Gauge Theory and Generalized Appell Functions

Jan Manschot

Trinity College Dublin
Coláiste na Tríonóide, Baile Átha Cliath
The University of Dublin

Number Theory and Physics
IHP, Amphitheater Hermite
May 24th, 2016

Appell function (1884)

SUR LES

FONCTIONS DOUBLEMENT PÉRIODIQUES

DE

TROISIÈME ESPÈCE[1],

Par M. P. APPELL,

MAITRE DE CONFÉRENCES A L'ÉCOLE NORMALE.

Ce Mémoire a pour objet l'étude des fonctions doublement périodiques de troisième espèce et plus particulièrement la décomposition de ces fonctions en éléments simples. Il se termine par quelques re-

Paul Émile Appell (1855-1930)

Source: IHP, 2nd floor

Source: fr.wikipedia.org

Appell function (1884)

“élément simple:”

$$A(u, v; \tau) = e^{\pi i u} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{2\pi i n v} q^{n(n+1)/2}}{1 - e^{2\pi i u} q^n}$$

with $q = e^{2\pi i \tau}$, $\tau \in \mathbb{H}$, $v \in \mathbb{C}$ and $u \in \mathbb{C}/\{\mathbb{Z}\tau + \mathbb{Z}\}$

Mock theta functions

Ramanujan's mock theta function (1920):

$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \dots$$

Watson (1936):

$$f(q) = \frac{2}{(q)_\infty} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}$$

Modular completion

Zwegers (2002):

Completion:

$$\widehat{A}(u,v) = A(u,v) + \frac{i}{2}\theta_1(v)\,R(u-v)$$

with

$$R(u) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \Big(\mathrm{sgn}(n) - E((n+\mathrm{Im}(u)/y)\sqrt{2y})\Big) \times(-1)^{n-\frac{1}{2}} e^{-2\pi i u n} q^{-n^2/2}$$

Properties:

- $\widehat{A}(u,v)$ transforms as a Jacobi form of weight 1 under $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$
- $$\frac{A(u+z,v+z)}{\theta_1(v+z)} - \frac{A(u,v)}{\theta_1(v)} = \frac{\eta^3\,\theta_1(u+v+z)\,\theta_1(z)}{\theta_1(u)\,\theta_1(v)\,\theta_1(u+z)\,\theta_1(v+z)}$$

Other applications

- Vacuum character of $\mathcal{N} = 2, 4$ superconformal algebra Eguchi, Taormina (1988) $\Longrightarrow$ K3 Mathieu moonshine Eguchi, Ooguri, Tachikawa (2011)
- Characters of admissible $\hat{sl}(m|n)$-representations Kac, Wakimoto (2000), Semikhatov, Taormina, Tipunin (2003)
- Mirror symmetry of elliptic curves Polishchuk (1998)

Generalized Appell functions

$$A_{Q,\{\boldsymbol{m}_j\}}(\boldsymbol{u},\boldsymbol{v};\tau) := \sum_{\boldsymbol{k}\in\mathbb{Z}^{n_+}} \frac{q^{\frac{1}{2}Q(\boldsymbol{k})}e^{2\pi i\boldsymbol{v}\cdot\boldsymbol{k}}}{\prod_{j=1}^{n_-}(1-e^{2\pi i u_j}q^{\boldsymbol{m}_j\cdot\boldsymbol{k}})}$$

with:

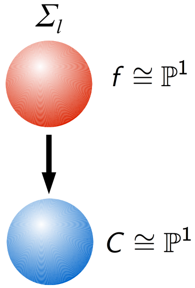
- Q: positive definite n_+-dimensional quadratic form
- $\boldsymbol{v} \in \mathbb{C}^{n_+}$
- $\boldsymbol{u} = (u_1,\ldots,u_{n_-}) \in \mathbb{C}^{n_-}/\{\text{poles}\}$
- $\boldsymbol{m}_j \in \mathbb{Z}^{n_+}$ for $j = 1,\ldots,n_-$

I will explain in this talk how specializations of $A_{Q,\{\boldsymbol{m}_j\}}$ occur as building blocks of:

- (*for physicists*) the partition function of $\mathcal{N} = 4$ supersymmetric topologically twisted Yang-Mills theory
- (*for mathematicians*) generating functions of invariants of moduli spaces of semi-stable vector bundles

