Number Theory and Physics

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Elliptic multiple zeta values

and superstring one-loop amplitudes

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based on arXiv:1412.5535 & 1507.02254 with J. Brödel, C. Mafra, N. Matthes

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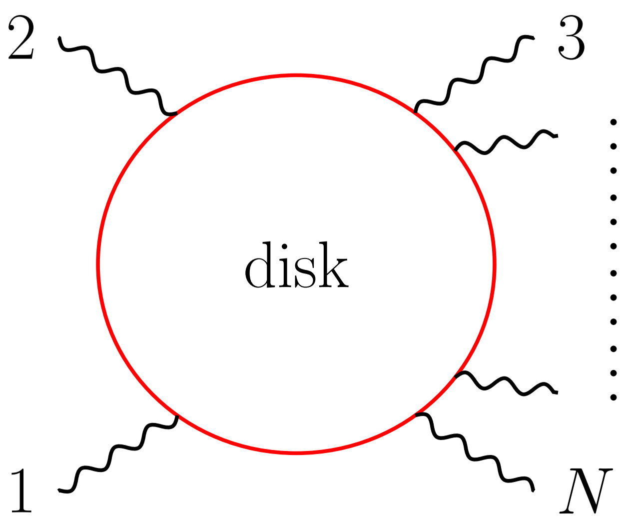
Motivation: Open-string scattering amplitudes @ tree level and one loop

$\Longrightarrow$ iterated integrals over the worldsheet boundaries

genus zero

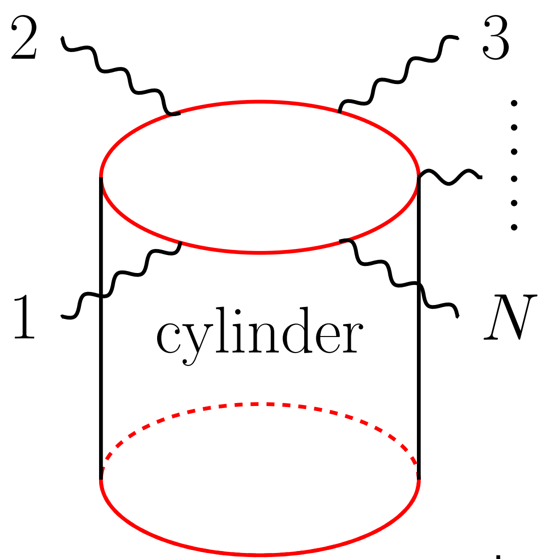


peel off YM trees $\downarrow$ expand in α'

$$\int\limits_{0\leq z_i\leq z_{i+1}\leq 1} \prod \frac{\mathrm{d}z_j}{z_j - n_j} = \text{MZVs}$$

[earlier work]

genus one



half a torus of modular parameter τ

$$\frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$$

$\downarrow$

$$\int\limits_{0\leq z_i\leq z_{i+1}\leq 1} \prod f^{(n_j)}(z_j)\mathrm{d}z_j = \text{eMZVs}$$

[this talk]

Outline

I. Definition of eMZVs and $f^{(n)}$

[Brown, Levin, Enriquez, ...]

II. Superstring one-loop amplitudes

[Brödel, Mafra, Matthes, OS 1412.5535]

III. eMZV relations & indecomposables

[Brödel, Matthes, OS 1507.02254]

I. 1 Iterated integrals: genus-zero versus elliptic

Recall that multiple polylogarithms **[Goncharov, see Pierre Vanhove's talk]**

$$G(a_1, a_2, \ldots, a_r; z) \equiv \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_r; t) , \qquad G(; z) \equiv 1$$

specialize to multiple zeta values (MZVs) upon $z \to 1$ and $a_j \to \{0, 1\}$:

$$\zeta(a_1, a_2, \ldots, a_r) \equiv (-1)^{\sum_{j=1}^r a_j} \int\limits_{0 \leq z_i \leq z_{i+1} \leq 1} \frac{\mathrm{d}z_1}{z_1 - a_1} \frac{\mathrm{d}z_2}{z_2 - a_2} \cdots \frac{\mathrm{d}z_r}{z_r - a_r}$$

Both are said to have weight $r \ \equiv$ number of integrations.

Can recursively bypass clashes $a_j = z$ via partial fraction:

$$G(a, 0, z; z) = G(0, 0, a; z) - G(0, a, a; z) - \zeta_2 G(a; z)$$

resting on: $$\frac{1}{(z-a)(z-b)} = \frac{1}{a-b} \left(\frac{1}{z-a} - \frac{1}{z-b} \right)$$

I. 1 Iterated integrals: genus-zero versus elliptic

Elliptic iterated integrals with suitable $f^{(n)}$, $n \in \mathbb{N}$ **[Brown, Levin]**

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ a_1 & a_2 & \dots & a_r \end{smallmatrix};z\right) \equiv \int_0^z \mathrm{d}t \, f^{(n_1)}(t-a_1)\, \Gamma\left(\begin{smallmatrix} n_2 & \dots & n_r \\ a_2 & \dots & a_r \end{smallmatrix};t\right) \ , \qquad \Gamma(;z) \equiv 1$$

specialize to elliptic multiple zeta values (eMZVs) upon $z \rightarrow 1$ and $a_j \rightarrow 0$:

$$\omega(n_1,n_2,\ldots,n_r) \equiv \int_{0\leq z_i \leq z_{i+1} \leq 1} f^{(n_1)}(z_1)\,\mathrm{d}z_1 \, f^{(n_2)}(z_2)\,\mathrm{d}z_2 \ \cdots \ f^{(n_r)}(z_r)\,\mathrm{d}z_r \, .$$

Both are said to have length r and weight $\sum_{j=1}^r n_j$. **[Enriquez 1301.3042]**

