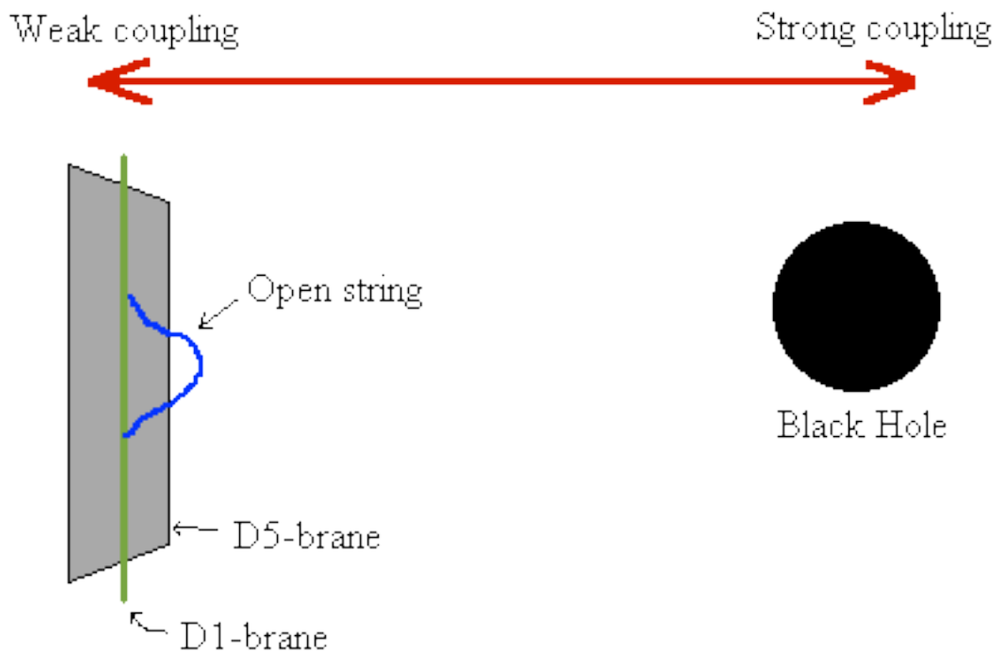
Arithmetic geometry of BPS States



work in progress with
Arnav Tripathy

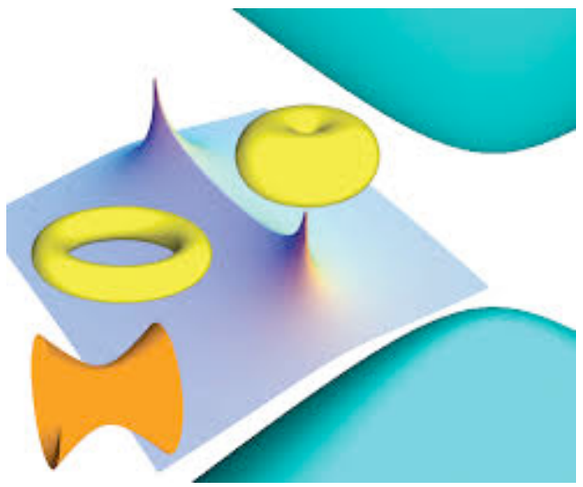
This talk will be about highly speculative work that is still in progress.

We are trying to find relations between natural BPS counts in 4d N=4 string vacua, and simple objects arising in arithmetic algebraic geometry.

Many of the facts I'll describe are known to experts, but there are some new observations, and the interpretations are still missing.

The basic philosophy of the project is that the ways we normally talk about string vacua - in terms of supergravity solutions, or 2d worldsheet CFTs - are perhaps not the best ways to specify the relevant data.

It is at least possible that instead, auxiliary geometric objects (encoding properties of the physics not immediately visible in the above formulations) are more basic, and will eventually be revealed as the more fundamental underlying description.

