BPS-states and automorphic representations of exceptional groups

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"*Number theory and physics*"

IHP, Paris

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Talk based on our recent papers:

[1511.04265] w/ Fleig, Gustafsson, Kleinschmidt

[1412.5625] w/ Gustafsson, Kleinschmidt

[1312.3643] w/ Fleig, Kleinschmidt

and work in progress with Gourevitch and Sahi

Outline

1. Motivation from string theory

2. Eisenstein series and automorphic representations

3. Minimal representations of exceptional groups

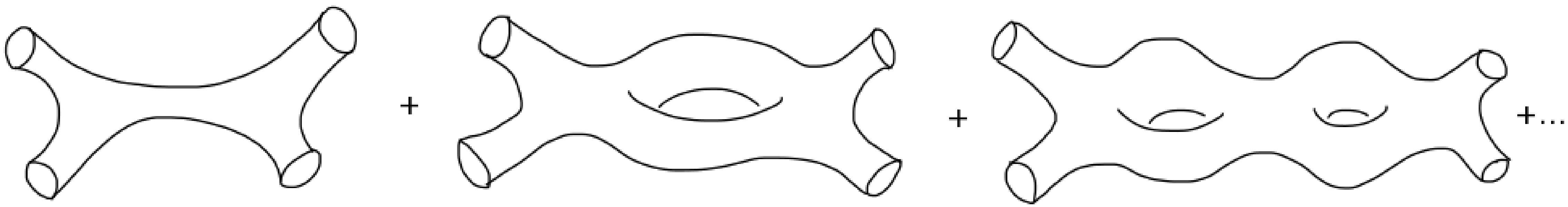
4. Next-to-minimal representations

5. Open problems and conjectures

1. Motivation from string theory

String amplitudes

Understand the structure of **string interactions**



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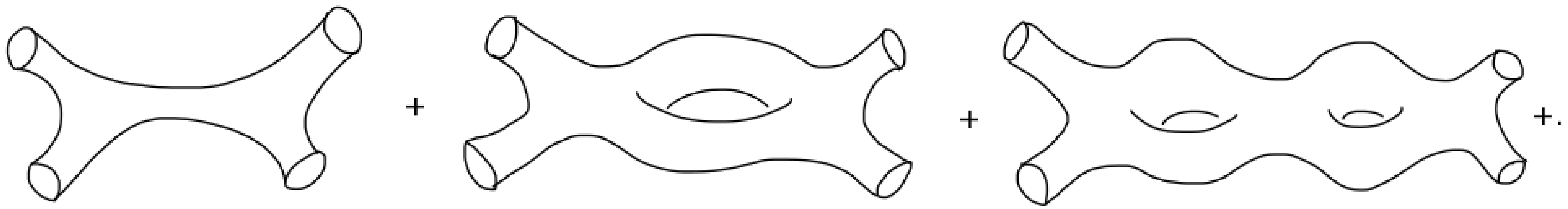

Strongly constrained by **symmetries**!

- supersymmetry
- U-duality

→ amplitudes have intricate **arithmetic structure** $G(\mathbb{Z})$

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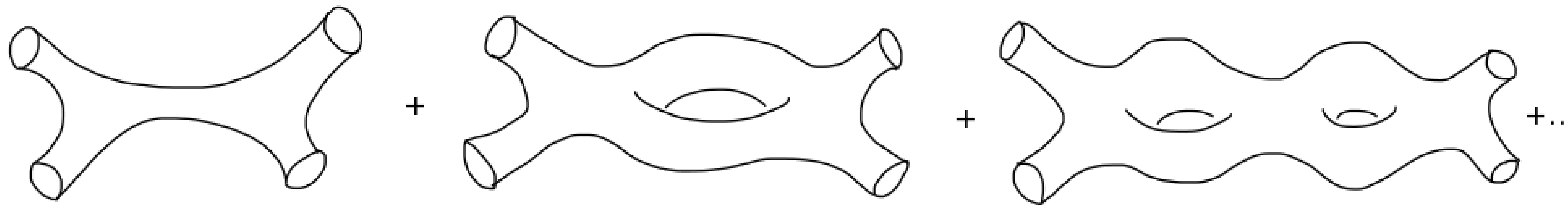

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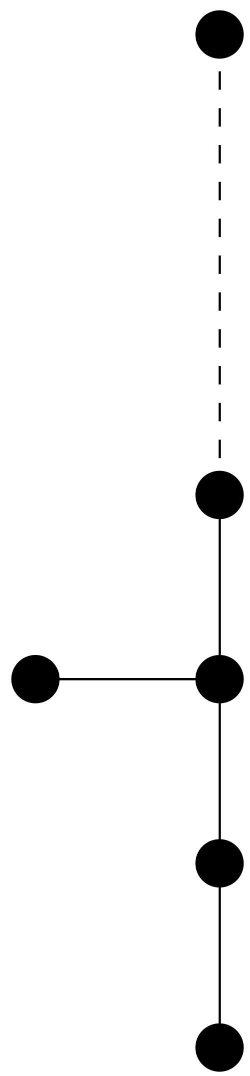
Symmetry constrains interactions, leads to insights about:

- ultraviolet properties of gravity
- non-perturbative effects (black holes, instantons)
- novel mathematical predictions from physics

Toroidal compactifications yield the famous chain of **U-duality groups**

[Cremmer, Julia][Hull, Townsend]