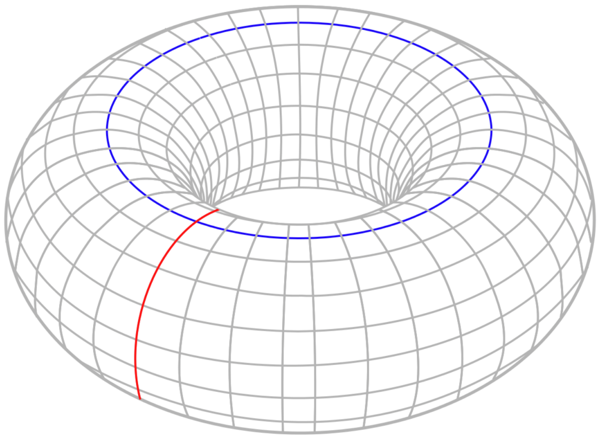
Can recursively bypass clashes $a_j = z$ via Fay relations ($C_{p,q} \in \mathbb{Z}$):

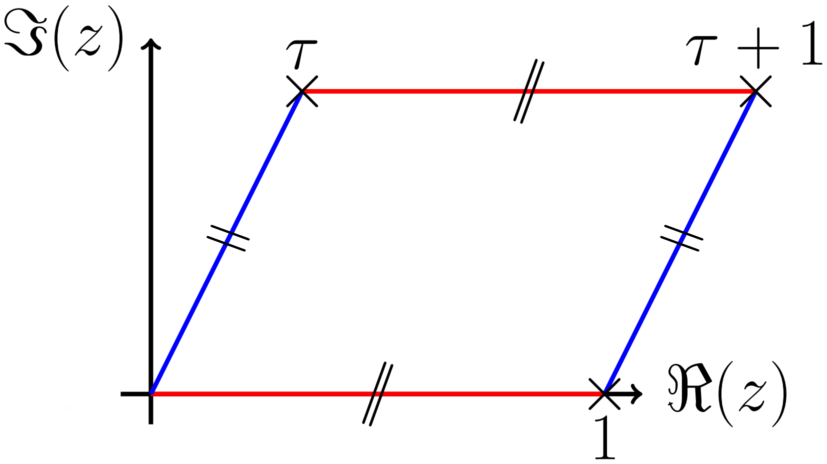
$$\Gamma\left(\begin{smallmatrix} 1 & 1 \\ z & 0 \end{smallmatrix};z\right) = 2\Gamma\left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix};z\right) + \Gamma\left(\begin{smallmatrix} 2 & 0 \\ 0 & 0 \end{smallmatrix};z\right) - 2\Gamma\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix};z\right) + \zeta_2$$

from $$f^{(m)}(z-a)f^{(n)}(z-b) = \sum_{p+q=m+n} C_{p,q} f^{(p)}(z-a) f^{(q)}(a-b) + \begin{matrix} (m,a) \\ \updownarrow \\ (n,b) \end{matrix}$$

I. 2 $f^{(n)}$ – doubly periodic extension of $\frac{\mathrm{d}z}{z}$ with Fay relation

Parametrization of elliptic curve $\equiv$ torus





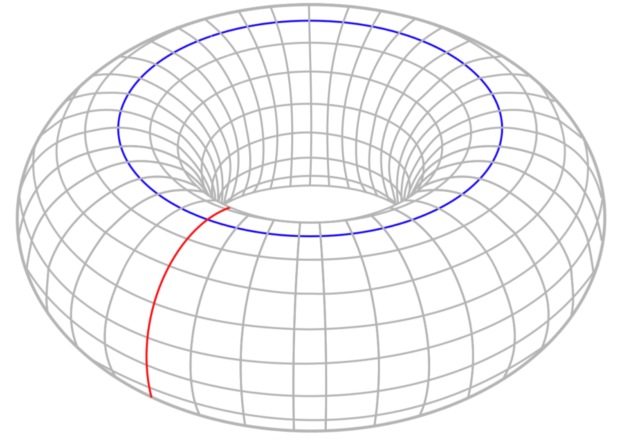
Jacobi θ-function takes role of the identity map on the torus

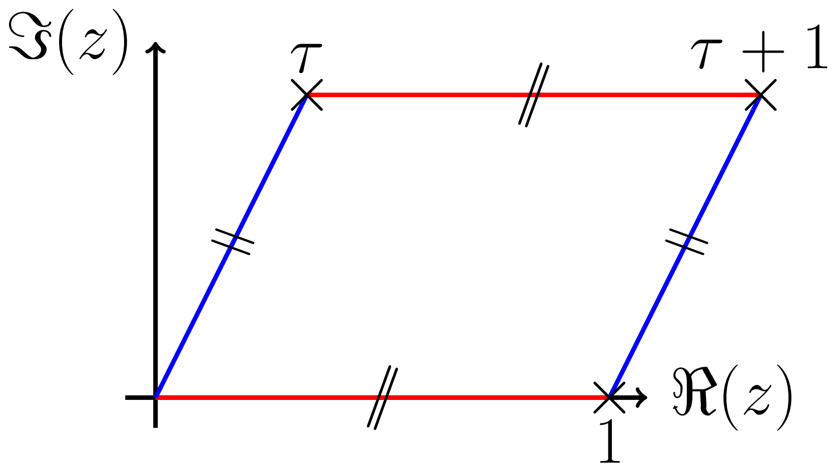
$$\theta(z,\tau) \equiv \sin(\pi z) \prod_{n=1}^{\infty} (1 - e^{2\pi i(n\tau + z)})(1 - e^{2\pi i(n\tau - z)})$$

$$\text{uplift} \quad \frac{1}{z} \;\rightarrow\; \partial_z \ln\theta(z,\tau) + \; 2\pi i \, \frac{\Im(z)}{\Im(\tau)} \;\equiv f^{(1)}(z,\tau)$$