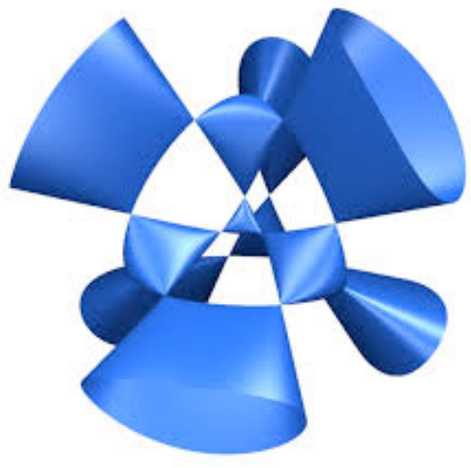


c.f. Seiberg-Witten curve in N=2 theories

1. Introduction

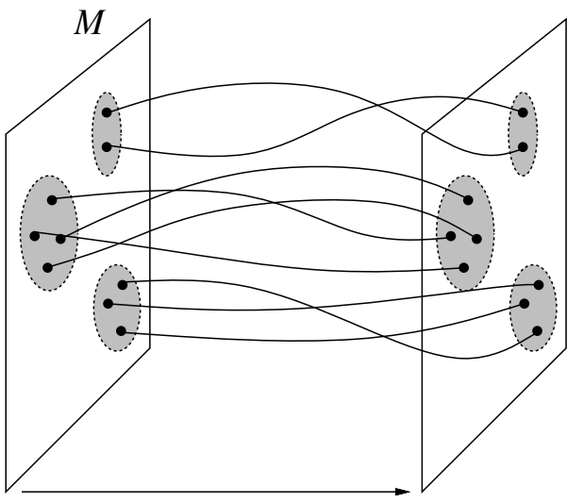
Let me begin with a success story in string theory, that brings many of the relevant objects into play.

We might be interested in counting BPS black holes, carrying enough gauge charges to have finite horizon area in general relativity:



One of the first successes: D1-D5 black holes on a K3 surface.

The 1/2-BPS state counts on K3 come from many duality frames, each giving different pictures of the final result.



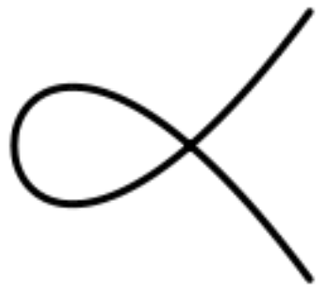
In the sector with charge n, a common picture involves the sigma model with target $\mathrm{Hilb}^{\mathrm{n}}(K3)$.

The resulting count of supersymmetric ground states (Witten index of the sigma models) yields:

$$q^{-1}\sum_n \chi(\mathrm{Hilb}^{\mathrm{n}}(K3))\ q^n = \frac{1}{\eta^{24}(q)}$$

Vafa

This formula has alternate derivations. For instance, in a different duality frame, where the BPS state involves D2-branes wrapping curves in K3, the relevant index localizes on counting nodal curves:



$$q^{-1}\sum_g(\chi(\mathcal{M}_g^H))q^g = \frac{1}{\eta^{24}(q)}$$

Yau, Zaslow

Perhaps the most immediate derivation involves the duality frame of the heterotic string on T^4 :

$$\left(q^{-\frac{1}{24}}\prod_{n=1}^{\infty}\frac{1}{1-q^n}\right)^{24} = \frac{1}{\eta^{24}(q)}$$

24 light cone cone left moving oscillators

Dabholkar, Harvey

The story for 1/4-BPS states, which include the black holes with macroscopic horizon area in string units, is more involved.

The low-energy theory of type II on $K3 \times T^2$ has 28 abelian gauge fields. Therefore, states are characterized by electric and magnetic charge vectors $\vec{Q}_e, \vec{Q}_m$.

Up to U-duality transformations, then, the states carry three kinds of charges, labelled by:

$$\vec{Q}_e \cdot \vec{Q}_e,\ \vec{Q}_m \cdot \vec{Q}_m,\ \vec{Q}_e \cdot \vec{Q}_m$$

Following the work of Dijkgraaf, Verlinde, Verlinde, one can think of obtaining the generating function as a function of three chemical potentials via the sum:

$$\sum_n p^{n-1} Z_{EG}(\mathrm{Hilb^n(K3)}; q, y)$$

Up to a correction factor of the Jacobi form $\phi_{10,1}$, this is the (inverse) Igusa cusp form Φ_{10}.

A direct interpretation of the appearance of the Siegel modular form, has been proposed in various works.

Gaiotto; Dabholkar, Gaiotto;
Banerjee, Sen, Srivastava

The genus 2 surface appears as an M theory lift of a web of string junctions.