Rational ruled surface

Restriction to the rational ruled surface Σ_1 in this talk



Rational ruled surface
$\pi : \Sigma_1 \to \mathbb{P}^1$

- $H_2(\Sigma_1, \mathbb{Z})$: generators C, f
- Intersection numbers: $C^2 = -1,\ f^2 = 0,\ C \cdot f = 1$
  $\Rightarrow$ signature (1,1) lattice
- self-dual Kähler form $J_{m,n} = m(C+f) + nf$ with $m, n \geq 0$

Yang-Mills theory

Gauge group $G = U(N)$

- Connection 1-form: $A \in \Omega^1(S, \mathfrak{g})$
- Field strength: $F = dA + A \wedge A \in \Omega^2(S, \mathfrak{g})$
- Action: $\mathcal{S}[A] = -\frac{1}{g^2} \int_S \mathrm{Tr}\, F \wedge *F + \frac{i\theta}{8\pi^2} \int_S \mathrm{Tr}\, F \wedge F$

Chern classes $c_i \in H^{2i}(S, \mathbb{Z})$:

$$c_1 = \frac{i}{2\pi} \mathrm{Tr}\, F, \qquad c_2 - \frac{1}{2} c_1^2 = \frac{1}{8\pi^2} \mathrm{Tr}\, F \wedge F$$

Hermitean Yang-Mills solutions:

- minimize action $\mathcal{S}[A]$ for fixed c_1, c_2
- $\int_S F \wedge J$ is proportional to the identity matrix and $F^{(2,0)} = F^{(0,2)} = 0$
- solution space $\mathcal{M}_J(\gamma)$ with $\gamma = (N, c_1, c_2) = (r, c_1, c_2)$

Topologically twisted $\mathcal{N} = 4$ $U(N)$ Yang-Mills on S

- Path integral:

$$Z_r(\tau; J) = \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}X \, e^{-\mathcal{S}[A,\psi,X]}$$

with $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$

- $Z_N(\tau; J)$ localizes on HYM solutions

Vafa, Witten (1994)

- Then

$$e^{-\mathcal{S}[A]} = q^{\frac{1}{2r}(c_1)_+^2} \bar{q}^{\Delta(\gamma) - \frac{1}{2r}(c_1)_-^2}$$

with $\Delta(\gamma) = c_2 - \frac{r-1}{2r} c_1^2$ and $c_{1+} = \frac{c_1 \cdot J}{|J|}$

- Coefficient: Euler number $\chi(\mathcal{M}_J(\gamma))$

Theta function decomposion

$$Z_r(\tau; J) = \sum_{c_1 \in H^2(S,\mathbb{Z}/r\mathbb{Z})} \overline{h_{r,c_1}(\tau; J)}\, \Theta_{r,c_1}(\tau)$$

with

$$h_{r,c_1}(\tau; J) = \sum_{c_2} \chi(\mathcal{M}_J(\gamma))\, q^{\Delta(\gamma)}$$

From YM theory to vector bundles

Computation of $h_{N,c_1}(\tau)$ for rational surfaces is possible using methods for complex surfaces Gottsche, Nakajima, Yoshioka,...

Donaldson-Uhlenbeck-Yau theorem:

$$\text{HYM} \quad \text{connections} \Leftrightarrow \begin{cases} - & \text{stable} \quad \text{holomorphic vector} \quad \text{bundles} \\ - & \text{reducible} \quad \text{connections} \end{cases}$$

Slope stability

Choose polarization (Kähler form) $J \in C(S)$, and define the slope:

$$\mu_J(E) := \frac{c_1(E) \cdot J}{r(E)}$$

Definition:

A bundle E is stable if for every subbundle $E' \subsetneq E$,
$\mu_J(E') < \mu_J(E)$ (semi-stable $\Rightarrow \mu_J(E') \leq \mu_J(E)$)

Invariants of moduli spaces

- Let $\mathcal{I}(\gamma, w; J)$ be the virtual Poincaré polynomial of the moduli stack $\mathfrak{M}_J(\gamma)$ of semi-stable vector bundles

  Joyce (2008)

