde{\eta}(\tau) := \sum_{n=0}^{\infty} n \left(\frac{12}{n}\right) q^{\frac{n^2}{24}}$ $\qquad \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n) = \sum_{n=0}^{\infty} \left(\frac{12}{n}\right) q^{\frac{n^2}{24}}$

- $\tau \to 1/N$: $\quad \varphi(1/N) = \widetilde{\eta}(1/N)$
- Nearly modularity of $\widehat{\eta}(z) := \frac{\sqrt{3\mathrm{i}}}{2\pi} \int_{\overline{z}}^{\mathrm{i}\infty} \frac{\eta(\tau)}{(\tau - z)^{3/2}} d\tau \quad (z \in \mathbb{H}^-)$

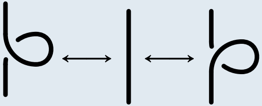
$$\widehat{\eta}(z) + (\mathrm{i}z)^{-\frac{3}{2}} \widehat{\eta}(-1/z) = \frac{\sqrt{3\mathrm{i}}}{2\pi} \int_0^{\mathrm{i}\infty} \frac{\eta(\tau)}{(\tau - z)^{3/2}} d\tau$$

- Limiting values coincide: $\quad \widetilde{\eta}(1/N) = \widehat{\eta}(1/N)$

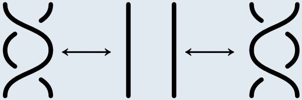
Quantum Knot Invariant

Reidemeister moves

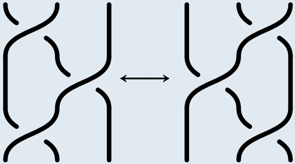
- R I:



- R II:



- R III:



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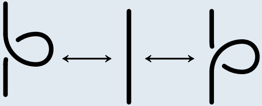

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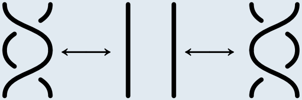

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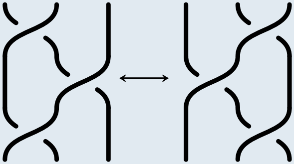
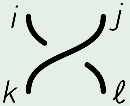


Any link can be given as the closure of tangles.

R-matrix $R^{ij}_{k\ell}$

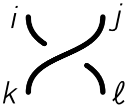


colored Jones polynomial $J_N(K;q)$

The Jones polynomial $J_2(K;q)$ for knot K

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & q^2 & 0 \\ 0 & q^2 & q-q^3 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \qquad i,j,k,\ell \in \{0,1\}$$

$$R = \left(R^{ij}_{k\ell} \right)$$



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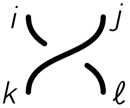
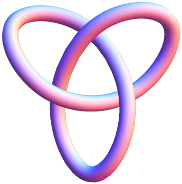


the N-colored Jones polynomial $J_N(K;q)$

$$R^{ij}_{k\ell} = \sum_{m=0}^{\min(N-1-i,j)} \delta_{\ell-i,m}\delta_{j-k,m}(-1)^m \frac{(q)_\ell (q)_{N-1-k}}{(q)_i (q)_m (q)_{N-1-j}} \times q^{-m(i-j)-\frac{m^2}{2}-\frac{Nm}{2}+\left(i-\frac{N-1}{2}\right)\left(j-\frac{N-1}{2}\right)} \qquad i,j,k,\ell \in \mathbb{Z}_N$$

colored Jones polynomial for trefoil 3_1



K.Habiro, G.Masbaum, T.Le, ...

$$(x)_n = \prod_{i=1}^{n}(1 - xq^{i-1})$$

$J_N(3_1;q)$

$$= q^{1-N}\sum_{n=0}^{\infty} q^{-nN}(q^{1-N})_n = \sum_{n=0}^{\infty} q^n (q^{1-N})_n (q^{1+N})_n \Big|_{q\to\frac{1}{q}}$$

N	1	2	3	...
$J_N(3_1;1/q)$	1	$q+q^3-q^4$	$q^2+q^5-q^7+q^8-q^9-q^{10}+q^{11}$	...