D	G	K	$G(\mathbb{Z})$
10	$\mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z})$
9	$\mathrm{SL}(2,\mathbb{R})\times\mathbb{R}^+$	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z})$
8	$\mathrm{SL}(3,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(3)\times\mathrm{SO}(2)$	$\mathrm{SL}(3,\mathbb{Z})\times\mathrm{SL}(2,\mathbb{Z})$
7	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{SO}(5)$	$\mathrm{SL}(5,\mathbb{Z})$
6	$\mathrm{Spin}(5,5,\mathbb{R})$	$\big(\mathrm{Spin}(5)\times\mathrm{Spin}(5)\big)/\mathbb{Z}_2$	$\mathrm{Spin}(5,5,\mathbb{Z})$
5	$\mathrm{E}_6(\mathbb{R})$	$\mathrm{USp}(8)/\mathbb{Z}_2$	$\mathrm{E}_6(\mathbb{Z})$
4	$\mathrm{E}_7(\mathbb{R})$	$\mathrm{SU}(8)/\mathbb{Z}_2$	$\mathrm{E}_7(\mathbb{Z})$
3	$\mathrm{E}_8(\mathbb{R})$	$\mathrm{Spin}(16)/\mathbb{Z}_2$	$\mathrm{E}_8(\mathbb{Z})$

Physical couplings are given by **automorphic forms** on

$$G(\mathbb{Z})\backslash G(\mathbb{R})/K$$

Green, Gutperle, Sethi, Vanhove, Kiritsis, Pioline, Obers, Kazhdan, Waldron, Basu, Russo, Cederwall, Bao, Nilsson, D.P., Lambert, West, Gubay, Miller, Fleig, Kleinschmidt,…

Higher-derivative action in type II string theory on tori

$$\int d^{10-n}x\sqrt{G}\left[(\alpha')^3 f_0(g)\mathcal{R}^4 + (\alpha')^5 f_4(g)\partial^4\mathcal{R}^4 + \cdots\right]$$

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contraction of four Riemann tensors

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- ⟶ $f_0(g), f_4(g)$ are functions of $g \in E_{n+1}(\mathbb{R})/K$
- ⟶ must be **invariant** under U-duality $E_{n+1}(\mathbb{Z})$
- ⟶ supersymmetry requires that they are **Laplacian eigenfunctions**
- ⟶ well-defined **weak-coupling expansions** as $g_s \to 0$

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defining properties of an **automorphic form!**

What is known?

The coefficient functions are Eisenstein series attached to certain **small automorphic representations** of G.

[Green, Miller, Vanhove][Pioline]

$\mathcal{R}^4$ 1/2-BPS	⟷	**minimal automorphic representation**
$\partial^4\mathcal{R}^4$ 1/4-BPS	⟷	**next-to-minimal automorphic representation**

Fourier coefficients of these functions reveal perturbative and non-perturbative quantum effects. **Very hard to compute!**

2. Eisenstein series and automorphic representations

Eisenstein series on semi-simple Lie groups

$G(\mathbb{R}) = B(\mathbb{R})K(\mathbb{R})$ semi-simple Lie group in its split real form

Borel subgroup:

$$B = \left\{ \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix} \right\} = AN$$

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defined by

$$\chi(b) = \chi(na) = \chi(a) = e^{\langle \lambda+\rho | H(a) \rangle}$$

$$H\ :\ A(\mathbb{R}) \to \mathfrak{h} = \mathrm{Lie}\, A(\mathbb{R})$$

$$H(a) = H\left(e^{\sum_{\alpha\in\Pi} y_\alpha H_\alpha}\right) = \sum_{\alpha\in\Pi} y_\alpha H_\alpha$$

$$\lambda \in \mathfrak{h}^\star \otimes \mathbb{C} \qquad\qquad \rho = \frac{1}{2}\sum_{\alpha>0}\alpha$$

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Extend to the whole group by: $\chi(g) = \chi(nak) = \chi(na)$

Eisenstein series on semi-simple Lie groups

Given this data the **Langlands Eisenstein series** is defined by:

$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} e^{\langle \lambda + \rho | H(\gamma g) \rangle}$$

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→ Converges absolutely on a subspace of $\mathfrak{h}^\star \otimes \mathbb{C}$

Godement's domain

$\{\lambda | \langle \lambda, \alpha \rangle > 1, \forall \alpha \in \Pi\}$

→ Can be continued to a meromorphic function on all of $\mathfrak{h}^\star \otimes \mathbb{C}$

→ Automorphic form: $E(\lambda, \gamma g k) = E(\lambda, g)$

$\gamma \in G(\mathbb{Z})$ $\qquad$ $k \in K(\mathbb{R})$

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- Satisfies a functional equation in λ

- Eigenfunction of the Laplacian: $\Delta_{G/K} E(\lambda, g) = \frac{1}{2}(\langle \lambda | \lambda \rangle - \langle \rho | \rho \rangle) E(\lambda, g)$

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Example: $G = SL(2,\mathbb{R})$ $\qquad s \in \mathbb{C}$

$$E(s,g) = \sum_{(c,d)=1} \frac{y^s}{|c\tau + d|^{2s}} = \sum_{\gamma \in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} e^{\langle \lambda+\rho | H(\gamma g)\rangle}$$

$\lambda + \rho = 2s\Lambda$ $\qquad$ (fundamental weight: $\Lambda = \alpha/2$)

$H(a) = H(e^{yH_\alpha}) = yH_\alpha$ $\qquad$ $\langle \Lambda | H_\alpha \rangle = 1$

Fourier coefficients

The periodicity $f(\tau+1)=f(\tau)$ generalises to

$$E(\lambda, ng) = E(\lambda, g) \qquad n \in N(\mathbb{Z})$$

Much more complicated since $N(\mathbb{Z})$ is **non-abelian**.