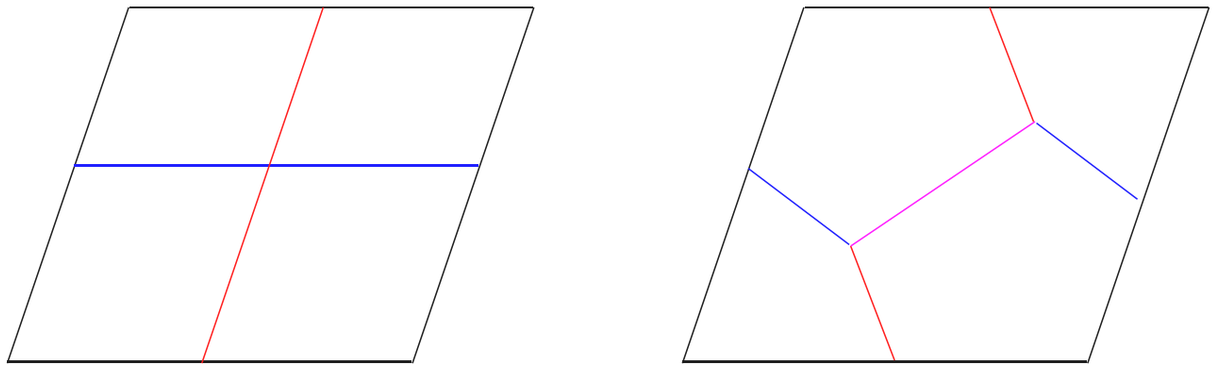


Figure 1: Intersection of strings of charges q_e(red) and q_m(blue) relaxes two supersymetric junctions joined by a string of charges $q_e + q_m$

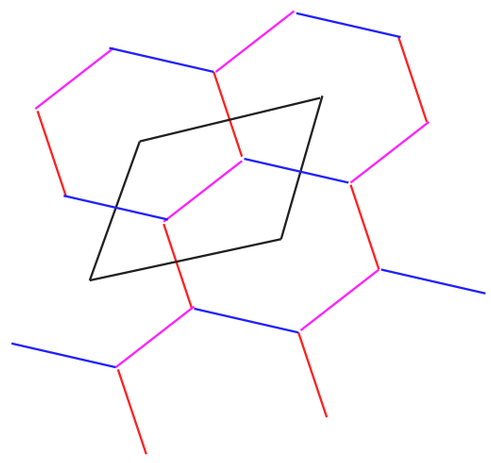


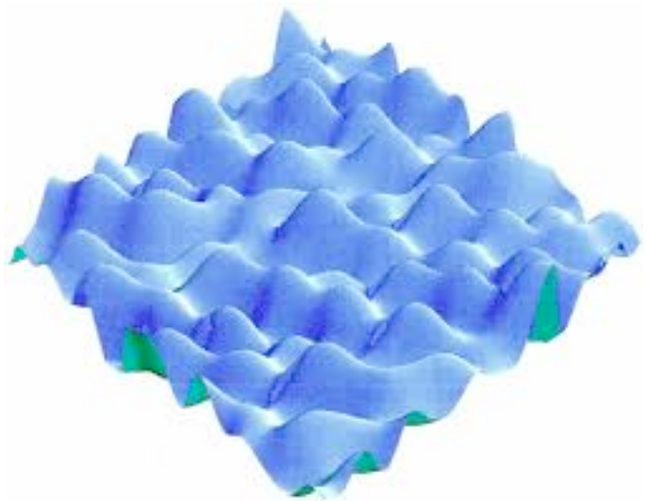
Figure 2: An honeycomb network can be quotiented to give the network on the torus

In this talk, I will describe two new sets of observations about these BPS state counts:

i) A potential relation to modularity conjectures for rigid Calabi-Yau n-folds, in a sense I'll explain

ii) Evolving interpretations in terms of curve counts on K3 surfaces and generalizations