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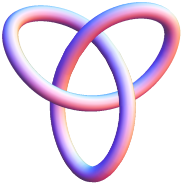

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Kontsevich–Zagier series @ $q = \zeta_N := e^{\frac{2\pi i}{N}}$

$$J_N(3_1; \zeta_N) = \zeta_N F(\zeta_N) = U(-1; \zeta_N^{-1})$$

$$F(q) := \sum_{n=0}^{\infty} (q)_n$$

$$U(x; q) = \sum_{n=0}^{\infty} (-xq)_n (-x^{-1} q)_n q^n$$

unimodal sequence

strongly unimodal sequence $\{a_1, \dots, a_s\}$ of size n & rank $= s - 2k + 1$

$$0 < a_1 < a_2 < \cdots < a_{k-1} < a_k > a_{k+1} > \cdots > a_s > 0 \qquad (a_1 + \cdots + a_s = n)$$

size $n = 5$

$\{a_1, \dots, a_s\}$	$\{5\}$	$\{1,4\}$	$\{4,1\}$	$\{1,3,1\}$	$\{2,3\}$	$\{3,2\}$
rank	0	−1	1	0	−1	1

generating function of $u(m,n) := \# \{\text{seq of size } n \,|\, \text{rank } m\}$

$$U(x;q) := \sum_{m,n} u(m,n) x^m q^n = \sum_{n=0}^{\infty} (-xq)_n (-x^{-1}q)_n q^{n+1} = q + q^2 + \dots$$

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Bryson–Ono–Pitman–Rhoades (2012): $\psi(\tau) := \mathrm{e}^{-\frac{\pi \mathrm{i}}{12}\tau} U(-1; \mathrm{e}^{2\pi \mathrm{i}\tau})$

$$\psi(\tau) + \frac{1}{(-\mathrm{i}\tau)^{3/2}} \psi\left(-\frac{1}{\tau}\right) = \frac{\sqrt{3\mathrm{i}}}{2\pi} \int_0^{\mathrm{i}\infty} \frac{\eta(w)}{(w+\tau)^{3/2}} dw + \frac{[\eta(\tau)]^2}{2\sqrt{\mathrm{i}}} \int_0^{\mathrm{i}\infty} \frac{[\eta(w)]^3}{(w+\tau)^{1/2}} dw$$

Volume Conjecture

Kashaev (1995), H.Murakami–J.Murakami (2000):

$$\lim_{N\to\infty} \frac{2\pi}{N} \log J_N(K; q = \zeta_N) = \mathrm{Vol}(S^3 \setminus K) + \mathrm{i}\, \mathrm{CS}(S^3 \setminus K)$$

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As $J_N(3_1; \zeta_N) = \zeta_N F(\zeta_N)$, we have in $N \to \infty$

$$\zeta_{24N} F(\zeta_N) \sim -\left(\frac{N}{\mathrm{i}}\right)^{3/2} \mathrm{e}^{-\frac{\pi \mathrm{i}}{12} N} F(\mathrm{e}^{-2\pi \mathrm{i} N}) + \sum_{k=0}^{\infty} \frac{T(k)}{k!} \left(\frac{\pi}{12 \mathrm{i} N}\right)^k$$

$$\frac{\sin(2x)\sin(3x)}{\sin(6x)} = \sum_{k=0}^{\infty} \frac{T(k)}{(2k+1)!} x^{2k+1}$$

agrees with

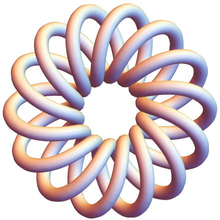
$$\begin{cases} \mathrm{Vol}(S^3 \setminus 3_1) = 0 \\ \mathrm{CS}(S^3 \setminus 3_1) = -\frac{\pi^2}{6} \end{cases}$$

Classification of Knots

Thurston (1978)

knots
- non-hyperblic
  - torus knots ($3_1, \ldots$)
  - satellite knots
- hyperbolic ($4_1, \ldots$)

torus knot $T_{(3,14)}$



satellite knot

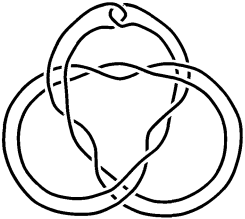
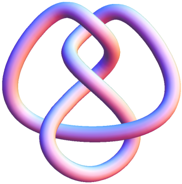


Figure-Eight Knot 4_1



K.Habiro, G.Masbaum, T.Le, ...

$$J_N(4_1;q) = \sum_{n=0}^{\infty} (-1)^n q^{-\frac{1}{2}n(n+1)} (q^{1-N})_n (q^{1+N})_n$$

N	1	2	3	...
$J_N(4_1;q)$	1	$1-q^{\pm1}+q^{\pm2}$	$3-q^{\pm1}-q^{\pm2}+2q^{\pm3}-q^{\pm4}-q^{\pm5}+q^{\pm6}$	...