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General structure:

$$E(\lambda, g) = E^{\text{const}}(\lambda, g) + \sum_{\psi} W_{\psi}(\lambda, g) + \cdots$$

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constant term
(zero-mode)
perturbative effects

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(non zero-modes)
non-perturbative effects

Perturbative terms - Langlands constant term formula

$$E^{\mathrm{const}}(\lambda, g) = \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} E(\lambda, ng)dn$$

"integrating out the axions"

Perturbative terms - Langlands constant term formula

$$E^{\text{const}}(\lambda, g) = \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} E(\lambda, ng)dn \quad = \sum_{w \in W(\mathfrak{g})} M(w,\lambda) e^{\langle w\lambda + \rho | g \rangle}$$

$\uparrow$

$$M(w,\lambda) = \prod_{\substack{\alpha>0 \\ w(\alpha)<0}} \frac{\xi(\langle \lambda | \alpha \rangle)}{\xi(1 + \langle \lambda | \alpha \rangle)}$$

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This can be generalised to **smaller unipotent subgroups**: $U \subset N$

$$\int_{U(\mathbb{Z})\backslash U(\mathbb{R})} E(\lambda, ug)du = \sum_{w \in W_U \backslash W(\mathfrak{g})} e^{\langle w\lambda + \rho | H(g) \rangle} M(w,\lambda) E^{G'}(w\lambda, g)$$

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The case of a **minimal unipotent** is relevant for string theory as it only leaves **one perturbative parameter,** e.g. the string coupling!

Eisenstein series for a Levi subgroup: $G' \subset G$

Perturbative limit - choices of unipotent subgroups

→ Decompactification limit

- perturbative parameter: radius of decompactified circle
- non-perturbative effects: KK-instantons, BPS-instantons

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→ M-theory limit

- perturbative parameter: volume of M-theory torus
- non-perturbative effects: M2- & M5-instantons

Example: $G = SO(5,5)$ type II string theory on T^4 [Green, Russo, Vanhove]

Higher-derivative coupling: $\int d^4x \sqrt{G} f_0(g) \mathcal{R}^4$

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Choose minimal unipotent for **string theory limit**: U $\big(L = SO(4,4)\big)$

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Constant term:

$$\int_{U(\mathbb{Z})\backslash U(\mathbb{R})} E(3\Lambda_1 - \rho, ug)du = \frac{2\zeta(3)}{g_s^2} + E^{SO(4,4)}$$

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Much more is known!

$$\int d^{11-n}x\sqrt{G}f_0(g)\mathcal{R}^4 \qquad f_0(g) = E(2s\Lambda_1 - \rho, g) \qquad s = 3/2$$

$$\int d^{11-n}x\sqrt{G}f_4(g)\mathcal{R}^4 \qquad f_4(g) = E(2s\Lambda_1 - \rho, g) \qquad s = 5/2$$

Successfully checked against perturbative string calculations for all

$$G = E_n(\mathbb{R}) \qquad n \leq 11$$

[Green, Gutperle][Kiritsis, Pioline][Obers, Pioline][Green, Vanhove]
[Green, Russo, Vanhove][Green, Miller, Vanhove][Pioline]
[Fleig, Kleinschmidt][Fleig, Kleinschmidt, D.P.]…

$\partial^6\mathcal{R}^4$ also works to some extent but more complicated story…

Non-perturbative terms

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Non-zero mode

$$W_{\psi}(g) = \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} E(\lambda, ng)\overline{\psi(n)}dn$$

Whittaker vector

$$\psi : N(\mathbb{Z})\backslash N(\mathbb{R}) \to U(1)$$

unitary character on $N(\mathbb{R})$
trivial on $N(\mathbb{Z})$

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$$\psi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) = \psi(e^{xE_{\alpha}}) = e^{2\pi i m x}$$

$x \in \mathbb{R}$ $\quad m \in \mathbb{Z}$

ψ generic $\longleftrightarrow$ $m \neq 0$

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$$W_m(\tau) = \int_0^1 E(s, \tau + u)e^{-2\pi i m u}du$$

$x \in \mathbb{R}$ $\qquad$ $m \in \mathbb{Z}$

$$= \frac{2y^{1/2}}{\xi(2s)}|m|^{s-1/2}\mu_{1-2s}(m)K_{s-1/2}(2\pi|m|y)e^{2\pi i m x}$$

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In general, a function on the "abelianization" $[N, N]\backslash N \cong \prod_{\alpha\in\Pi} N_{\alpha}$

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$m_j \in \mathbb{Z}$ "instanton charges"

$x_j \in \mathbb{R}$ "axions"

if all $m_j \neq 0$ then ψ is **generic**

if some $m_j = 0$ then ψ is **degenerate**

Automorphic representations

Eisenstein series are attached to the (non-unitary) **principal series:**

$$I(\lambda) = \mathrm{Ind}_B^G \chi = \{ f : G \to \mathbb{C} \,|\, f(bg) = \chi(b) f(g), b \in B \}$$

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The theory of Eisenstein series then defines a map

$$E \,:\, I(\lambda) \longrightarrow \mathcal{A}(G(\mathbb{Z})\backslash G(\mathbb{R}))$$

from the principal series to **the space of automorphic forms on** $G(\mathbb{R})$

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G acts on $\mathcal{A}(G(\mathbb{Z})\backslash G(\mathbb{R}))$ by **right-translation:**

$$[\rho(h)f](g) = f(gh)$$

The irreducible constituents in the decomposition of $\mathcal{A}(G(\mathbb{Z})\backslash G(\mathbb{R}))$ under this action are called **automorphic representations**

[Gelfand, Graev, Piatetski-Shapiro][Langlands]...

Automorphic representations

There is an important notion of "size" of an automorphic representation, called the **Gelfand-Kirillov dimension**.