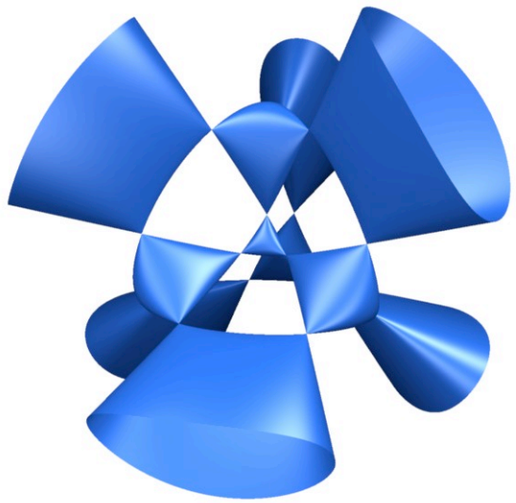
II. More 4d N=4 compactifications



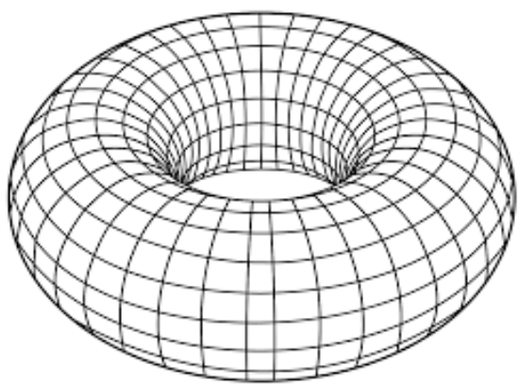
To gather data, it will be useful
to describe more than one connected
component of the moduli space of
4d N=4 string vacua.

In addition to the familiar component arising from
heterotic strings on T^6 and their duals,
there are a number of easily constructed
generalizations -- CHL strings.

I will not detail their construction here, but the basic idea is to start with type II on $K3 \times T^2$ and



orbifold by $\mathbb{Z}_p$ here



accompany by shift here

in such a way as to preserve N=4 supersymmetry.

There is a standard table of resulting theories. The simplest cases arise for primes p such that (p-1) | 24:

c.f. Jatkar, Sen

p	rank	$\mathcal{M}_{\text{dilaton}}$
–	28	$SL(2,Z)\backslash H$
2	20	$\Gamma_1(2)\backslash H$
3	16	$\Gamma_1(3)\backslash H$
5	12	$\Gamma_1(5)\backslash H$
7	10	$\Gamma_1(7)\backslash H$

There are many other examples that work in a slightly more intricate way, but these will suffice for our discussion today.

For each of these connected components in the space of vacua, there is a simple coupling function which counts the 1/2 BPS states.

$$\int d^4x\ \Phi(\tau,\bar{\tau})\left(R_{\mu\nu\kappa\rho}R^{\mu\nu\kappa\rho}-4R_{\mu\nu}R^{\mu\nu}+R^2\right)$$

It is one of the lowest order higher-derivative terms relevant for computing corrections to black hole entropies.

For the standard N=4 vacua, one finds:

$$\Phi(\tau,\bar{\tau}) \;=\; \ln\left(f(\tau)f(\bar{\tau})\cdot(\mathrm{Im}(\tau))^{14}\right)$$

$$f(\tau)=\Delta(\tau)=(\eta(\tau))^{24}\ .$$

The function f governing the correction term, literally counts the 1/2 BPS states.

In the CHL vacua we've discussed, similar terms arise:

p	$f(\tau)$	weight
$-$	$\eta^{24}(\tau)$	12
2	$\eta^8(\tau)\eta^8(2\tau)$	8
3	$\eta^6(\tau)\eta^6(3\tau)$	6
5	$\eta^4(\tau)\eta^4(5\tau)$	5
7	$\eta^3(\tau)\eta^3(7\tau)$	3

I would like to discuss some observations about these functions and moduli spaces.

III. Arithmetic underlying BPS counts?

(with A. Tripathy)

Let us move briefly to another story which many of you know better than me. Consider some simple elliptic curve, like

$$y^2 + y = x^3 - x^2 - 10x - 20$$

You might be interested in the counts of points on this curve, over $\mathbb{F}_p$.

Suppose this number of points is N_p .

Then, let

$$a_p = p + 1 - N_p$$

You extend this definition to non-prime coefficients in a standard way:

$$a_p \cdot a_{p^r} = a_{p^{r+1}} - p \cdot a_{p^{r-1}}$$

and to more composite index coefficients via

$$a_{mn} = a_m \cdot a_n, \quad \gcd(m, n) = 1$$

These definitions have been for "primes of good reduction"; the coefficient is modified in a simple way at primes of bad reduction (i.e. when the curve is singular viewed as a curve over $\mathbb{F}_p$).