(4_1 is amphichiral)

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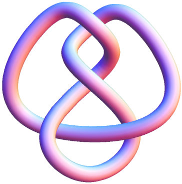

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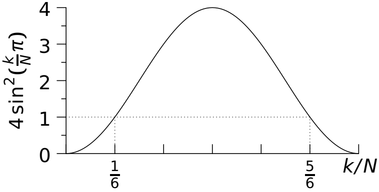
(4_1 is amphichiral)

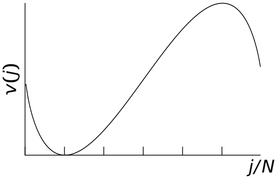
$$J(\zeta_N) := J_N(4_1;\zeta_N) = \sum_{j=1}^{N-1} \prod_{k=1}^{j} 4\sin^2\left(\frac{k}{N}\pi\right)$$

$$J_N(4_1;\zeta_N) = \sum_{j=1}^{N-1} \prod_{k=1}^{j} 4\sin^2\left(\frac{k}{N}\pi\right)$$

$$u(k) = 4\sin^2\left(\frac{k}{N}\pi\right)$$



$$v(j) = \prod_{k=1}^{j} u(k)$$



$$v(\tfrac{5}{6}N) < J_N(4_1;\zeta_N) < N\,v(\tfrac{5}{6}N)$$

$$\lim_{N\to\infty} \frac{1}{N} \log J_N(4_1;\zeta_N) = \lim_{N\to\infty} \frac{1}{N} \log v(\tfrac{5}{6}N) = 4\underbrace{\int_0^{\frac{5}{6}\pi} \log(2\sin x)\,dx}_{\text{Lobachevsky}}$$

$$= 2.02988\cdots$$

Volume Conjecture: 4_1

$$\lim_{N\to\infty} \frac{2\pi}{N} \log J_N(4_1;\zeta_N) = 2.02988\cdots = 2D(\mathrm{e}^{\pi\mathrm{i}/3}) = \mathrm{Vol}(S^3 \setminus 4_1)$$

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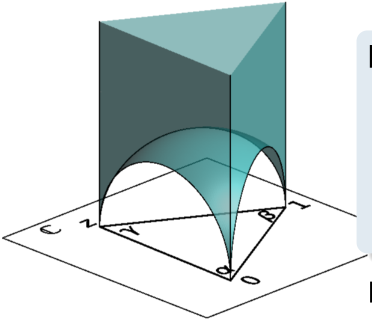
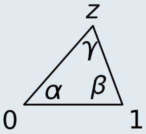


upper half-space: $ds^2 = \dfrac{dx^2+dy^2+dz^2}{z^2}$

hyperbolic ideal tetrahedron $\triangle$:

$$\mathrm{Vol}(\triangle) = D(z)$$



Bloch–Wigner function

$$D(z) := \mathrm{Im}\left(\mathrm{Li}_2(z)\right) + \arg(1-z)\cdot\log|z|$$

Asymptotic Expansion: 4_1

$$\text{knots}\begin{cases}\text{non-hyperblic} & \begin{cases}\text{torus knots}\\ \text{satellite knots}\end{cases}\\ \text{hyperbolic } (4_1,\dots) & \end{cases}$$

Garoufalidis–Zagier

$$J_N(4_1;\zeta_N) \simeq N^{\frac{3}{2}}\,\mathrm{e}^{2D(\mathrm{e}^{\pi \mathrm{i}/3})\frac{N}{2\pi}}\,\frac{1}{\sqrt[4]{3}}\left(1+\frac{11}{36\sqrt{3}}\,\frac{\pi}{N}+\frac{697}{7776}\,\frac{\pi^2}{N^2}+\cdots\right)$$

with a power series in $\frac{\pi}{\sqrt{3}N}$ with rational coefficients.

K		$J_N(K;q)$
torus knots	$\cdots\cdots$	similar to MΘF (shadow is wt $1/2$)
hyperboilc	$\cdots\cdots$	beyond MΘF

Colored Jones Polynomial: Torus Knots

$$J_N(T_{(s,t)};\zeta_N) = \zeta_N^{\frac{s^2t^2-s^2-t^2}{4st}}\, \widetilde{\Phi}_{s,t}^{s-1,1}(1/N)$$

$$\widetilde{\Phi}_{s,t}^{n,m}(\tau) := -\frac{1}{2}\sum_{k=0}^{\infty} k\chi_{s,t}^{n,m}(k)q^{\frac{k^2}{4st}}$$

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Character of $\mathscr{M}(s,t)$, $c = 1 - \frac{6(s-t)^2}{st}$: $\quad \mathrm{ch}_{s,t}^{n,m}(\tau) = \dfrac{\Phi_{s,t}^{n,m}(\tau)}{\eta(\tau)}$