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$\mathrm{GKdim} =$ "smallest number of variables on which the functions depend"

For example, in the case of the infinite-dimensional Hilbert space of square-integrable functions in $\mathbb{R}^n$ we have

$$\mathrm{GKdim}(L^2(\mathbb{R}^n)) = n$$

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For the **principal series** we have:

$$\mathrm{GKdim}(I(\lambda)) = \dim_{\mathbb{R}} B\backslash G = \dim_{\mathbb{R}} N$$

$$= (\dim_{\mathbb{R}} G - \mathrm{rank}\, G)/2$$

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This is important for physics, since we have the rough correspondence:

number of independent **physical charges** (e.g. electric, magnetic) $\longleftrightarrow$ **Gelfand-Kirillilov dimension** of the associated automorphic representation

Adelic framework

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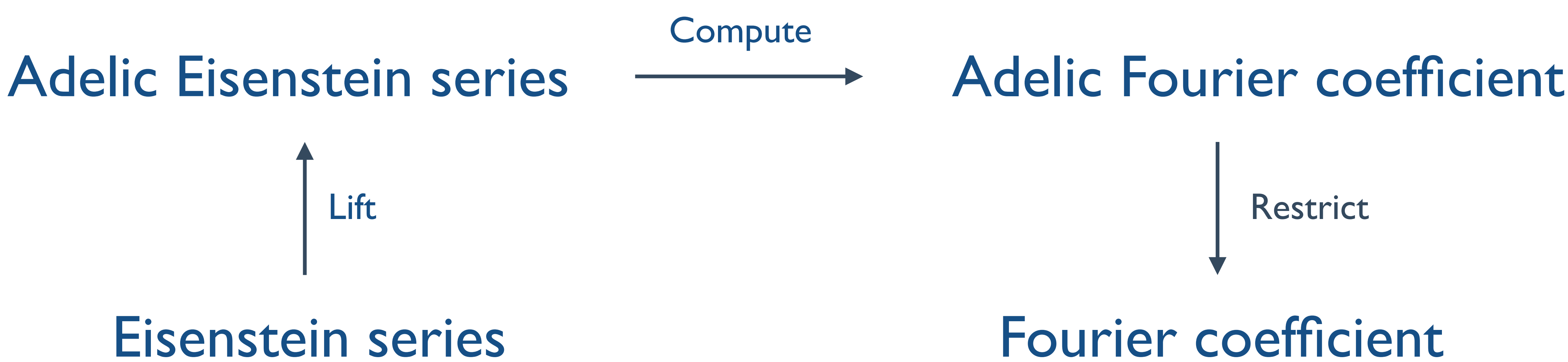

Adelic framework

For each **prime number** p

Euclidean norm

$\mathbb{Q} \to \mathbb{R}$ *real numbers*

$\mathbb{Q} \to \mathbb{Q}_p$ p-adic numbers

p-adic norm

$$\mathbb{Q}_\infty = \mathbb{R}$$

Adelic framework

For each **prime number** *p*

Euclidean norm

$\mathbb{Q} \to \mathbb{R}$ — *real numbers*

$\mathbb{Q} \to \mathbb{Q}_p$ — *p*-adic numbers

p-adic norm

$$\mathbb{Q}_\infty = \mathbb{R}$$

completions of $\mathbb{Q}$

Adelic framework

For each **prime number** *p*

Euclidean norm

$\mathbb{Q} \to \mathbb{R}$ *real numbers*

$\mathbb{Q} \to \mathbb{Q}_p$ *p*-adic numbers

p-adic norm

$$\mathbb{Q}_\infty = \mathbb{R}$$

The adeles are then defined as

$$\mathbb{A} = \mathbb{R} \times \prod_{p\,\mathrm{prime} < \infty}{}^{'} \mathbb{Q}_p \qquad x = (x_\infty; x_2, x_3, x_5, \ldots) \in \mathbb{A}$$

Adelic framework

For each **prime number** *p*

Euclidean norm

$\mathbb{Q} \to \mathbb{R}$ *real numbers*

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global   local

$$x = (x_\infty; x_2, x_3, x_5, \ldots) \in \mathbb{A}$$

$$\mathbb{Q} \hookrightarrow \mathbb{A}$$

$$q \mapsto (q; q, q, q, \ldots)$$

$\mathbb{Q} \subset \mathbb{A}$ **discrete embedding**

analogous to: $\mathbb{Z} \subset \mathbb{R}$

Adelic framework

For each **prime number** p

Euclidean norm

$\mathbb{Q} \to \mathbb{R}$ *real numbers*

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$$\mathbb{Q} \hookrightarrow \mathbb{A}$$

$$q \mapsto (q; q, q, q, \dots)$$

$$\mathbb{Q} \subset \mathbb{A}$$

much easier to work with since $\mathbb{Q}$ is a **field**.

analogous to: $\mathbb{Z} \subset \mathbb{R}$

Adelic framework

(completed) **Riemann zeta function**:

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

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$$= \int_{\mathbb{R}} e^{-\pi x^2} |x|^s dx \prod_{p\,\mathrm{prime}<\infty} \int_{\mathbb{Q}_p} \gamma_p(x)|x|_p^s dx$$

Adelic framework

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$$= \int_{\mathbb{R}} e^{-\pi x^2} |x|^s dx \prod_{p \text{ prime} < \infty} \int_{\mathbb{Q}_p} \gamma_p(x) |x|_p^s dx$$

$$= \int_{\mathbb{A}} \gamma_{\mathbb{A}}(x) |x|_{\mathbb{A}}^s dx$$

In his famous thesis, Tate gave elegant new proofs of the **functional equation and analytic continuation** of $\xi(s)$ using these techniques

Adelic framework

arithmetic groups	$G(\mathbb{Z}) \subset G(\mathbb{R})$	$\longrightarrow$	$G(\mathbb{Q}) \subset G(\mathbb{A})$
space of automorphic forms	$\mathcal{A}\big(G(\mathbb{Z})\backslash G(\mathbb{R})\big)$	$\longrightarrow$	$\mathcal{A}\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$
	$\bigcup$		$\bigcup$
Eisenstein series	$\sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} e^{\langle \lambda+\rho \vert H(\gamma g)\rangle}$	$\longrightarrow$	$\sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} e^{\langle \lambda+\rho \vert H(\gamma g)\rangle}$

Adelic Eisenstein series

- G split, simply-laced semisimple Lie group over $\mathbb{Q}$
- $B = AN$ **Borel subgroup**
- quasi-character: $\chi : B(\mathbb{A}) \to \mathbb{C}^\times$
- **induced representation:** $\mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}\chi = \prod_p \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi_p$
- $f_\chi \in \mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}\chi$ **unique spherical section:**

$$f_\chi(g) = f_\chi(nak) = \chi(na) = \chi(a)$$

$$f_\chi = \prod f_{\chi_p}$$

Adelic Eisenstein series

Associated to this data we construct the **Eisenstein series**

$$E(f_\chi, g) = \sum_{\gamma \in B(\mathbb{Q}) \backslash G(\mathbb{Q})} f_\chi(\gamma g) \qquad\qquad g \in G(\mathbb{A})$$

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$$E(f_\chi, g) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} f_\chi(\gamma g) \qquad g \in G(\mathbb{A})$$

It is also convenient to represent this in the following form:

$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} e^{\langle \lambda + \rho | H(\gamma g) \rangle} \qquad \lambda \in \mathfrak{h}^\star \otimes \mathbb{C}$$

It converges absolutely in the Godement range of λ.