I. 2 $f^{(n)}$ – doubly periodic extension of $\frac{\mathrm{d}z}{z}$ with Fay relation

Parametrization of elliptic curve $\equiv$ torus





Jacobi θ-function takes role of the identity map on the torus

$$\theta(z,\tau) \equiv \sin(\pi z) \prod_{n=1}^{\infty} (1 - e^{2\pi i (n\tau + z)})(1 - e^{2\pi i (n\tau - z)})$$

$$\text{uplift } \frac{1}{z} \rightarrow \partial_z \ln\theta(z,\tau) + \underbrace{2\pi i \frac{\Im(z)}{\Im(\tau)}}_{\text{sacrifice holomorphicity for } f^{(1)}(z,\tau) = f^{(1)}(z+\tau,\tau)} \equiv \overbrace{f^{(1)}(z,\tau) = f^{(1)}(z+1,\tau)}^{\text{poles like both } \frac{1}{z} \text{ and } \frac{1}{1-z}}$$

I. 2 $f^{(n)}$ – doubly periodic extension of $\frac{\mathrm{d}z}{z}$ with Fay relation

To go beyond

$$\text{uplift} \ \ \frac{1}{z} \ \rightarrow \ f^{(1)}(z,\tau) \equiv \partial_z \ln \theta(z,\tau) + 2\pi i \, \frac{\Im(z)}{\Im(\tau)} \ ,$$

check partial fraction $\Rightarrow$ new function $f^{(2)}$ without pole

$$\begin{aligned} \frac{1}{(z-a)(z-b)} + \text{cyc}(z,a,b) \ &= \ 0 \\ &\updownarrow \\ f^{(1)}(z-a)f^{(1)}(z-b) + \text{cyc}(z,a,b) \ &= \ f^{(2)}(z-a) + \text{cyc}(z,a,b) \end{aligned}$$

- similar identity for $f^{(1)}(z-a)f^{(2)}(z-b)$ yields new function $f^{(3)}$, etc.
- formally adjoin $f^{(0)} \equiv 1$
- drop the second argument τ of θ and $f^{(n)}$ here and henceforth

I. 2 $f^{(n)}$ – doubly periodic extension of $\frac{\mathrm{d}z}{z}$ with Fay relation

Generating fct.: non-holomorphic version of Kronecker-Eisenstein series

$$\Omega(z,\alpha) \equiv \exp\left(2\pi i \alpha \frac{\Im(z)}{\Im(\tau)}\right) \frac{\theta'(0)\theta(z+\alpha)}{\theta(z)\theta(\alpha)} = \sum_{n=0}^{\infty} \alpha^{n-1} f^{(n)}(z)$$

[Kronecker, Brown, Levin]

- doubly-periodic $\Omega(z,\alpha) = \Omega(z+\tau,\alpha) = \Omega(z+1,\alpha)$ thanks to $\exp(\ldots)$
- e.g. $f^{(2)}(z) \equiv \frac{1}{2}\left\{\left(\partial \ln\theta(z) + 2\pi i \frac{\Im(z)}{\Im(\tau)}\right)^2 + \partial^2 \ln\theta(z) - \frac{\theta'''(0)}{3\theta'(0)}\right\}$
- $\mathrm{Res}_{z=0}\Omega(z,\alpha) = 1 \ \Rightarrow$ no other poles than $f^{(1)}(z) \sim \frac{1}{z}$
- partial fraction generalizes to Fay relation:

$$\Omega(z_1,\alpha_1)\Omega(z_2,\alpha_2) = \Omega(z_1,\alpha_1+\alpha_2)\Omega(z_2-z_1,\alpha_2) + (1 \leftrightarrow 2)$$

II. One-loop superstring amplitude

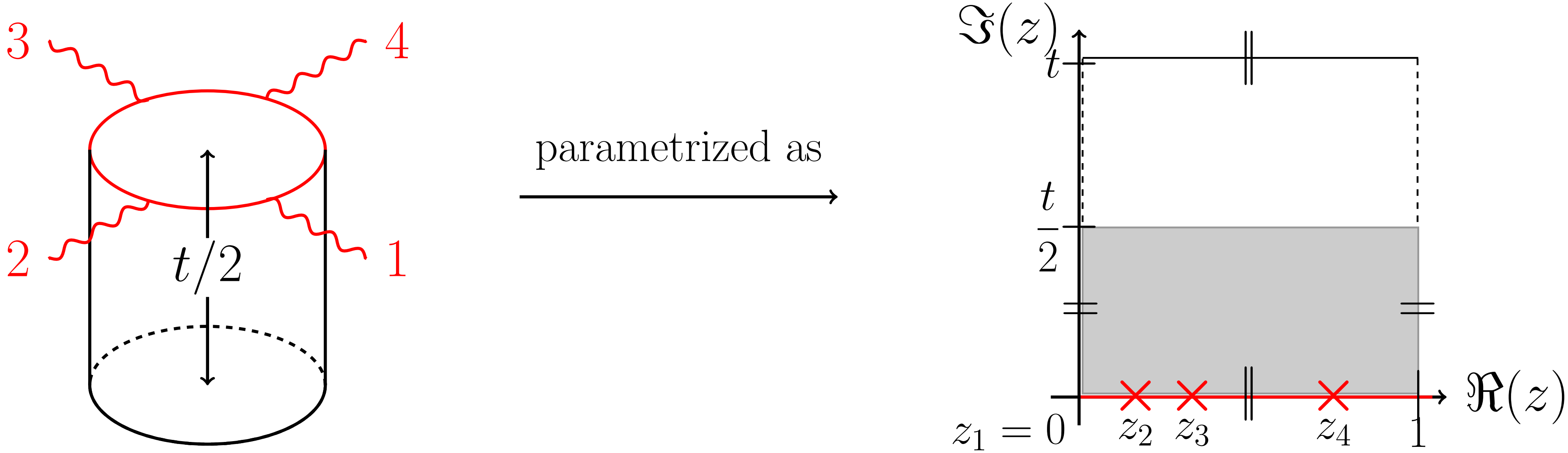
Focus on planar cylinder diagram at four-points:

$$A^{\text{1-loop}}_{\text{string}}(1,2,3,4) = s_{12}s_{23}A^{\text{tree}}_{\text{YM}}(1,2,3,4)\int_0^\infty \mathrm{d}t\; I_{1234}(s_{ij},\tau=it)$$

$$I_{1234}(s_{ij},\tau) = \int_0^1 \mathrm{d}z_4 \int_0^{z_4}\mathrm{d}z_3\int_0^{z_3}\mathrm{d}z_2\;\exp\Big(\sum_{i<j}^4 s_{ij}P(z_i-z_j,\tau)\Big)\Big|_{z_1=0}$$

[Brink, Green, Schwarz 1982]

with dim'less $s_{ij}\equiv\alpha'(k_i+k_j)^2$ and worldsheet propagator $\partial P = f^{(1)}$.