For the particular curve I mentioned, 11 is a prime of bad reduction.

Then if we gather the resulting coefficients together

$$f(q) \equiv \sum a_n q^n$$

we find an elegant result:

$$f(q) = \eta^2(\tau)\eta^2(11\tau)$$

This is a cusp form of weight 2 for $\Gamma_0(11)$!

This is a particular example of a famous relation between elliptic curves and weight 2 cusp forms following from Taniyama-Shimura (now the modularity theorem):

Taniyama-Shimura Conjecture: Let E be an elliptic curve whose equation has integer coefficients, let N be the so-called conductor of E and, for each n, let a_n be the number appearing in the L-function of E . Then there exists a modular form of weight two and level N which is an eigenform under the Hecke operators and has a Fourier series $\sum a_n q^n$.

Here are some natural questions:

Are there analogous results for modular forms associated to counting points on higher dimensional varieties?

Do these point counts have any interesting physical interpretation, say related to the appearance of automorphic forms in various string theory BPS state counts?

c.f. Candelas, de la Ossa; Schimmrigk

There are only partial answers to the first question, e.g.:

RIGID CALABI-YAU THREEFOLDS OVER $\mathbb{Q}$ ARE MODULAR

FERNANDO Q. GOUVÊA AND NORIKO YUI

Abstract. The proof of Serre's conjecture on Galois representations over finite fields allows us to show, using a method due to Serre himself, that all rigid Calabi-Yau threefolds defined over $\mathbb{Q}$ are modular.

But there is quite a bit of lore coming from the Langlands program, and various explorers have uncovered examples of relations in specific cases.

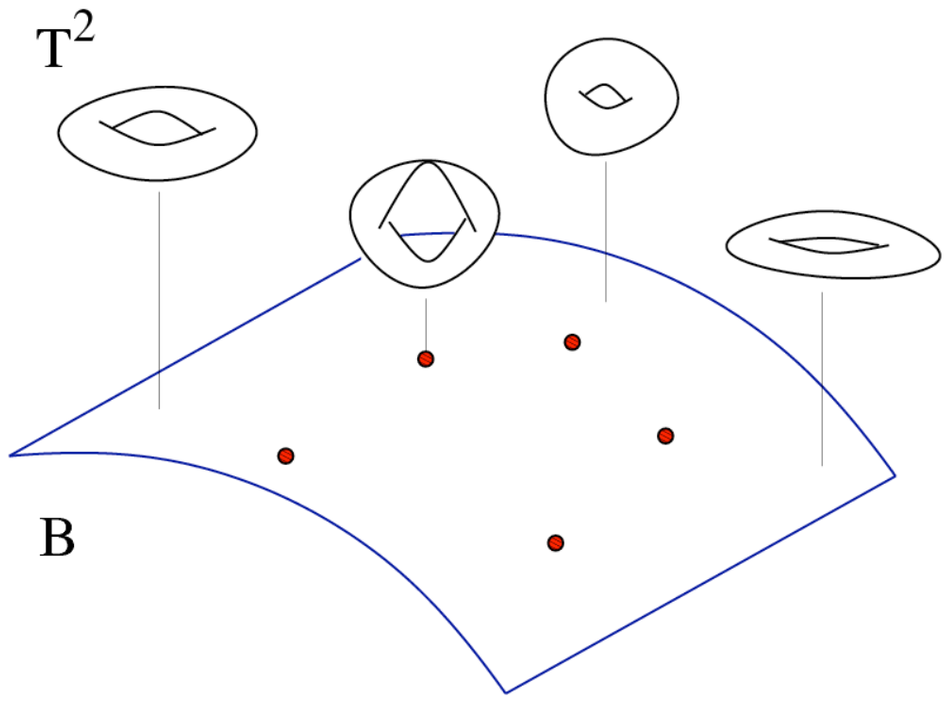
The basic result is that motives with the same structure as "rigid" CY manifolds, should be modular (in the sense that their point counts give automorphic forms for SL(2,Z) or congruence subgroups thereof).