Adelic Eisenstein series

Associated to this data we construct the **Eisenstein series**

$$E(f_\chi, g) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} f_\chi(\gamma g) \qquad\qquad g \in G(\mathbb{A})$$

For a **unitary character** $\psi : N(\mathbb{Q})\backslash N(\mathbb{A}) \to U(1)$

we have the **Whittaker-Fourier coefficient**

$$W_\psi(f_\chi, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(f_\chi, ng)\overline{\psi(n)}dn$$

It is a well-known that this is Eulerian: [Langlands]

$$W_\psi(f_\chi, g) = W_\infty(f_{\chi_\infty}, g_\infty) \times \prod_{p<\infty} W_p(f_{\chi_p}, g_p)$$

with $g_\infty \in G(\mathbb{R})$, $g_p \in G(\mathbb{Q}_p)$ and

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with $g_\infty \in G(\mathbb{R}),\ g_p \in G(\mathbb{Q}_p)$ and

$$W_\infty(f_{\chi_\infty}, g_\infty) = \int_{N(\mathbb{R})} f_{\chi_\infty}(ng_\infty)\overline{\psi_\infty(n)}dn$$

$$W_p(f_{\chi_p}, g_p) = \int_{N(\mathbb{Q}_p)} f_{\chi_p}(ng_p)\overline{\psi_p(n)}dn$$

can be computed using the CS-formula

More general Fourier coefficients

- $P = LU$ **standard parabolic** of G

Example:

$$P = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{pmatrix} \right\} = LU$$

More general Fourier coefficients

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More general Fourier coefficients

- $P = LU$ **standard parabolic** of G
- **unitary character** $\psi_U : U(\mathbb{Q})\backslash U(\mathbb{A}) \to U(1)$
- We then have the U -**Fourier coefficient:**

$$F_{\psi_U}(f_\chi, g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(f_\chi, ug)\overline{\psi_U(u)}du$$

much less is known in general in this case…

$$F_{\psi_U}(f_\chi, g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(f_\chi, ug)\overline{\psi_U(u)}du$$

- These are not Eulerian in general, no CS-formula…
- It is sufficient to determine the coefficient for one representative in each Levi orbit of ψ_U
- Each Levi orbit is contained in some complex nilpotent G -orbit
- It is fruitful to restrict to small automorphic representations.

3. Minimal representations of exceptional groups

Minimal automorphic representations

Definition: *An automorphic representation*

$$\pi = \bigotimes_{p \leq \infty} \pi_p$$

is minimal if each factor π_p *has smallest non-trivial Gelfand-Kirillov dimension.*

[Joseph][Brylinski, Kostant][Ginzburg, Rallis, Soudry][Kazhdan, Savin]. . . .

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[Joseph][Brylinski, Kostant][Ginzburg, Rallis, Soudry][Kazhdan, Savin]....

Automorphic forms $\varphi \in \pi_{min}$ are characterised by having **very few non-vanishing Fourier coefficients**. [Ginzburg, Rallis, Soudry]

Maximal parabolic subgroups

Now consider the case when $P = LU$ is a **maximal parabolic**

This implies that U only contains **a single simple root** α

Now choose a representative in the Levi orbit which is only sensitive to this simple root:

$$\psi_U = \psi\big|_U = \psi_\alpha$$

This is non-trivial only on the simple root space N_α

Theorem [Miller-Sahi]: *Let* G *be a split group of type* E_6 *or* E_7 *Then any Fourier coefficient of* $\varphi \in \pi_{min}$ *of* G *is completely determined by the maximally degenerate Whittaker coefficient*

$$W_{\psi_\alpha}(\varphi, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi_\alpha(n)}dn$$

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$$W_{\psi_\alpha}(\varphi, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi_\alpha(n)}dn$$

Can one use this to calculate

$$F_{\psi_U}(\varphi, g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(\varphi, ug)\overline{\psi_U(u)}du$$

in terms of W_{ψ_α} **?**

Is there a relation between the **degenerate Whittaker coefficient**:

$$W_{\psi_\alpha}(\varphi, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi_\alpha(n)}dn$$

and the U **-coefficient**:

$$F_{\psi_U}(\varphi, g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(\varphi, ug)\overline{\psi_U(u)}du$$

?

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$$W_{\psi_\alpha}(\varphi, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi_\alpha(n)}dn$$

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$$F_{\psi_U}(\varphi, g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(\varphi, ug)\overline{\psi_U(u)}du \qquad ?$$

A priori they live on **different spaces!**

$$W_{\psi_\alpha}(nak) = \psi_\alpha(n)W_{\psi_\alpha}(a) \qquad F_{\psi_U}(ulk) = \psi_\alpha(u)F_{\psi_U}(l)$$

Is there a relation between the degenerate Whittaker coefficient:

$$W_{\psi_\alpha}(\varphi, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi_\alpha(n)}dn$$

and the U -coefficient:

$$F_{\psi_U}(\varphi, g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(\varphi, ug)\overline{\psi_U(u)}du \qquad ?$$

Conjecture [Gustafsson, Kleinschmidt, D.P.]:

For $\varphi \in \pi_{min}$ these two functions are equal.