Analytic α'-dependence from expanding the exponentials

$$I_{1234}(s_{ij},\tau) = \int_0^1 \mathrm{d}z_4 \int_0^{z_4} \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2 \prod_{i<j}^{4} \sum_{n_{ij}=0}^{\infty} \frac{\big[s_{ij} P(z_i - z_j,\tau)\big]^{n_{ij}}}{n_{ij}!}\Big|_{z_1=0}$$

and integrating $P(z_i - z_j) = \int_{z_j}^{z_i} \mathrm{d}x \; f^{(1)}(x - z_j)$ order by order in α'.

Each monomial in s_{ij} is accompanied by eMZVs, e.g.

$$s_{ij}^0 \;\leftrightarrow\; \int_0^1 \mathrm{d}z_4 \; f^{(0)}(z_4) \int_0^{z_4} \mathrm{d}z_3 \; f^{(0)}(z_3) \int_0^{z_3} \mathrm{d}z_2 \; f^{(0)}(z_2) = \omega(0,0,0)$$

after formally inserting $f^{(0)} = 1$ as well as

$$\left.\begin{array}{c} s_{12} \\ \\ s_{13} \end{array}\right\} \leftrightarrow \int_0^1 \mathrm{d}z_4 \; f^{(0)} \int_0^{z_4} \mathrm{d}z_3 \; f^{(0)} \int_0^{z_3} \mathrm{d}z_2 \; f^{(0)} \left\{\begin{array}{l} \int_0^{z_2} \mathrm{d}x \; f^{(1)}(x) \\ \\ \int_0^{z_3} \mathrm{d}x \; f^{(1)}(x) \end{array}\right.$$

$$\Longrightarrow \quad s_{12} \;\leftrightarrow\; \omega(1,0,0,0)\;, \qquad s_{13} \;\leftrightarrow\; \underbrace{\omega(1,0,0,0)}_{\text{from } 0 \leq x \leq z_2} + \underbrace{\omega(0,1,0,0)}_{\text{from } z_2 \leq x \leq z_3}$$

At higher order ...

$$s_{12}s_{23} \;\leftrightarrow\; \int_0^1 \mathrm{d}z_4 \int_0^{z_4} \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2 \left(\int_{z_3}^{z_2} \mathrm{d}x \; f^{(1)}(x - z_3) \right) \left(\int_0^{z_2} \mathrm{d}y \; f^{(1)}(y) \right)$$

$$= \; - \int_0^1 \mathrm{d}z_4 \int_0^{z_4} \mathrm{d}z_3 \; \Gamma\left(\begin{smallmatrix} 1 & 0 & 1 \\ z_3 & 0 & 0 \end{smallmatrix} ; z_3 \right)$$

... need Fay relations

$$\Gamma\left(\begin{smallmatrix} 1 & 0 & 1 \\ z_3 & 0 & 0 \end{smallmatrix} ; z_3 \right) = 2\Gamma\left(\begin{smallmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \end{smallmatrix} ; z_3 \right) + \Gamma\left(\begin{smallmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \end{smallmatrix} ; z_3 \right) - 2\Gamma\left(\begin{smallmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{smallmatrix} ; z_3 \right) + \zeta_2 \Gamma\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} ; z_3 \right) \; .$$

Above example then integrates to

$$s_{12}s_{23} \;\leftrightarrow\; -2\omega(2,0,0,0,0) - \omega(0,2,0,0,0) + 2\omega(1,1,0,0,0) - \zeta_2\omega(0,0,0)$$

$$= -\omega(1,0,0,0,1)$$

after using eMZV relations at length five.

After using momentum conservation for s_{ij}, first orders simplify to

$$I_{1234}(s_{ij}) \;=\; \omega(0,0,0) \;-\; 2\omega(0,1,0,0)\,(s_{12}+s_{23}) \;+\; 2\omega(0,1,1,0,0)\,\big(s_{12}^2+s_{23}^2\big)$$

$$-\;2\omega(0,1,0,1,0)\,s_{12}s_{23} \;+\; \beta_5\,(s_{12}^3+2s_{12}^2s_{23}+2s_{12}s_{23}^2+s_{23}^3) \;+\; \beta_{2,3}\,s_{12}s_{23}(s_{12}+s_{23}) \;+\; \ldots$$

with shorthands **[Brödel, Mafra, Matthes, OS 1412.5535]**

$$\begin{aligned}
\beta_5 \;&=\; \frac{4}{3}\Big[\omega(0,0,1,0,0,2)+\omega(0,1,1,0,1,0)-\omega(2,0,1,0,0,0)-\zeta_2\omega(0,1,0,0)\Big]\\
\beta_{2,3} \;&=\; \frac{1}{3}\omega(0,0,1,0,2,0)-\frac{3}{2}\omega(0,1,0,0,0,2)-\frac{1}{2}\omega(0,1,1,1,0,0)\\
&\quad -2\omega(2,0,1,0,0,0)-\frac{4}{3}\omega(0,0,1,0,0,2)-\frac{10}{3}\zeta_2\omega(0,1,0,0)\;.
\end{aligned}$$

Choice of indecomposable eMZVs requires guidance,

in particular at higher α'-order $\leftrightarrow$ weight $\leftrightarrow$ length ...

also: n-point amplitude naturally compatible with eMZV language !