(s,t) coprime, $0 < n < s$, $0 < m < t$

$$\Phi_{s,t}^{n,m}(\tau) := \sum_{k=0}^{\infty} \chi_{s,t}^{n,m}(k)\,q^{\frac{k^2}{4st}}$$

$$\chi_{s,t}^{n,m}(k) := \begin{cases} 1 & k = \pm(nt - ms) \mod 2st \\ -1 & k = \pm(nt + ms) \mod 2st \\ 0 & \text{otherwise} \end{cases}$$

$\Phi_{2,3}^{1,1}(\tau) = \eta(\tau)$ $\qquad$ Rogers–Ramanujan $\quad \Phi_{2,5}^{1,1}(\tau) = q^{9/40}(q,q^4,q^5;q^5)_\infty$, $\quad \Phi_{2,5}^{1,2}(\tau) = q^{1/40}(q^2,q^3,q^5;q^5)_\infty$

Asymptotic Expandion: Torus Knots

$$\Phi_{s,t}^{n,m}(\tau+1) = \mathrm{e}^{\frac{(nt-ms)^2}{2st}\pi \mathrm{i}}\Phi_{s,t}^{n,m}(\tau)$$

$$\Phi_{s,t}^{n,m}\left(-\tfrac{1}{\tau}\right) = \sqrt{\tfrac{\tau}{\mathrm{i}}}\sum_{n',m'}(\mathbf{S}_{s,t})_{n,m}^{n',m'}\,\Phi_{s,t}^{n',m'}(\tau)$$

$$(\mathbf{S}_{s,t})_{n,m}^{n',m'} = \sqrt{\tfrac{8}{st}}(-1)^{nm'+mn'+1}\sin\left(nn'\tfrac{t}{s}\pi\right)\sin\left(mm'\tfrac{s}{t}\pi\right)$$

H-Kirillov (2003)

$$\widetilde{\Phi}_{s,t}^{n,m}(1/N)+\left(\tfrac{N}{\mathrm{i}}\right)^{\frac{3}{2}}\sum_{n',m'}(\mathbf{S}_{s,t})_{n,m}^{n',m'}\,\phi_{s,t}^{n',m'}\,\mathrm{e}^{-\frac{(n't-m's)^2}{2st}\pi \mathrm{i}N} \simeq \sum_{k=0}^{\infty}\frac{T_{s,t}^{n,m}(k)}{k!}\left(\frac{\pi}{2st\mathrm{i}N}\right)^k$$

$$\phi_{s,t}^{n,m} = \begin{cases}(s-n)m & \text{if } nt > ms\\ n(t-m) & \text{if } nt < ms\end{cases}$$

q-Hypergeometric Expression for Torus Knots

$$J_N(T_{(2,2t+1)};q) = q^{t(1-N)} \sum_{k_1,\cdots,k_t\geq 0} (q^{1-N})_{k_t} q^{-Nk_t} \prod_{i=1}^{t-1} q^{k_i(k_i+1-2N)} \begin{bmatrix} k_{i+1} \\ k_i \end{bmatrix}_q$$

$$F_t(q) := q^t \sum_{k_1,\cdots,k_t\geq 0} (q)_{k_t} q^{k_1^2+\cdots+k_{t-1}^2+k_1+\cdots+k_{t-1}} \prod_{i=1}^{t-1} \begin{bmatrix} k_{i+1} \\ k_i \end{bmatrix}_q$$

H–Lovejoy (2014): $F_t(\zeta_N) = U_t(-1;\zeta_N^{-1})$

$$U_t(x;q) := q^{-t} \sum_{k_t\geq\cdots\geq k_1\geq 1} (-xq)_{k_t-1}(-x^{-1}q)_{k_t-1} q^{k_t} \prod_{i=1}^{t-1} q^{k_i^2} \begin{bmatrix} k_{i+1}+k_i-i+2\sum_{j=1}^{i-1} k_j \\ k_{i+1}-k_i \end{bmatrix}_q$$

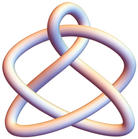
$$= -q^{-\frac{t}{2}-\frac{1}{8}} \frac{(xq)_\infty (x^{-1}q)_\infty}{[(q)_\infty]^2} \left(\sum_{\substack{r,s\geq 0 \\ r\not\equiv s(2)}} - \sum_{\substack{r,s<0 \\ r\not\equiv s(2)}} \right) \frac{(-1)^{\frac{r-s-1}{2}} q^{\frac{1}{8}r^2+\frac{4t+3}{4}rs+\frac{1}{8}s^2+\frac{2+t}{2}r+\frac{t}{2}s}}{1-xq^{\frac{r+s+1}{2}}}$$