- They determine rational and integer BPS invariants

$$\bar{\Omega}(\gamma, w; J) := \sum_{\substack{\gamma_1+\cdots+\gamma_\ell=\gamma \\ \mu_J(\gamma_i)=\mu_J(\gamma),\forall i}} \frac{(-1)^{\ell-1}}{\ell} \prod_{i=1}^{\ell} \mathcal{I}(\gamma_i, w; J)$$

and

$$\begin{aligned} \Omega(\gamma; J) &= \sum_{m|\gamma} \frac{\mu(m)}{m} \bar{\Omega}(\gamma/m, -(-w)^m; J) \\ &= \frac{w^{-d(\gamma)/2} \sum_{k=0}^{d(\gamma)} b_k(\mathcal{M}_J(\gamma))\, w^k}{w - w^{-1}}, \end{aligned}$$

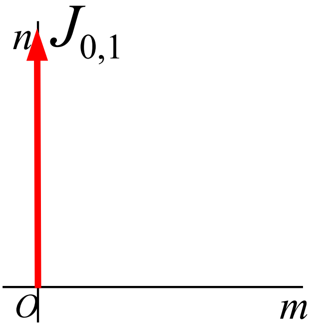
with b_k conjecturably Betti numbers of the intersection cohomology of $\mathcal{M}_J(\gamma)$

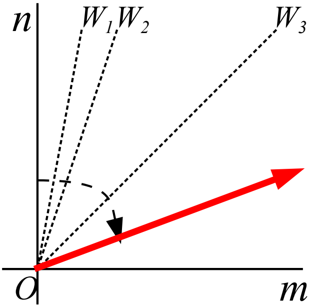
Generating functions:

- $$H_{r,c_1}(z;J) := \sum_{c_2} \mathcal{I}(\gamma, w; J)\, q^{\Delta(\gamma)}$$

- $$h_{r,c_1}(z;J) := \sum_{c_2} \bar{\Omega}(\gamma, w; J)\, q^{\Delta(\gamma)}$$

Outline of the computation





1. Determine invariants for boundary polarization

2. Wall-crossing

Boundary polarization

Conjecture: JM (2011)

$$H_{r,c_1}(z; J_{0,1}) = \begin{cases} 0 & \text{if } c_1 \cdot f \neq 0 \mod r \\ H_r(z) & \text{if } c_1 \cdot f = 0 \mod r \end{cases}$$

where $H_r(z,\tau)$ is the infinite product:

$$H_r(z) := \frac{i\,(-1)^{r-1}\,\eta^{2r-3}}{\theta_1(2z)^2\,\theta_1(4z)^2\ldots\theta_1((2r-2)z)^2\,\theta_1(2rz)},$$

with $\eta = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$, $\theta_1(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}}(-1)^r q^{\frac{r^2}{2}} w^r$, $w = e^{2\pi i z}$

Argument: bundles on $f \cong \mathbb{P}^1$ are equivalent to direct sums $\Rightarrow$ unstable for $c_1 \cdot f \neq 0 \mod r$

Proofs: $r=1$: Gottsche (1990), $r=2$: Yoshioka (1995), general r: Mozgovoy (2013)

Joyce wall-crossing formula

$$\mathcal{I}(\gamma, w; J') = \sum_{\sum_{i=1}^{\ell} \gamma_i = \gamma, r_i \geq 1} S(\{\gamma_i\}, J, J')\, w^{-\sum_{i<j}(r_i c_{1,j} - r_j c_{1,i})\cdot K_S} \times \prod_{i=1}^{\ell} \mathcal{I}(\gamma_i, w; J)$$

with

$$S(\{\gamma_i\}, J, J') = \frac{1}{2^{\ell-1}} \times \prod_{i=2}^{\ell} \left[\mathrm{sgn}(\mu_J(\gamma_{i-1} - \gamma_i) - v) - \mathrm{sgn}(\mu_{J'}(\textstyle\sum_{j=1}^{i-1} \gamma_j - \sum_{j=i}^{\ell} \gamma_j) - v)\right]$$

with $0 < v \ll 1$

Explicit expressions I

Rank 1:

$$H_{1,c_1}(z;J) = H_1(z)$$

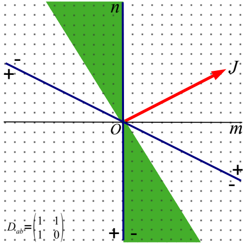
Gottsche (1990)