III. 1 eMZV relations and indecomposables

The following MZVs are believed to be indecomposable over $\mathbb{Q}$:

weight w	0	1	2	3	4	5	6	7	8	9	10	11	12
indec. MZVs	1	$\emptyset$	ζ_2	ζ_3	$\emptyset$	ζ_5	$\emptyset$	ζ_7	$\zeta_{3,5}$	ζ_9	$\zeta_{3,7}$	ζ_{11}, $\zeta_{3,3,5}$	$\zeta_{3,9}$, $\zeta_{1,1,4,6}$

MZVs satisfy shuffle and stuffle relations, and eMZVs obey

- shuffle $\omega(n_1,\ldots,n_r)\omega(k_1,\ldots,k_s) = \omega\big((n_1,\ldots,n_r) \sqcup\!\sqcup (k_1,\ldots,k_s)\big)$
- reflection: $\omega(n_1,n_2,\ldots,n_r) = (-1)^{n_1+n_2+\ldots+n_r}\omega(n_r,\ldots,n_2,n_1)$
- Fay rel's $\omega(n_1,n_2,\ldots) \leftrightarrow \omega(n_1{+}j,n_2{-}j,\ldots)$ such as $\omega(0,5)=\omega(2,3)$

all of which preserve the weight $\sum_{j=1}^r n_j$ of $f^{(n_j)}$ integrands.

$\longrightarrow$ Which eMZVs remain indecomposable w.r.t. $\mathbb{Q}[\mathrm{MZV}]$?

However, $f^{(0)}$ @ zero weight $\Rightarrow$ $\exists\, \infty$ eMZVs $\omega(n, \underbrace{0, \ldots, 0}_{\text{any number}})$ @ weight n

$\Rightarrow$ organize relations by length r:

length $r = 1$: only constant eMZVs:
$$\omega(n) = \begin{cases} -2\,\zeta_n & : \ n \text{ even} \\ 0 & : \ n \text{ odd} \end{cases}$$

length $r = 2$: shuffle and reflection reduce even-weight eMZVs to $r = 1$:
$$\omega(n_1, n_2)\,\Big|_{n_1+n_2 \text{ even}} = \begin{cases} 2\,\zeta_{n_1}\,\zeta_{n_2} & : \ n_1, n_2 \text{ both even} \\ 0 & : \ n_1, n_2 \text{ both odd} \end{cases}$$

odd-weight $\omega(n_1, n_2)$ depend on τ and can be reduced to $\omega(0, 2p-1)$:
$$\omega(n_1, n_2)\,\Big|_{n_1+n_2 \text{ odd}} = (-1)^{n_1}\omega(0, n_1+n_2) + 2\delta_{n_1,1}\zeta_{n_2}\omega(0,1) - 2\delta_{n_2,1}\zeta_{n_1}\omega(0,1)$$
$$+\,2\left\{ \sum_{p=1}^{\lceil \frac{1}{2}(n_2-3)\rceil} \binom{n_1+n_2-2p-2}{n_1-1} \zeta_{n_1+n_2-2p-1}\,\omega(0, 2p+1) - (n_1 \leftrightarrow n_2) \right\}$$

any length r: eMZVs with $(-1)^{\text{length}} = (-1)^{\text{weight}}$ reducible to lower length

length $r=3$: $\lceil \frac{w}{6} \rceil$ indecomposable $\omega(n_1,n_2,n_3)$ @ even $w=\sum_{j=1}^{3} n_j$:

w	2	4	6	8	10	12	14
indec. eMZVs at $r=3$	$\omega(0,0,2)$	$\omega(0,0,4)$	$\omega(0,0,6)$	$\omega(0,0,8)$ $\omega(0,3,5)$	$\omega(0,0,10)$ $\omega(0,3,7)$	$\omega(0,0,12)$ $\omega(0,3,9)$	$\omega(0,0,14)$ $\omega(0,3,11)$ $\omega(0,5,9)$

at $w=8$, for instance, shuffle, reflection & Fay imply

$$\omega(0,6,2) = -\frac{21}{2}\zeta_8 + 2\,\omega(0,3)\,\omega(0,5) - 14\,\zeta_6\,\omega(0,0,2) - 6\,\zeta_4\,\omega(0,0,4) - \frac{9}{2}\omega(0,0,8) - \frac{2}{5}\omega(0,3,5)$$

Further eMZV relations available @ `https://tools.aei.mpg.de/emzv` any length r: eMZVs with $(-1)^{\text{length}} = (-1)^{\text{weight}}$ reducible to lower length

length $r = 3$: $\left\lceil \frac{w}{6} \right\rceil$ indecomposable $\omega(n_1, n_2, n_3)$ @ even $w = \sum_{j=1}^{3} n_j$:

w	2	4	6	8	10	12	14
indec. eMZVs at $r = 3$	$\omega(0,0,2)$	$\omega(0,0,4)$	$\omega(0,0,6)$	$\omega(0,0,8)$ $\omega(0,3,5)$	$\omega(0,0,10)$ $\omega(0,3,7)$	$\omega(0,0,12)$ $\omega(0,3,9)$	$\omega(0,0,14)$ $\omega(0,3,11)$ $\omega(0,5,9)$

length $r = 4$: conjecture $\left\lfloor \frac{1}{2} + \frac{1}{48}(w+5)^2 \right\rfloor$ indecomp's @ odd $w = \sum_{j=1}^{4} n_j$:

w	1	3	5	7	9
indec. eMZVs at $r = 4$	$\omega(0,0,1,0)$	$\omega(0,0,0,3)$	$\omega(0,0,0,5)$ $\omega(0,0,2,3)$	$\omega(0,0,0,7)$ $\omega(0,0,2,5)$ $\omega(0,0,4,3)$	$\omega(0,0,0,9)$, $\omega(0,0,4,5)$ $\omega(0,0,2,7)$, $\omega(0,1,3,5)$

summary @ length $r \leq 7$: indecomposable eMZVs @ $(-1)^{\text{weight}} \neq (-1)^{\text{length}}$

$r \backslash w$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
2	1		1		1		1		1		1		1		1		1		1		1		1
3		1		1		1		2		2		2		3		3		3		4		4	
4	1		1		2		3		4		5		7		8		10		12		14		16
5		1		2		4		6		9		13		17		23		30		37		47	
6	1		2		4		8		13		22		31		45		?		?		?		?
7		1		4		8		16		29		48		?		?		?		?		?	