Proof: *In progress with [Gourevitch, Gustafsson, Kleinschmidt, D.P., Sahi]*

Example: Let $G = SL(3, \mathbb{A})$ [Gustafsson, Kleinschmidt, D.P.]

$$\psi_\alpha(x) = \psi_\alpha(e^{2\pi i(uE_\alpha + vE_\beta)}) = e^{2\pi i n u}, \qquad n \in \mathbb{Q},\ u \in \mathbb{A}$$

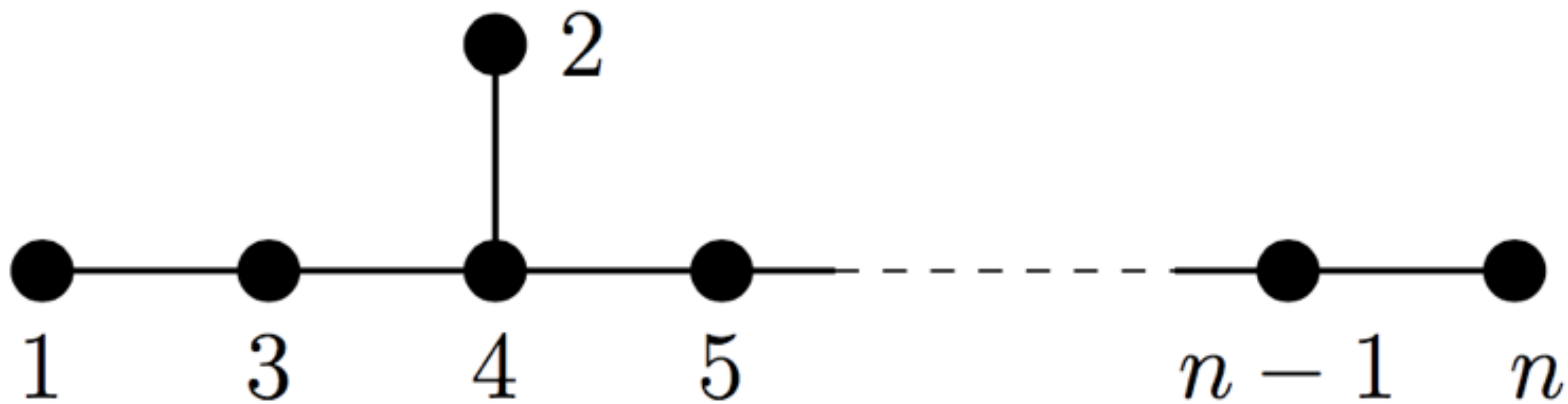
$$U = \left\{ \begin{pmatrix} 1 & u_1 & u_2 \\ & 1 & \\ & & 1 \end{pmatrix} : u_i \in \mathbb{A} \right\}$$

In this case we find the following equality

$$F_{\psi_{U_{m,n}}}(\varphi, g) = W_{\psi_n}\left(\varphi, \begin{pmatrix} -1 & 0 & \\ 0 & 0 & -1 \\ 0 & -1 & m/n \end{pmatrix} g\right)$$

so the functions are **equal up to a Levi translate** of the argument!

Exceptional groups



Functional dimension of minimal representations:

$$\mathrm{GKdim}\,\pi_{min} = \begin{cases} 11, & E_6 \\ 17, & E_7 \\ 29, & E_8 \end{cases}$$

Automorphic realization

Consider the Borel-Eisenstein series on $G(\mathbb{A})$

$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} e^{\langle \lambda + \rho | H(\gamma g) \rangle}$$

Now fix the weight to

$$\lambda = 2s\Lambda_1 - \rho$$

where Λ_1 is the fundamental weight associated to node 1.

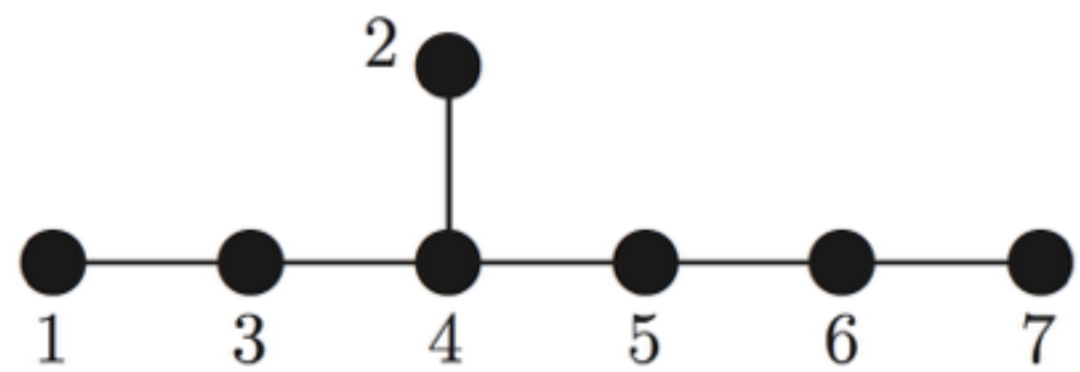
Theorem [Ginzburg,Rallis,Soudry][Green,Miller,Vanhove]

For $G = E_6, E_7, E_8$ *the Eisenstein series* $E(2s\Lambda - \rho, g)$ *evaluated at* $s = 3/2$ *is attached to the representation* π_{min} *with wavefront set* $WF(\pi_{min}) = \overline{\mathcal{O}_{min}}$.

This theorem yields an explicit automorphic realisation of the minimal representation.

Our aim is to use this to calculate Fourier coefficients associated with maximal parabolic subgroups.