A special exception occurs in d=2: the only non-trivial Calabi-Yau space is K3. It is not rigid, but K3 surfaces which are singular (in the sense that their Picard number is 20!) are thought to be modular.

Let us return to our list of N=4 models.

* For each of our test cases, the moduli space of the axio-dilaton $\mathcal{M}_{\rm dilaton}$ is of genus zero.

* There is a standard construction associated with such moduli spaces, yielding something called a Kuga-Sato variety.



Here is the list again:

p	$f(\tau)$	weight	k
—	$\eta^{24}(\tau)$	12	10
2	$\eta^8(\tau)\eta^8(2\tau)$	8	6
3	$\eta^6(\tau)\eta^6(3\tau)$	6	4
5	$\eta^4(\tau)\eta^4(5\tau)$	5	3
7	$\eta^3(\tau)\eta^3(7\tau)$	3	1

We've added a column for k, which is found by subtracting 2 from the weight of the form governing the R^2 correction.

Now, consider the Kuga-Sato variety obtained by taking the k-fold fiber product of the elliptic curve above each point on $\mathcal{M}_{\text{dilaton}}$.

We claim that there is a Calabi-Yau manifold birational to this space, whose point counts yield the modular form $f(\tau)$ counting half-BPS states, for each model.

That is, the BPS states in the CHL model with a given k, are tied to arithmetic geometry of a Calabi-Yau k+1-fold which is an elliptic fibration over $\mathcal{M}_{\text{dilaton}}$.

For this construction to yield (rigid) Calabi-Yau spaces, it was important that the modular curve for the relevant S-duality group be of genus zero.

Example:

For k=1, we get an elliptic fibered K3 surface.

A K3 surface whose Frobenius traces correctly give the desired modular form was studied by Ahlgren-Ono-Penniston. It is a double cover of $\mathbb{P}^2$ branched over 6 lines:

$$X_\lambda \quad : \quad s^2 = xy(x+1)(y+1)(x+\lambda y)$$

with $\lambda = -64$. (Other values seem to enjoy a similar reationship with CHL models I'm not discussing here).

Example:

MODULAR FORMS AND CALABI-YAU VARIETIES

KAPIL PARANJAPE[1] AND DINAKAR RAMAKRISHNAN[2]

Paranjape and Ramakrishnan have found several examples of Calabi-Yau manifolds (via the construction I mentioned) whose zeta functions are related to the CHL counting functions appropriately.

Most basically, Δ itself arises from counting points on a Calabi-Yau eleven-fold birational to the 10-fold Kuga-Sato fiber product over $SL(2,Z)\backslash H$.

Example:

This example is distinct from the others in that it isn't an elliptic fibration over the axio-dilaton moduli space. Rather (in a case not appearing in the table, corresponding to the prime p = 11), we choose a CHL string with k=0 -- giving rise to a weight 2 cusp form as $f(\tau)$.

In this model, the axio-dilaton moduli space is:

$$\mathcal{M}_{\text{dilaton}} = \Gamma_0(11)\backslash H$$

And the particular modular form which arises is our friend:

$$\eta^2(\tau)\eta^2(11\tau)$$

IV. Lifts and curve counts

It is an interesting fact that the 1/4 BPS counting functions for each of these CHL strings, can be obtained as "lifts" of the 1/2 BPS counting functions.

Let us start with the canonical example. The 1/2 BPS states of the heterotic string have counting function $\eta^{24}(\tau)$.

There are a couple of canonical "lifts" in the modularity literature that turn this into the 1/4 BPS counting function $\Phi_{10}(\tau, \sigma, z)$.

The first lift, discussed by Skoruppa, takes a cusp form of weight k to a Jacobi form of weight k-2 and index 1:

$$\eta^{24}(\tau) \to \eta^{18}(\tau)\theta_1(\tau,z)^2 \equiv \phi_{10,1}(\tau,y)$$