- strong evidence that shuffle, reflection & Fay $\Rightarrow$ **any** eMZV relation
- will derive these numbers from combinatorial principles

[Brödel, Matthes, OS 1507.02254]

III. 2 Direct computation of eMZVs

eMZVs satisfy ODE w.r.t. modular parameter $q = e^{2\pi i\tau}$: **[Enriquez 1301.3042]**

$$-4\pi^2 q\frac{\mathrm{d}}{\mathrm{d}q}\, \mathrm{eMZV}\begin{pmatrix} \text{weight } w \\ \text{length } \ell \end{pmatrix} \ \sim \ \mathrm{G}_k(q) \ \times \ \mathrm{eMZV}\begin{pmatrix} \text{weight } w{-}k{+}1 \\ \text{length } \ell{-}1 \end{pmatrix}$$

recursive in length, generating *Eisenstein series* $\mathrm{G}_k(q)$ in each step:

$$\mathrm{G}_k(q) \ = \ \begin{cases} 0 & : \ k \text{ odd} \\ 2\zeta_k + \frac{2(2\pi i)^k}{(k-1)!}\sum_{m,n=1}^{\infty} m^{k-1} q^{mn} & : \ k \text{ even} \end{cases}$$

Supplement the above ODE in q by boundary values: **[Enriquez 1301.3042]**

$$\lim_{q\to 0} \mathrm{eMZV}\begin{pmatrix} \text{weight } w \\ \text{length } \ell \end{pmatrix} \ \sim \ (2\pi i)^{\pm k} \ \times \ \mathrm{MZVs}\big(\text{weight } w \mp k\big)$$

generating series for eMZV degenerates to Drinfeld associator

III. 2 Direct computation of eMZVs

eMZVs satisfy ODE w.r.t. modular parameter $q = e^{2\pi i\tau}$: **[Enriquez 1301.3042]**

$$-4\pi^2 q \frac{\mathrm{d}}{\mathrm{d}q}\, \mathrm{eMZV}\begin{pmatrix} \text{weight } w \\ \text{length } \ell \end{pmatrix} \sim \mathrm{G}_k(q) \times \mathrm{eMZV}\begin{pmatrix} \text{weight } w{-}k{+}1 \\ \text{length } \ell{-}1 \end{pmatrix}$$

recursively get q-expansion from *truncated Eisenstein series* G^0_k:

$$\mathrm{G}^0_k(q) \sim \begin{cases} -1 & : \; k=0 \\ \sum_{m,n=1}^{\infty} m^{k-1} q^{mn} & : \; k = 2,4,6,\ldots \end{cases}$$

$\Rightarrow$ eMZVs are $\mathbb{Q}[\mathrm{MZV}]$ linear combination of iterated Eisenstein integrals

$$\gamma(k_1,k_2,\ldots,k_r) \equiv \frac{1}{(4\pi^2)^r} \int_{0\leq q_i \leq q_{i+1} \leq q} \mathrm{G}^0_{k_1}(q_1)\frac{\mathrm{d}q_1}{q_1}\, \mathrm{G}^0_{k_2}(q_2)\frac{\mathrm{d}q_2}{q_2} \cdots \mathrm{G}^0_{k_r}(q_r)\frac{\mathrm{d}q_r}{q_r}$$

special cases of iterated Shimura integrals **[Manin 2005; Brown 2014]**

Proof of linear independence **[Matthes 1601.05743]**

q-expansion of Eisenstein integrals easy to compute

$$\int_0^q (q')^N \, \frac{\mathrm{d}q'}{q'} \; = \; \frac{q^N}{N}$$

and turns out to resemble MZVs as nested sums

$$\gamma(k_1, 0^{p_1-1}, k_2, 0^{p_2-1}, \ldots, k_r, 0^{p_r-1}) \sim \sum_{0<n_1<n_2<\cdots<n_r} \frac{\sigma_{k_1-1}(n_1)\sigma_{k_2-1}(n_2-n_1)\ldots\sigma_{k_r-1}(n_r-n_{r-1})q^{n_r}}{n_1^{p_1} n_2^{p_2}\ldots n_r^{p_r}}$$

$$\zeta(1, 0^{p_1-1}, 1, 0^{p_2-1}, \ldots, 1, 0^{p_r-1}) \sim \sum_{0<n_1<n_2<\cdots<n_r} \frac{1}{n_1^{p_1} n_2^{p_2}\ldots n_r^{p_r}} \; = \; \zeta_{p_1,p_2,\ldots,p_r}$$

Divisor sums $\sigma_k(n) \equiv \sum_{d|n} d^k$ obstruct analogues of stuffle-relations

$\Rightarrow$ no relations among $\gamma(\ldots)$ beyond shuffle $\gamma(W_1)\gamma(W_2) = \gamma(W_1 \sqcup\!\sqcup W_2)$

[Matthes 1601.05743]

Length-two example: indecomposable eMZVs $\omega(0,k)$ @ $k = 3, 5, 7, \ldots$

$$\omega(0,k) = \; k\gamma(k+1) \; = \; \text{rational} \times \pi^{k-1} \sum_{m,n=1}^{\infty} \frac{m^{k-1}}{n} q^{mn}$$

III. 3 Back to counting indecomposable eMZVs

Naively: indecomposable eMZVs $\leftrightarrow$ shuffle-independent Eisenstein int's?