Example: $G = E_7$



Consider the **3-grading** of the Lie algebra

$$\mathfrak{e}_7 \;=\; \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \;=\; \mathbf{27} \oplus (\mathfrak{e}_6 \oplus \mathbf{1}) \oplus \mathbf{27}$$

The space $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ is the Lie algebra of a maximal parabolic $P = LU$ with 27-dim unipotent U and Levi $L = E_6 \times GL(1)$

The degenerate Whittaker vector associated with α_1 is given by:

[Fleig, Kleinschmidt, D.P.]

$$W_{\psi_k}(3/2, a) = |k|^{3/2}\sigma_{-3}(k)K_{3/2}(2\pi|k|a)$$

where $a \in A \subset E_7$ and

$$\sigma_s(k) = \sum_{d|k} d^s$$

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$$\sigma_s(k) = \sum_{d|k} d^s$$

We now want to relate this to the U - Fourier coefficient

$$F_{\psi_U}(3/2, g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(3/2, ug)\overline{\psi_U(u)}du$$

This captures **instantons in the decompactification limit** of II/T^6 !

Claim: [Pioline][Gustafsson, Kleinschmidt, D.P.][Bossard, Verschinin]

$$F_{\psi_U}(3/2; h, r) = |k|^{3/2}\sigma_{-3}(k)K_{3/2}(2\pi r|k| \times ||h^{-1}\vec{x}||)$$

where $h \in E_6$, $r \in GL(1)$ and $\vec{x} \in \mathbb{Z}^{27}$

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where $h \in E_6$, $r \in GL(1)$ and $\vec{x} \in \mathbb{Z}^{27}$

Proof: ***To appear by** [Gustafsson, Gourevitch, Kleinschmidt, D.P., Sahi]*

This gives the **complete abelian Fourier expansion** of the minimal representation

Physically the vector $\vec{x}$ corresponds to the **instanton charges** of the 27 vector fields in D=5.

4. Next-to-minimal representations

Properties of π_{ntm}

No multiplicity one theorem known for π_{ntm} .

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No multiplicity one theorem known for π_{ntm} .

Theorem [Green, Miller, Vanhove]: *Let* $G = E_6, E_7, E_8$
The Eisenstein series

$$E(s,g) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} e^{\langle 2s\Lambda_1 | H(\gamma g)\rangle}$$

evaluated at $s = 5/2$ *is a spherical vector in* π_{ntm} .

Whittaker coefficients attached to π_{ntm}

Theorem [Fleig, Kleinschmidt, D.P.]:

The abelian term of the Fourier expansion of $E(5/2, g)$ *is given by*

$$\sum_{\substack{\psi:N(\mathbb{Q})\backslash N(\mathbb{A})\to U(1)\\ \psi\neq 1}} W_\psi(5/2, na) = \sum_{\alpha\in\Pi}\sum_{\psi_\alpha} c_\alpha(a) W_{\psi_\alpha}(5/2, na)$$

$$+ \sum_{\substack{\alpha,\beta\in\Pi\\ [E_\alpha,E_\beta]=0}} \sum_{\psi_{\alpha,\beta}} c_{\alpha,\beta}(a) W_{\psi_{\alpha,\beta}}(5/2, na)$$

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$$+ \sum_{\substack{\alpha, \beta \in \Pi \\ [E_\alpha, E_\beta] = 0}} \sum_{\psi_{\alpha,\beta}} c_{\alpha,\beta}(a) W_{\psi_{\alpha,\beta}}(5/2, na)$$

Bala-Carter type $2A_1$ (product of two K-Bessel functions)

Conjecture [Gustafsson, Kleinschmidt, D.P.]:

Let G be a semisimple, simply laced Lie group.

Then all Fourier coefficients of $\varphi \in \pi_{ntm}$ are completely determined by degenerate Whittaker vectors of the form

$$W_{\psi_\alpha}(\varphi, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi_\alpha(n)}dn$$

$$W_{\psi_{\alpha,\beta}}(\varphi, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi_{\alpha,\beta}(n)}dn$$

where (α, β) are commuting simple roots.

Proof. *In progress with [Gustafsson, Gourevitch, Kleinschmidt, D.P., Sahi]*

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where (α, β) are commuting simple roots.

This will allow us to **extract instanton effects** from $\partial^4\mathcal{R}^4$ couplings!

5. Conjectures and open problems

Spherical vectors for Kac-Moody groups

Spherical vectors for Kac-Moody groups

Let $G = E_9, E_{10}, E_{11}$, The Eisenstein series $E(3/2, g)$ is conjecturally a spherical vector in π_{min} and has partial Fourier expansion [Fleig, Kleinschmidt, D.P.]

$$E(3/2, g) = E_0 + \sum_{\alpha \in \Pi} \sum_{\psi_\alpha} c_\alpha(a) W_{\psi_\alpha}(3/2, na) + \text{“non-ab”}$$

where $W_{\psi_\alpha}(3/2, na) = \prod_{p \leq \infty} W_p(3/2, na)\,.$

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where $W_{\psi_\alpha}(3/2,na) = \prod_{p\leq\infty} W_p(3/2,na)\,.$

Conjecture: *The minimal representation of* E_9, E_{10}, E_{11} *factorises:*

$$\pi_{min} = \otimes_p \pi_{min,p}$$

and $W_p(3/2,na)$ *is (the abelian limit of) a spherical vector in* $\pi_{min,p}$.

This generalises earlier results by Kazhdan, Savin, Polishchuk et. al.