3-manifolds

Lickorish (1962), Wallace (1960):
Any closed oriented 3-manifold can be constructed from integral surgery on link $L \subset S^3$

1. Take a knot exterior $S^3 \setminus N(K)$
2. Glue $D^2 \times S^1$ back in $N(K)$ by $h : \partial D^2 \times S^1 \to \partial N(K)$
3. For meridian m and longitude ℓ of $\partial N(K)$, a p/q-surgery is

$$h : \partial D^2 \times \text{point} \mapsto pm + q\ell$$



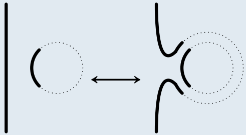
Kirby Calculus

Kirby Moves (1978)

- K I:

$$L \longleftrightarrow L \sqcup \bigcirc \pm 1$$

- K II:



A dotted circle means an any strand (possibly) linked with others.

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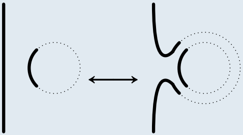

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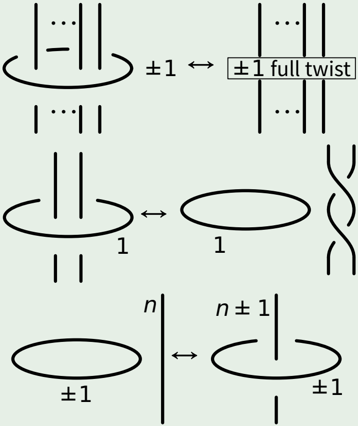
locally,

Fenn–Rourke (1979)



Reshetikhin–Turaev Invariant

M: p_j/q_j-surgery on the j-th component of n-component link L

$$\tau_N(M) = \mathrm{e}^{\frac{\pi \mathrm{i}}{4}\frac{N-2}{N}\left(\sum_j \Phi(U^{(p_j,q_j)}) - 3\,\mathrm{sgn}(\mathbf{L})\right)} \sum_{k_1,\dots,k_n=1}^{N-1} J_{k_1,\dots,k_n}(L) \prod_{j=1}^{n} \rho(U^{(p_j,q_j)})_{k_j,1}$$

$$U^{(p_j,q_j)} = \begin{pmatrix} p_j & r_j \\ q_j & s_j \end{pmatrix} \in SL_2(\mathbb{Z}); \quad \Phi\begin{pmatrix} p & r \\ q & s \end{pmatrix} = \begin{cases} \frac{p+s}{q} - 12 s(p,q) & \text{for } q \neq 0 \\ \frac{r}{s} & \text{for } q = 0 \end{cases}$$

$\mathrm{sgn}(\mathbf{L})$: signature of linking matrix $\mathbf{L}_{j,k} = \mathrm{lk}(j,k) + p_j/q_j \cdot \delta_{j,k}$

$$\rho\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{a,b} = \sqrt{\frac{2}{N}} \sin\left(\frac{ab\pi}{N}\right); \quad \rho\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}_{a,b} = \mathrm{e}^{\frac{a^2}{2N}\pi \mathrm{i} - \frac{1}{4}\pi \mathrm{i}} \delta_{a,b}$$

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$$\tau_N(S^3) = 1 \qquad \tau_N(S^2 \times S^1) = \sqrt{\frac{N}{2}} \frac{1}{\sin(\pi/N)}$$

Chern–Simons Partition Function

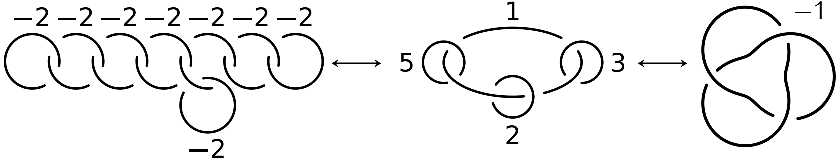
Witten (1989)

$$Z_k(M) = \int e^{2\pi i k \, \mathrm{CS}(A)} dA \qquad \mathrm{CS}(A) = \frac{1}{8\pi^2} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

is related to Reshetikhin–Turaev (1991)

$$Z_k(M) = \frac{\tau_{k+2}(M)}{\tau_{k+2}(S^2 \times S^1)}$$

Poincaré Homology Sphere $\Sigma(2,3,5)$

- $\pi_1(M) = \langle x, y \,|\, (xy)^2 = x^3 = y^5 \rangle$; $\quad H_1(M;\mathbb{Z}) = 0$
- 