Rank 2: using wall-crossing formula

$$H_{2,\beta C-\alpha f}(z;J_{m,n}) = \delta_{\beta,0}\, H_2(z) \qquad |q^{\frac{1}{4}}| < |w| < |q^{-\frac{1}{4}}|$$

$$+\, H_1(z)^2 \sum_{(a,b)=-(\alpha,\beta)\ \mathrm{mod}\ 2} \tfrac{1}{2}\left(\,\mathrm{sgn}(b-\varepsilon) - \mathrm{sgn}(bn-am-\varepsilon)\,\right) w^{-b+2a}\, q^{\frac{1}{4}b^2+\frac{1}{2}ab}$$

Indefinite theta function:



Hoppe, Spindler (1980); Ellingsrud, Strømme (1987), Yoshioka (1994/5), Gottsche, Zagier (1996)

Explicit expressions II

Specialize to $J = J_{1,0}$, $\beta = 1$:

Geometric sum $\Longrightarrow$

$$H_{2,C-\alpha f}(z; J_{1,0}) = H_1(z)^2 \sum_{b \in 1 \bmod 2\mathbb{Z}} \frac{w^{2\alpha - b} q^{\frac{1}{4}b^2 + \frac{1}{2}\alpha b}}{1 - w^4 q^b}$$

Yoshioka (1994/5), Bringmann, JM (2010)

Specialization of Appell function $A(u, v)$

Example:

JM (2014)

$$H_{3,0}(z;J_{1,0}) = H_3(z) + 2H_1(z)H_2(z)\sum_{k\in\mathbb{Z}}\frac{w^{-6k}q^{3k^2}}{1-w^6q^{3k}}$$
$$+H_1(z)^3\sum_{k_1,k_2\in\mathbb{Z}}\frac{w^{-2(k_1+2k_2)}q^{k_1^2+k_2^2+k_1k_2}}{(1-w^4q^{2k_1+k_2})(1-w^4q^{k_2-k_1})}$$

- Specialization of $A_{Q,\{\boldsymbol{m}_j\}}$ with Q the quadratic form of the A_2 root lattice
- Two terms in denominator
- Function cannot be written as product of two Appell functions

$$H_{r,0}(z; J_{1,0}) = \sum_{r_1+\cdots+r_\ell,\, r_i\in\mathbb{N}^*} \Psi_{(r_1,\ldots,r_\ell),0}(z) \prod_{j=1}^{\ell} H_{r_j}(z)$$

with

$$\Psi_{(r_1,\ldots,r_\ell),0}(z) \quad = \quad \sum_{\substack{r_1 b_1+\cdots+r_\ell b_\ell=0 \\ b_i\in\mathbb{Z}}} \frac{w^{\sum_{j<i} r_i r_j (b_i-b_j)}}{\prod_{i=2}^{\ell}(1 - w^{2(r_i+r_{i-1})} q^{b_{i-1}-b_i})}$$
$$\times q^{\sum_{i=1}^{\ell} \frac{r_i(r-r_i)}{2r} b_i^2 - \frac{1}{r}\sum_{i<j} r_i r_j b_i b_j}$$

The contribution of $r_1 = \cdots = r_\ell = 1$ gives the quadratic form of the A_{r-1} root lattice

S-duality

- Electric-magnetic duality:
  Montonen, Olive (1977)

$$S : \begin{cases} (F, *F) \to (*F, F) \\ \mathfrak{g} \to \mathfrak{g}^L \\ \tau \to -1/\tau \end{cases}$$

- Translations:

$$\frac{1}{8\pi^2} \int \mathrm{Tr}\, F \wedge F \in \mathbb{Z}$$

$\Rightarrow \quad T : \tau \to \tau + 1$ is a symmetry of the theory

- $S + T$ generate $SL_2(\mathbb{Z})$ S-duality group

$\Rightarrow Z_r(\tau; J)$ should transform as modular form
Vafa, Witten (1994)

Modularity is understood for $r = 2$ using properties of $A(u, v)$

- $$Z_r(\tau, \rho; J) = \sum_{c_1 \mod r} \overline{h_{r,c_1}(\tau; J)}\, \Theta_{r,c_1}(\tau, \rho)$$

  with $\rho \in \mathbb{C}^{b_2(S)}$

- $$\mathcal{D}\widehat{Z}_2(\tau, \rho) = \frac{\sqrt{K_S^2}}{16\pi i y^{\frac{3}{2}}} Z_1(\tau, \rho)^2$$

  with $\mathcal{D} = \partial_\tau + \frac{i}{8\pi}\partial^2_{\rho_+}$

  Vafa, Witten (1994); Minahan, Nemeschansky, Vafa, Warner (1996); JM (2011)