$$\mathrm{eMZV}\begin{pmatrix} \text{weight } \sum_j k_j - r \\ \text{length } r+1 \end{pmatrix} \stackrel{?}{\longleftrightarrow} \frac{\{\gamma(k_1,k_2,\ldots,k_r) \text{ @ } k_j = 0,2,4,\ldots\}}{\text{shuffle-relations } \gamma(W_1)\gamma(W_2) = \gamma(W_1 \sqcup\!\sqcup W_2)}$$

Correct at length two $\omega(0,k) \leftrightarrow \gamma(k+1)$, but $\exists$ problems at length three:

$$\text{e.g.} \quad \underbrace{\omega(0,0,12),\ \omega(0,3,9)}_{2\,\times\,\text{indecomposable}} \quad \leftrightarrow \quad \underbrace{\gamma(0,14),\ \gamma(2,12),\ \gamma(4,10),\ \gamma(6,8)}_{4\,\times\,\text{shuffle-independent}}$$

Additional selection rules arise from derivation algebra $\{\epsilon_0,\epsilon_2,\epsilon_4,\ldots\}$

"dual" to Eisenstein series $\mathrm{G}_0, \mathrm{G}_2, \mathrm{G}_4, \ldots$ subject to relations

$$[\epsilon_2,\epsilon_k] = 0\ , \qquad [\epsilon_{10},\epsilon_4] - 3\,[\epsilon_8,\epsilon_6] = 0\ , \qquad \text{etc.}$$

Explains above mismatch "$2\times\omega(\ldots)$ vs. $4\times\gamma(\ldots)$"

Generating series of eMZVs: elliptic associator [Enriquez 1301.3042]

$$A(q) \equiv \sum_{r\geq 0}(-1)^r \sum_{n_1,n_2,\ldots,n_r\geq 0} \omega(n_1,n_2,\ldots,n_r)\,\mathrm{ad}_x^{n_r}(y)\ldots\mathrm{ad}_x^{n_2}(y)\mathrm{ad}_x^{n_1}(y)$$

Derivations $\epsilon_0, \epsilon_2, \epsilon_4, \ldots$ act on non-commutative variables x, y via

$$\epsilon_{2n}(x) = (\mathrm{ad}_x)^{2n}(y)\ , \qquad \epsilon_0(y) = 0\ , \qquad \text{[Nakamura, Tsunogai 1995, \ldots]}$$

$$\epsilon_{2n}(y) = \sum_{0\leq j<n}(-1)^j[(\mathrm{ad}_x)^j(y), (\mathrm{ad}_x)^{2n-1-j}(y)] \quad @\ n>0$$

satisfy various commutator relations and enter the associator's ODE

$$q\frac{\mathrm{d}}{\mathrm{d}q}A(q) = \frac{1}{4\pi^2}\left(\sum_{n=0}^{\infty}(2n-1)\mathrm{G}_{2n}(q)\epsilon_{2n}\right)A(q)\,.$$

$\Rightarrow$ more precise version of "ϵ_{2n} are dual to G_{2n}"

$\Rightarrow$ commutator relations among ϵ_{2n} impose selection rules on $\gamma(\ldots)$

Hence, more accurate picture is

$$\mathrm{eMZV}\binom{\text{weight } \sum_j k_j - r}{\text{length } r+1} \longleftrightarrow \frac{\{\gamma(k_1, k_2, \ldots, k_r) \text{ @ } k_j = 0, 2, 4, 6, \ldots\}}{\{\text{shuffle-relations}\} \times \{\epsilon_k\text{-relations}\}}$$

For instance, $[\epsilon_2, \epsilon_k] = 0$ obstructs non-trivial appearance of $k_j = 2$ and

$$\omega(0, n) \leftrightarrow \gamma(n+1)\,, \qquad \omega(0,3,5) \leftrightarrow \gamma(4,6)$$

$$\omega(0, 0, n) \leftrightarrow \gamma(n+2, 0)\,, \qquad \omega(0,3,7) \leftrightarrow \gamma(4,8)$$

$$\left.\begin{array}{c} \omega(0,3,9) \\ \\ \omega(0,5,7) \end{array}\right\} \begin{array}{c} \leftrightarrow \quad 81\gamma(10,4) + 35\gamma(8,6) \\ \\ \text{selected by } [\epsilon_{10}, \epsilon_4] - 3[\epsilon_8, \epsilon_6] = 0 \end{array}$$

Cusp forms control ϵ_k-relations, e.g. **[Brown; Hain Matsumoto; Pollack]**

wt. 12 cusp form $\leftrightarrow$ $0 = [\epsilon_{10}, \epsilon_4] - 3[\epsilon_8, \epsilon_6]$, **[Ihara, Takao 1993; Schneps 2006]**

wt. 16 cusp form $\leftrightarrow$ $0 = 2[\epsilon_{14}, \epsilon_4] - 7[\epsilon_{12}, \epsilon_6] + 11[\epsilon_{10}, \epsilon_8]$

At length $\ell = 4$, for instance,

$$\text{eMZV}\begin{pmatrix}\text{weight } \sum_j k_j - 3 \\ \text{length } 4\end{pmatrix} \longleftrightarrow \frac{\{\gamma(k_1,k_2,k_3) \text{ @ } k_j = 0,2,4,6,\ldots\}}{\{\text{shuffle-relations}\} \times \{\epsilon_k\text{-relations}\}}$$

indecomposable eMZVs governed by derived relations, e.g.

$$0 = \big[\epsilon_k \ , \ [\epsilon_{10},\epsilon_4] - 3[\epsilon_8,\epsilon_6]\,\big] = \big[\epsilon_k \ , \ 2[\epsilon_{14},\epsilon_4] - 7[\epsilon_{12},\epsilon_6] + 11[\epsilon_{10},\epsilon_8]\,\big]$$

and irreducible relations controlled by cusp forms **[Pollack 2009]**

$$\begin{aligned} 0 \ = \ & 80\,[\epsilon_{12},[\epsilon_4,\epsilon_0]] + 16\,[\epsilon_4,[\epsilon_{12},\epsilon_0]] - 250\,[\epsilon_{10},[\epsilon_6,\epsilon_0]] \\ & -125\,[\epsilon_6,[\epsilon_{10},\epsilon_0]] + 280\,[\epsilon_8,[\epsilon_8,\epsilon_0]] - 462\,[\epsilon_4,[\epsilon_4,\epsilon_8]] - 1725\,[\epsilon_6,[\epsilon_6,\epsilon_4]] \end{aligned}$$