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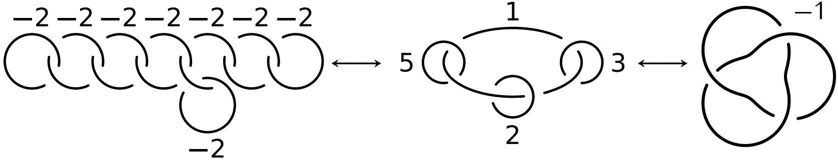
- 

Lawrence–Rozansky (1999):

$$W(\zeta_N) := \zeta_N(\zeta_N - 1)\,\tau_N(M) = \frac{e^{\frac{\pi i}{4}}}{\sqrt{240N}} \sum_{\substack{n=0 \\ N \nmid n}}^{60N-1} \zeta_{120N}^{-n^2-1} \frac{(\zeta_{6N}^{n} - \zeta_{6N}^{-n})(\zeta_{10N}^{n} - \zeta_{10N}^{-n})}{\zeta_{4N}^{n} + \zeta_{4N}^{-n}}$$

$$= 1 - \frac{1}{2}\chi_0^*(\zeta_N)$$

$$\chi_0^*(q) = \sum_{n=0}^{\infty} q^n (q^n)_n$$

Ramanujan's 5th Order Mock Theta Functions

$$\chi_0(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1})_n} \qquad \chi_1(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1})_{n+1}}$$

$$\chi_0^*(q) = 2 - \sum_{n=0}^{\infty} \frac{q^{-n}}{(q^{-n-1};q^{-1})_n}$$

$$\chi_1^*(q) = \sum_{n=0}^{\infty} \frac{q^{-n}}{(q^{-n-1};q^{-1})_{n+1}} = \sum_{n=0}^{\infty} q^{n+1}(q^{n+1})_n$$

WRT Invariant for $\Sigma(2,3,5)$

Lawrence–Zagier (1999): $\zeta_N(\zeta_N - 1)\,\tau_N(M) = 1 + \frac{1}{2}\zeta_{120N}^{-1}\,\widetilde{\Theta}_+(1/N)$

wt $3/2$ unary theta series

$$\begin{cases} \Theta_+(\tau) = q^{\frac{1}{120}}(1 + 11q + 19q^3 + 29q^7 - 31q^8 - \cdots) \\ \Theta_-(\tau) = q^{\frac{49}{120}}(7 + 13q + 17q^2 + 23q^4 - 37q^{11} - \cdots) \end{cases}$$

$$\begin{pmatrix} \Theta_+(-1/\tau) \\ \Theta_-(-1/\tau) \end{pmatrix} = \left(\frac{\tau}{\mathrm{i}}\right)^{\frac{3}{2}} \frac{2}{\sqrt{5}} \begin{pmatrix} \sin(\pi/5) & \sin(2\pi/5) \\ \sin(2\pi/5) & -\sin(\pi/5) \end{pmatrix} \begin{pmatrix} \Theta_+(\tau) \\ \Theta_-(\tau) \end{pmatrix}$$

the Eichler integral

$$\begin{cases} \widetilde{\Theta}_+(\tau) = q^{\frac{1}{120}}(1 + q + q^3 + q^7 - q^8 - \cdots) \\ \widetilde{\Theta}_-(\tau) = q^{\frac{49}{120}}(1 + q + q^2 + q^4 - q^{11} - \cdots) \end{cases}$$

Asymptotic Expansion

Lawrence–Zagier (1999): in $N \to \infty$

$$\widetilde{\Theta}_+(1/N) \sim \sqrt{\frac{N}{\mathrm{i}}}\,\frac{2}{\sqrt{5}}\left(\sin(\pi/5)\,\mathrm{e}^{-\frac{1}{60}\pi \mathrm{i} N} + \sin(2\pi/5)\,\mathrm{e}^{-\frac{49}{60}\pi \mathrm{i} N}\right) + \sum_{k=0}^{\infty} \frac{L(-2k,\chi_+)}{k!}\left(\frac{\pi \mathrm{i}}{60N}\right)^k$$

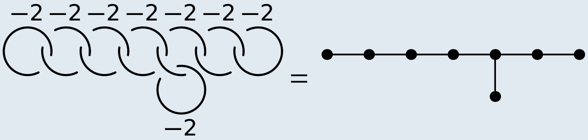
where

$$2\,\frac{\cosh(5x)\cosh(9x)}{\cosh(15x)} = -\sum_{k=0}^{\infty} \frac{L(-2k,\chi_+)}{(2k)!}\,x^{2k}$$