The Maass lift from Jacobi to Siegel forms then takes

$$\phi_{10,1} \to \Phi_{10}$$

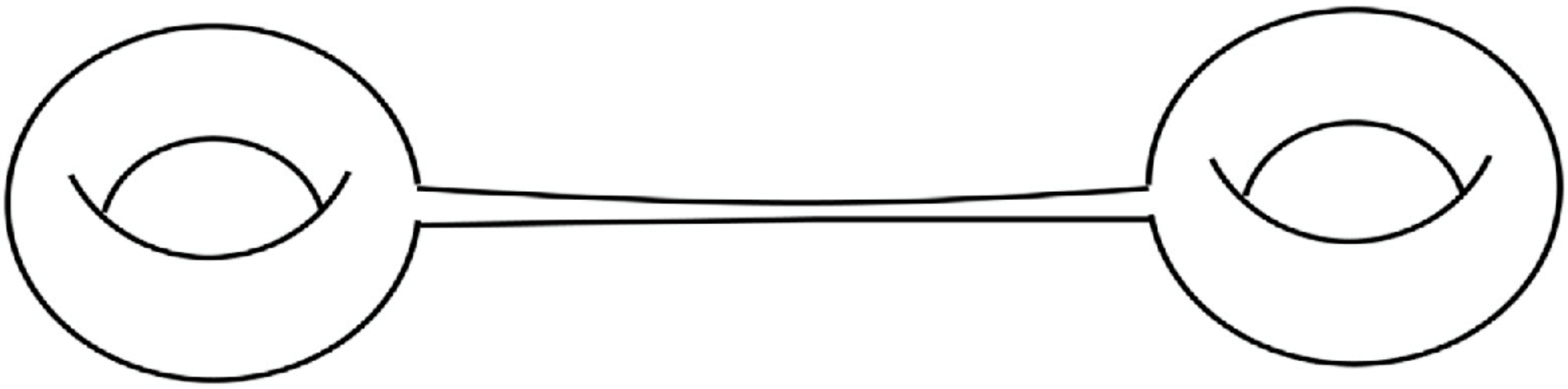
$$\phi_{10,1} = \sum_{n,\ell} c(n,\ell) q^n y^\ell$$

$$\Phi_{10}(q,y,p) = \sum g(n,\ell,m) q^n y^\ell p^m$$

$$g(n,\ell,m) = \sum_{d|(n,\ell,m)} d^9 c(\frac{n\ell}{d^2}, \frac{m}{d})$$

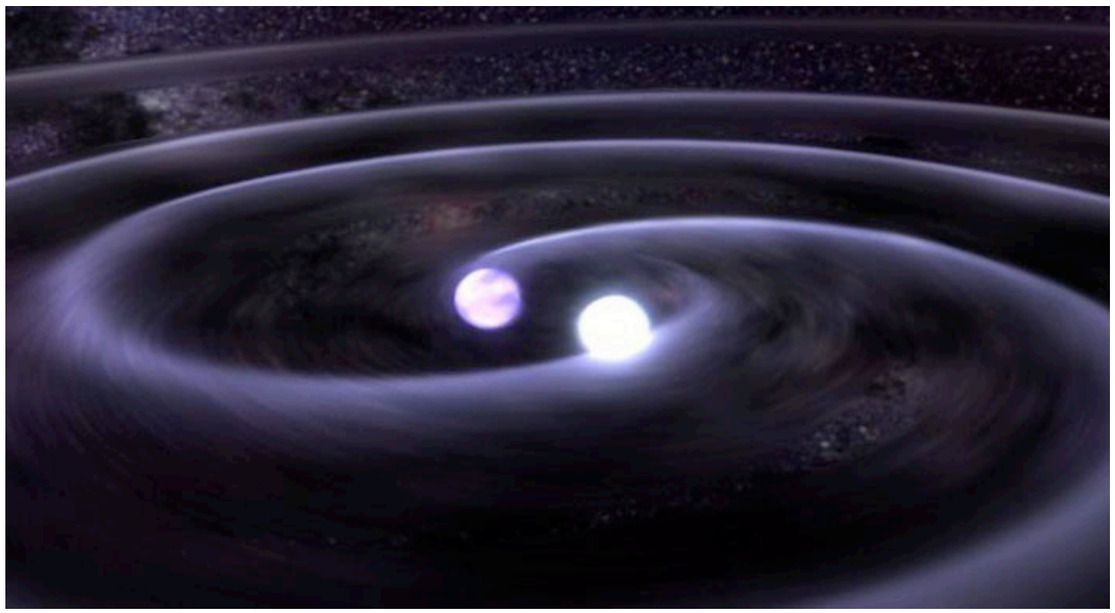
This pattern repeats for each of the CHL strings I've described. The 1/2 BPS function "lifts" to a quarter BPS counting function.