Modularity is also expected for $r > 2$

Blow-up formula

Let $\phi : \Sigma_1 \to \mathbb{P}^2$

A semi-stable vector bundle on Σ_1 is related to one on the projective plane $\mathbb{P}^2$ by an elementary transformation

$$\Rightarrow \; H_{r,\phi^* c_1 - kC}(z; C+f) = B_{r,k}(z)\, H_{r,c_1}(z; \mathbb{P}^2)$$

with

$$B_{r,k}(z) = \eta(\tau)^{-r} \sum_{\substack{\sum_{i=1}^{r} a_i = 0 \\ a_i \in \mathbb{Z} + \frac{k}{r}}} q^{-\sum_{i<j} a_i a_j} w^{\sum_{i<j} a_i - a_j}$$

Vafa, Witten (1994), Yoshioka (1994, 1996), Li, Qin (1999), Göttsche (1999)

Rank 3 for $\mathbb{P}^2$

$h_{3,H}(z,\tau;\mathbb{P}^2)$:

c_2	b_0	b_2	b_4	b_6	b_8	b_{10}	b_{12}	b_{14}	b_{16}	b_{18}	b_{20}	b_{22}	b_{24}	b_{26}	χ
2	1	1													3
3	1	2	5	8	10										42
4	1	2	6	12	24	38	54	59							333
5	1	2	6	13	28	52	94	149	217	273	298				1968
6	1	2	6	13	29	56	108	189	322	505	744	992	1200	1275	9609

$h_{3,0}(z,\tau;\mathbb{P}^2)$:

c_2	b_0	b_2	b_4	b_6	b_8	b_{10}	b_{12}	b_{14}	b_{16}	b_{18}	b_{20}	b_{22}	b_{24}	b_{26}	b_{28}	χ
3	1	1	2	2	2	2										18
4	1	2	5	9	15	19	22	23	24							216
5	1	2	6	12	25	43	70	98	125	142	154	156				1512
6	1	2	6	13	28	53	99	165	264	383	515	631	723	774	795	8109

Rank 4 for $\mathbb{P}^2$

$h_{4,2}(z,\tau;\mathbb{P}^2)$

c_2	b_0	b_2	b_4	b_6	b_8	b_{10}	b_{12}	b_{14}	b_{16}	b_{18}	b_{20}	χ
4	1	1	1									6
5	1	2	6	10	17	21	24					162
6	1	2	6	13	27	49	84	126	173	211	231	1846

In agreement with partial results in the literature:

- $r = 3,\ c_1 = H,\ 2 \leq c_2 \leq 4$

  Yoshioka (1996)

- χ for $r = 3,\ c_1 = H,\ c_2 \geq 2$

  Kool (2009), Weist (2009)

- b_2 for general (r, c_1, c_2)

  Le Potier (1981), Drezet (1988)

We arrive at a A_{r-1} generalization

$$\mu_{Q,\{\boldsymbol{m}_j\}}(\boldsymbol{u},\boldsymbol{v}) = \frac{A_{Q,\{\boldsymbol{m}_j\}}(\boldsymbol{u},\boldsymbol{v})}{\Theta_Q(\boldsymbol{v})}$$

of the classical Appell-Lerch sum

$$\mu(u,v) = \frac{A(u,v)}{\theta_1(v)}$$

Identities for generalized Appell functions

Blow-up formula $\Rightarrow$ identities among $H_{r,c_1}(z;J_{1,0})$

For example

$$\frac{H_{r,C+f}(z;J_{1,0})}{B_{r,0}(z)} = \frac{H_{r,f}(z;J_{1,0})}{B_{r,1}(z)}$$

- $r=2$: follow from known property of $A(u,v)$
- $r=3$: proven by Bringmann, JM, Rolen (2015) using only properties of the q-series. Implies identities among $A_{Q,\{\boldsymbol{m}_j\}}$ and gives independent proof of $H_3(z)$.

Conclusions

Partition functions of $\mathcal{N} = 4$, $U(r)$ Yang-Mills theory on a large class of 4-manifolds can be determined explicitly in terms of generalized Appell functions $A_{Q,\{\boldsymbol{m}_j\}}$

What are their modular properties?

Applications:

- Understanding of electric-magnetic duality
- Quantum black hole state counting/D3-instantons
  $\Rightarrow$ Sergei Alexandrov's talk at 4pm
- ...