Seifert Manifold

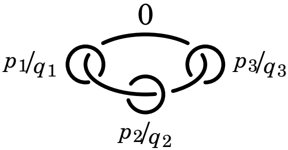
Poincaré homology sphere $\Sigma(2,3,5)$

- homology sphere $H_1(M;\mathbb{Z}) = 0$
- "E_8"



Brieskorn Homology Sphere $\Sigma(p_1,p_2,p_3)$

pairwise coprime $p_i \in \mathbb{Z}_{>1}$

- $M = \Sigma(p_1,p_2,p_3) = \{z_1^{p_1} + z_2^{p_2} + z_3^{p_3} = 0\} \cap S^5$
- 

$$\frac{q_1}{p_1} + \frac{q_2}{p_2} + \frac{q_3}{p_3} = \frac{1}{p_1 p_2 p_3}$$

- $\pi_1(M) = \langle x_1, x_2, x_3, h \,|\, h \text{ center}, x_i^{p_i} = h^{-q_i}, x_1 x_2 x_3 = 1 \rangle$

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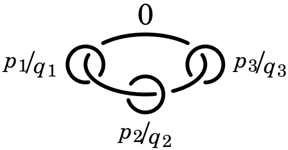
- 

  $\frac{q_1}{p_1} + \frac{q_2}{p_2} + \frac{q_3}{p_3} = \frac{1}{p_1 p_2 p_3}$
- $\pi_1(M) = \langle x_1, x_2, x_3, h \,|\, h \text{ center}, x_i^{p_i} = h^{-q_i}, x_1 x_2 x_3 = 1 \rangle$

Rozansky (1995): $P = p_1 p_2 p_3;$

$$\zeta_N^{\frac{\phi}{4} - \frac{1}{2}} (\zeta_N - 1)\, \tau_N(M) = \frac{\sqrt{\mathrm{i}}}{2\sqrt{2PN}} \sum_{\substack{n=0 \\ N \nmid n}}^{2PN-1} \frac{\zeta_{4PN}^{-n^2}}{\zeta_{2N}^{n} - \zeta_{2N}^{-n}} \prod_{j=1}^{3} \left(\zeta_{2p_j N}^{n} - \zeta_{2p_j N}^{-n} \right)$$

$$\phi = 3 - \frac{1}{P} + 12 \sum_j s\left(\frac{P}{p_j}, p_j\right)$$

Brieskorn Homology Sphere $\Sigma(p_1,p_2,p_3)$ II

$$\zeta_N^{\frac{\phi}{4}-\frac{1}{2}}(\zeta_N-1)\,\tau_N(M)=\frac{1}{2}\widetilde{\Phi}_{p_1,p_2,p_3}^{(1,1,1)}(1/N)$$

the Eichler integral of wt 3/2 unary theta series satisfying

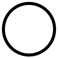
$$\Phi_{p_1,p_2,p_3}^{(\ell_1,\ell_2,\ell_3)}(\tau+1)=\mathrm{e}^{\frac{p_1p_2p_3}{2}\left(1+\sum_j\frac{\ell_j}{p_j}\right)^2\pi\mathrm{i}}\,\Phi_{p_1,p_2,p_3}^{(\ell_1,\ell_2,\ell_3)}(\tau)$$

$$\Phi_{p_1,p_2,p_3}^{(\ell_1,\ell_2,\ell_3)}(-1/\tau)=\left(\frac{\tau}{\mathrm{i}}\right)^{\frac{3}{2}}\sum_{\ell_1',\ell_2',\ell_3'}\mathbf{S}_{\ell_1,\ell_2,\ell_3}^{\ell_1',\ell_2',\ell_3'}\,\Phi_{p_1,p_2,p_3}^{(\ell_1',\ell_2',\ell_3')}(\tau)$$

$$\mathbf{S}_{\ell_1,\ell_2,\ell_3}^{\ell_1',\ell_2',\ell_3'}=\pm\sqrt{\frac{32}{P}}\prod_{j=1}^{3}\sin\left(P\frac{\ell_j\ell_j'}{p_j^2}\pi\right)$$

$$0<\ell_j<p_j$$

Lens Space $L(p,q)$

-  $-p/q$ $\qquad H_1(M;\mathbb{Z}) \cong \mathbb{Z}_p$

Jeffrey (1992)

$$\zeta_N^{-3s(q,p)-\frac{1}{2}}(\zeta_N - 1)\,\tau_N(M) = \sum_{n=1}^{p} \mathrm{e}^{2\pi \mathrm{i}\frac{q}{p}n^2 N} \sum_{\varepsilon=\pm 1} \frac{\varepsilon}{\sqrt{p}} \mathrm{e}^{2\pi \mathrm{i}\frac{q+\varepsilon}{p}n} \zeta_{2Np}^{\varepsilon}$$