The intuition is perhaps the following. The Siegel form is defined on a genus two surface. In a suitable limit



the form becomes a product:

$$\Phi_{10}(\tau, \sigma, z \to 0) \sim z^2 \eta^{24}(\tau) \eta^{24}(\sigma)$$



In this limit, the quarter BPS black hole is degenerating to a widely separated pair of 1/2 BPS black holes.

The "Skoruppa lift" is a kinematic factor accounting for center of mass degrees of freedom.

These facts should have echoes in other pictures of the system.

In algebraic geometry, the BPS states have found interpretation in terms of suitable curve counts on

$$K3 \times T^2 \ .$$

* The 1/2 BPS counting function, as we discussed earlier, counts nodal curves on K3.

* In a 5D picture on $K3 \times S^1$, there should be additional quantum numbers one can grade the state by.

In particular, a massive particle in 5D has a little group given by $SO(4) \simeq SU(2) \times SU(2)$.

So, there are two additional quantum numbers we can grade a 1/2 BPS state with, the SU(2) spins.

The first SU(2) was added quite some time ago by Katz, Klemm, Vafa. The (inverse) counting function is:

$$\phi_{KKV}(\tau, z) = q \prod_k (1-q^k)^{20} (1-q^k y)^2 (1-q^k y^{-1})^2$$

This is nothing but our friend $\phi_{10,1}$.

Much more recently, Katz-Klemm-Pandharipande have generalized this to include the second SU(2):

$$\phi_{KKP}(\tau,z,w) = q\prod_k (1-q^k)^{20}(1-q^k uy)(1-q^k uy^{-1})(1-q^k u^{-1}y)(1-q^k u^{-1}y^{-1})$$

The KKV functions lifts to the Igusa cusp form. This raises the question: what is the analogous 4 variable generalization of the KKP counting function?

1/4 BPS black holes on $K3\times T^2$ have four charges, as in addition to the electric and magnetic charges and their dot product, there is a spin.

We have some conjectures about this, which aren't quite ready for public appearance.

Here, I want to mention some further connections to curve counting.

On the modularity side, passing from 1/2 to 1/4 BPS states corresponds to a lift. What is the mirror of this lift in enumerative algebraic geometry?

CURVE COUNTING ON $K3 \times E$, THE IGUSA CUSP FORM χ_{10}, AND DESCENDENT INTEGRATION

G. OBERDIECK AND R. PANDHARIPANDE

These authors define $N_{g,\beta,d}$:

# of genus g holomorphic curves (or really orbits under action of the elliptic curve) in the class

$$\beta \in H^2(S, \mathbb{Z})$$

wrapping the torus d times.

Suppose that β_h is primitive, and that

$$\langle \beta_h, \beta_h \rangle = 2h - 2 \ .$$

Then the O-P conjecture is as follows. If you define the counting function

$$\mathsf{N}^{X\bullet}(u,q,\tilde{q}) = \sum_{g\in\mathbb{Z}}\sum_{h\geq 0}\sum_{d\geq 0}\mathsf{N}^{X\bullet}_{g,\beta_h,d}\; u^{2g-2}q^{h-1}\tilde{q}^{d-1}.$$

it satisfies:

$$N^{X\bullet} = -\frac{1}{\Phi_{10}}\ .$$

In fact, this function is a count of quarter BPS states in string compactification on $K3\times T^2$.

This basic structure:

* counts of 1/2 & 1/4 BPS states correspond to curve-counts in a suitable geometry

* adding wrappings of the base of a fibration maps to a "lift" on the modular side

is repeated in all of our examples.

For the CHL strings we mentioned, at a given (small) prime p the 1/2-BPS counting function was given by:

$$\eta^{k+2}(\tau)\eta^{k+2}(p\tau)$$

These lift to (higher level) genus 2 Siegel modular forms of weight k, for each p.

The type II geometries corresponding to these models are $\mathbb{Z}_p$ orbifolds of $K3 \times T^2$.

And, curve counts in the fiber,
and counts of curves also wrapping the base, give generating functions related by automorphic lifts.

As the final speaker, it is my duty and pleasure to thank the organizers:




Thank you!