Perfect matching with indecomposable eMZVs

[Brödel, Matthes, OS 1507.02254]

wt	1	2	3	4	5	6	7	8	9	10	11	12	13
$\ell=2$	1		1		1		1		1		1		1
$\ell=3$		1		1		1		2		2		2	
$\ell=4$	1		1		2		3		4		**5**		**7**

Further directions

- construct one-loop string amplitude from elliptic associator $A(q)$ in analogy to tree-level amplitudes from the Drinfeld associator

  [Brödel, OS, Stieberger, Terasoma 1304.7304]

- identify single-valued eMZVs and connect with closed strings

  [Pierre Vanhove's talk; D'Hoker, Green, Vanhove 1502.06698 & 1509.00363]

  [Zerbini 1512.05689; D'Hoker, Green, Gürdogan, Vanhove 1512.06779]

- eMZVs in one-loop string amplitudes of orbifold compactifications

  [OS, Berg, Buchberger 1603.05262]

- higher-genus counterparts of MZVs along with single-valued subset

Summary

partial fraction

shuffle

Fay relations

MPLs $G(a_1, \ldots, a_r; z)$

made of $\mathrm{d}\log|z - a_i|$

MZVs $\zeta(a_1, a_2, \ldots, a_r)$

in α'-expansion of

superstring trees

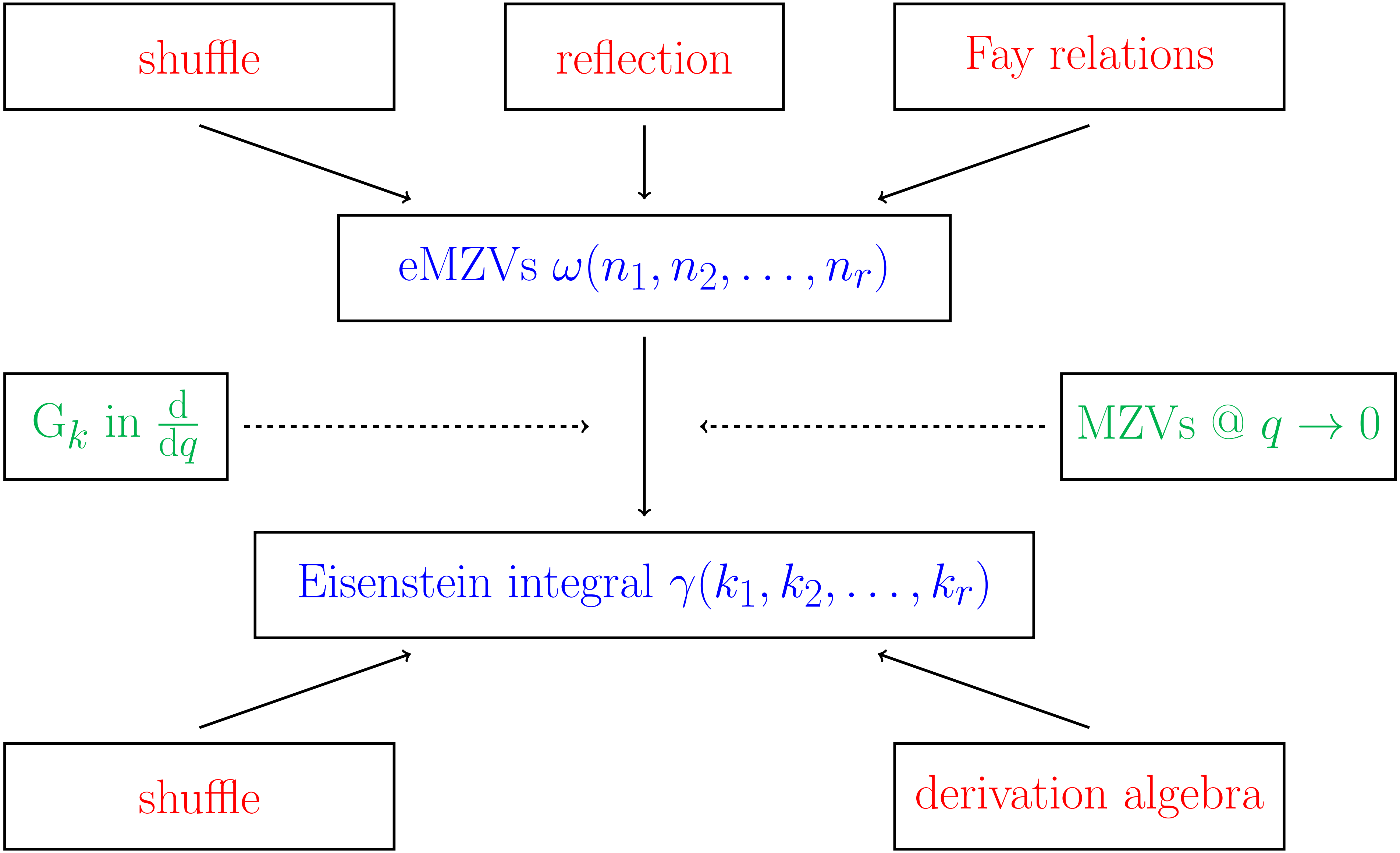
ell. it. int's $\Gamma\left(\begin{smallmatrix} n_1 & \ldots & n_r \\ a_1 & \ldots & a_r \end{smallmatrix}; z\right)$

made of $f^{(n_i)}(z_i - a_i)\mathrm{d}z_i$

eMZVs $\omega(n_1, n_2, \ldots, n_r)$

in α'-expansion of

1-loop string amplitudes



Various eMZV relations available at `https://tools.aei.mpg.de/emzv`