Black hole counting in string theory

Recall: string theory on T^6 has black hole solutions with charges $\gamma \in \mathbb{Z}^{56}$. For 1/2 BPS-states only charges in a 28-dimensional subspace $\mathcal{C} \subset \mathbb{Z}^{56}$ are realised.

$$\Omega(\gamma) = \text{number of BPS-black holes with charge}\, \gamma$$

Constraint: $\Omega(\gamma) = 0$ if $\gamma \notin \mathcal{C}$

Symmetry: $\Omega(\gamma)$ must be $E_7(\mathbb{Z})$-invariant

A **generating function** of these states takes the form

$$Z(l,u) = \sum_{\gamma=(x_1,\ldots,x_{56})\in\mathbb{Z}^{56}} \Omega(\gamma) c_\gamma(l) e^{2\pi i(x_1 u_1 \cdots x_{56} u_{56})}$$

where $l \in E_7(\mathbb{R})$ and $(u_1, \ldots, u_{56}) \in \mathbb{R}^{56}$ "chemical potentials"

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where $l \in E_7(\mathbb{R})$ and $(u_1,\dots,u_{56}) \in \mathbb{R}^{56}$ "chemical potentials"

This is precisely the structure of the abelian Fourier coefficients of an automorphic form φ on E_8
with respect to the **Heisenberg unipotent radical** $Q \subset E_8$

$$\sum_{\psi:Q(\mathbb{Q})\backslash Q(\mathbb{A})\to U(1)} F_{\psi_Q}(\varphi,l)\psi_Q(u)$$

If we take $\varphi \in \pi_{min}$ so $\mathrm{GKdim}(\pi_{min}) = 29$ then

$$F_{\psi_Q}(\varphi, g) = \int_{Q(\mathbb{Q})\backslash Q(\mathbb{A})} \varphi(ug)\overline{\psi_Q(u)}du = \prod_{p\leq\infty} F_{\psi,p}(\varphi, g)$$

vanishes unless ψ_Q lies in a 28-dimensional subspace of $\mathfrak{g}_1(\mathbb{Q})$.

[Kazhdan, Polishchuk]

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vanishes unless ψ_Q lies in a 28-dimensional subspace of $\mathfrak{g}_1(\mathbb{Q})$.

[Kazhdan, Polishchuk]

Conjecture:

The 1/2 BPS-states are counted by the p-adic spherical vectors in the minimal representation of E_8 *:*

$$\Omega(\gamma) = \prod_{p<\infty} F_{\psi_Q,p}(\pi_{min}, 1)$$

[Pioline][Gunaydin, Neitzke, Pioline, Waldron][Fleig, Gustafsson, Kleinschmidt, D.P.]

String theory on Calabi-Yau 3-folds

In general, very little is known about the duality group in this case.

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However, consider the case of X a **rigid** CY3-fold. $(h_{2,1}(X) = 0)$

Intermediate Jacobian of X is an **elliptic curve**:

$$H^3(X, \mathbb{R})/H^3(X, \mathbb{Z}) = \mathbb{C}/\mathcal{O}_d$$

ring of integers: $\mathcal{O}_d \subset \mathbb{Q}(\sqrt{-d})$ ($d > 0$ and square-free)

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Intermediate Jacobian of X is an **elliptic curve**:

$$H^3(X, \mathbb{R})/H^3(X, \mathbb{Z}) = \mathbb{C}/\mathcal{O}_d$$

ring of integers: $\mathcal{O}_d \subset \mathbb{Q}(\sqrt{-d})$ ($d > 0$ and square-free)

Conjecture: [Bao, Kleinschmidt, Nilsson, D.P., Pioline]

String theory on X is invariant under the **Picard modular group**

$$PU(2,1;\mathcal{O}_d) := U(2,1) \cap PGL(3,\mathcal{O}_d)$$

Theorem: [Bao, Kleinschmidt, Nilsson, D.P., Pioline]

The Borel Eisenstein series

$$E(\chi_s, P, g) = \sum_{\gamma \in P(\mathcal{O}_d)\backslash PU(2,1;\mathcal{O}_d)} \chi_s(\gamma g)$$

has Fourier coefficients

$$F_{\psi_U}(s, g) = \int_{U(\mathcal{O}_d)\backslash U} E(\chi_s, P, ug)\overline{\psi_U(u)}du$$

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$$= F_{\psi_U,\infty}(s,g) \times \prod_{p<\infty} F_{\psi_U,p}(s,1)$$

where

$$\prod_{p<\infty} F_{\psi_U,p}(s,1) = \sum_{\substack{\omega \in \mathcal{O}_d \\ \gamma/\omega \in \mathcal{O}_d^\star}} \left|\frac{\gamma}{\omega}\right|^{2s-2} \sum_{\substack{z \in \mathcal{O}_d \\ \gamma/(z\omega) \in \mathcal{O}_d^\star}} |z|^{4-4s}$$

Conjecture: [Bao, Kleinschmidt, Nilsson, D.P., Pioline]

The counting of BPS-black holes in string theory on X *with charges* $\gamma \in H_3(X, \mathbb{Z})$ *is given by the Fourier coefficient*

$$\Omega(\gamma) = \sum_{\substack{\omega \in \mathcal{O}_d \\ \gamma/\omega \in \mathcal{O}_d^\star}} \left|\frac{\gamma}{\omega}\right|^{2s-2} \sum_{\substack{z \in \mathcal{O}_d \\ \gamma/(z\omega) \in \mathcal{O}_d^\star}} |z|^{4-4s}$$

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The function $\Omega(\gamma)$ *counts the number of special Lagrangian submanifolds of* X *in the homology class* $[\gamma] \in H_3(X, \mathbb{Z})$.

Quaternionic discrete series

For string theory on Calabi-Yau 3-folds with $h_{1,1}(X) = 1$ we expect that the duality group is the **exceptional Chevalley group** $G_2(\mathbb{Z})$.

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$$SL(2, \mathbb{Z}) \subset G_2(\mathbb{Z})$$

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This is precisely the constraint satisfied by **Fourier coefficients of automorphic forms** attached to the **quaternionic discrete series** of $G_2(\mathbb{R})$. [Wallach][Gan, Gross, Savin]

Quaternionic discrete series

The **quaternionic discrete series** can be realised as [Gross, Wallach]

$$\pi_k = H^1(\mathcal{Z}, \mathcal{O}(-k)) \qquad\qquad k \geq 2$$

where $\mathcal{Z}$ is the twistor space:

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Open problem: Can one construct explicit **automorphic forms attached** to π_k in terms of **holomorphic functions** on $\mathcal{Z}$?

Final question: [Moore]

Is there a natural role for automorphic L-functions in BPS-state counting problems?