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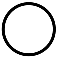
-  $-p/q$ $\qquad H_1(M;\mathbb{Z}) \cong \mathbb{Z}_p$

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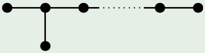
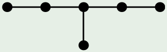
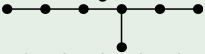
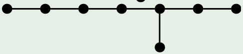
Decomposition:

$$\tau_N(M) = \sum_{\ell \in \mathrm{Tors}\, H_1(M;\mathbb{Z})} \mathrm{e}^{2\pi\mathrm{i}\lambda_M(\ell,\ell)N} \tau_N^{(\lambda_M(\ell,\ell))}(M)$$

linking pairing $\lambda_M : \mathrm{Tors}\, H_1(M;\mathbb{Z}) \otimes H_1(M;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$

ADE Singularities

Seifert manifold: $M = \{f(x,y,z) = 0\} \cap S^5$

	$f(x,y,z)$	M	$H_1(M;\mathbb{Z})$	λ_M
A_n	$x^{n+1} + y^2 + z^2$		$\mathbb{Z}_{n+1}$	$\left(\frac{1}{n+1}\right)$
D_{4K}	$x^{4K-1} + xy^2 + z^2$		$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$
D_{4K+1}	$x^{4K} + xy^2 + z^2$		$\mathbb{Z}_4$	$\left(\frac{3}{4}\right)$
D_{4K+2}	$x^{4K+1} + xy^2 + z^2$		$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\left(\frac{1}{2}\right) \oplus \left(\frac{1}{2}\right)$
D_{4K+3}	$x^{4K+2} + xy^2 + z^2$		$\mathbb{Z}_4$	$\left(\frac{1}{4}\right)$
E_6	$x^4 + y^3 + z^2$		$\mathbb{Z}_3$	$\left(\frac{2}{3}\right)$
E_7	$x^3 y + y^3 + z^2$		$\mathbb{Z}_2$	$\left(\frac{1}{2}\right)$
E_8	$x^5 + y^3 + z^2$		0	$\varnothing$

WRT Invariants for ADE

$\tau_N(M)$ for $M = D_k, E_{6,7,8}$: the Eichler integral of unitary theta series

WRT Invariants for ADE

$\tau_N(M)$ for $M = D_k, E_{6,7,8}$: the Eichler integral of unitary theta series

$M = D_4$:

$$(\zeta_N - 1)\,\tau_N(M) = 4\left(\tfrac{1}{2} - \zeta_{8N}^{-1}\,\widetilde{\Psi}\left(\tfrac{1}{N}\right)\right)$$

where

$$\widetilde{\Psi}(\tau) = q^{\frac{1}{8}}\left(1 - q + q^3 - q^6 + q^{10} - q^{15} + \cdots\right)$$
$$\Psi(\tau) = q^{\frac{1}{8}}\left(1 - 3q + 5q^3 - 7q^6 + 9q^{10} - 11q^{15} + \cdots\right) = [\eta(\tau)]^3$$

Ramanujan's Mock Theta Functions

order		M
3	$\phi(q)$	$E_7 \cong M(0; 2/-1, 3/1, 4/1)$
	$\omega(q)$	$D_5 \cong M(0; 2/-1, 2/1, 3/1)$
5	$\chi_0(q)$	$E_8 \cong \Sigma(2,3,5)$
7	$\mathscr{F}_0(q)$	$E_{10} \cong \Sigma(2,3,7)$
10	$\Psi(q)$	$D_7 \cong M(0; 2/-1, 2/1, 5/1)$

$$\pi_1(M(b; p_1/q_1, p_2/q_2, p_3/q_3)) = \left\langle x_1, x_2, x_3, h \;\middle|\; \begin{array}{l} h \text{ is center} \\ x_i^{p_i} = h^{-q_i} \\ x_1 x_2 x_3 = h^b \end{array} \right\rangle$$

Concluding Remarks

QMF arises from quantum invariants of knots/3-manifolds

- quantum invariants for torus knots/Seifert-fibered manifolds are like mock modular forms,
- quantum invariants for hyperbolic manifolds (e.g. figure-eight knot) are important for Volume Conjecture, and they are mysterious.