# Single Valued Elliptic Multizetas and String Theory

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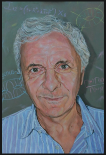
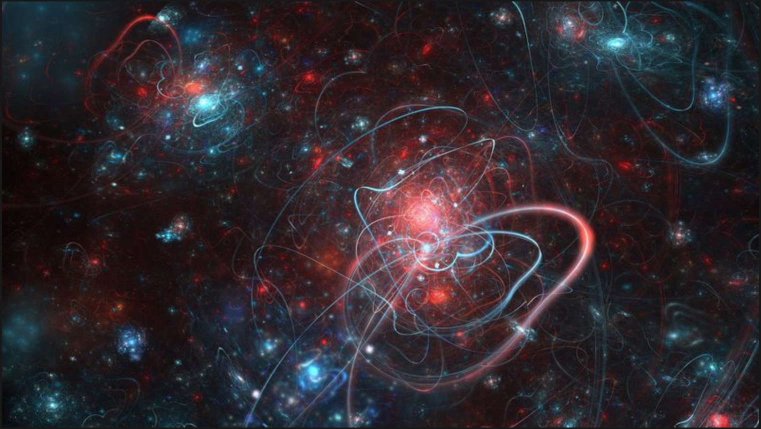
based on [1502.06698], [1509.00363], [1512.06779] with



# Part I

## String theory effective action



# Derivative expansion in string theory

Higher derivative corrections to the low-energy effective action of string theory have coupling depending on the moduli $\varphi^i$

$$\mathcal{L} = \int d^{10}x\, |-g|^{\frac{1}{2}} \left( \frac{e^{-2\phi}}{\alpha'^4} \mathcal{R} + \sum_{k \geqslant 0} f_k(\varphi^i)(\alpha' \partial^2)^k \alpha'^3 \mathcal{R}^4 + \cdots \right)$$

$\alpha'$ is the inverse tension of the string

# Derivative expansion in string theory

The structure of the effective action is constrained by

- unitarity, spectrum
- supersymmetry leading to differential equations on $f_k(\varphi^i)$
- duality symmetry acting on the scalars $\varphi^i$

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Determining these coefficients is extremely important for

- UV divergences of maximal supergravity
- $\alpha'$ corrections to black hole entropy [Sen,...]
- lead to $\alpha'$ corrections the vector- and hyper-multiplet geometry on CY 3-fold
- ...

# Type IIB superstring

The vacuum parametrized by $\Omega = C^{(0)} + ie^{-\phi}$ in the duality group coset $SL(2,\mathbb{R})/SO(2)$

The Einstein frame effective action $\ell_P^8 = \alpha'^4 \exp(2\phi)$

$$\mathcal{L}_{\mathrm{IIB}} = \int d^{10}x \, |-g|^{\frac{1}{2}} \left( \frac{1}{\ell_P^8} \mathcal{R} + \sum_{k \geqslant 0} f_k^0(\Omega) \, (\ell_P^2 \partial^2)^k \ell_P^6 \, \mathcal{R}^4 + \cdots \right)$$

The coefficients are modular forms for $SL(2,\mathbb{Z})$ with a $U(1)$ weight $w = -$ the $R$-symmetry charge of the coupling

$$f_k^w \left( \frac{a\Omega + b}{c\Omega + d} \right) = \left( \frac{c\Omega + d}{c\bar{\Omega} + d} \right)^w f_k(\Omega) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

# Supersymmetry protected couplings

The $\frac{1}{2}$- and $\frac{1}{4}$-BPS protected terms satisfy

$$4(\Im m\Omega)^2 \partial_\Omega \bar{\partial}_{\bar{\Omega}} f_k^w(\Omega) = \lambda_k^w f_k^w(\Omega); \quad k = 0, 2$$

Given by Eisenstein series $f_k^w(\Omega) = E^w_{\frac{3}{2}+k}(\Omega)$

$$E^w_{\frac{3}{2}+k}(\Omega) = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{(\Im m\Omega)^{\frac{3+k}{2}}}{(m\Omega+n)^{\frac{3+k}{2}+w}(m\bar{\Omega}+n)^{\frac{3+k}{2}-w}}$$

# Supersymmetry protected couplings

- $\mathcal{R}^4$ : $\frac{1}{2}$-BPS term one-loop exact

$$\Omega_2^{\frac{1}{2}} E_{\frac{3}{2}}(\Omega) = 2\zeta(3)\Omega_2^{2} + 4\zeta(2) + \text{non-pert.}$$

- $D^4\mathcal{R}^4$ : $\frac{1}{4}$-BPS term two-loop exact

$$\Omega_2^{-\frac{1}{2}} E_{\frac{5}{2}}(\Omega) = 2\zeta(5)\Omega_2^{2} + 0 + \frac{8}{3}\zeta(4)\Omega_2^{-2} + \text{non-pert.}$$

Only a finite number of perturbative term:
tree-level, one-loop, two-loop

# Supersymmetry protected couplings

The $\frac{1}{8}$-BPS is not an Eisenstein series

$$4\mathfrak{Im}(\Omega)^2\partial_\Omega\bar{\partial}_{\bar{\Omega}}f_3^0(\Omega) = 12f_3^0(\Omega) - 6(f_0^0(\Omega))^2$$

- $D^6\mathcal{R}^4$ : $\frac{1}{8}$-BPS three-loop exact

$$\Omega_2^{-1}f_3^0(\Omega) = \frac{2}{3}\zeta(3)^2\Omega_2^2 + \frac{4\zeta(2)\zeta(3)}{3} + 4\zeta(4)\Omega_2^{-2} + \frac{4\zeta(6)}{27}\Omega_2^{-4} + \text{non-pert.}$$

Only a finite number of perturbative term:
tree-level, one-loop, two-loop, three-loop

# Weak coupling expansion

The weak coupling expansion reads

$$\Omega_2^{-\alpha_k} f_k^w(\Omega) = \sum_{g \geqslant 0} a_g \Omega_2^{2-2g} + \text{non-pert.}$$

The power behaved terms are the perturbative contributions given by the analytic contribution from genus-$g$ four gravitons amplitudes in string theory

They are expressed as integrals over the moduli space of $\mathcal{M}_{g,n}$ with $n$ punctures at genus $g$ with period matrix $\tau$

$$a_g = \int_{\mathcal{M}_{g,n}} d\mu \, f(\tau)$$

# UV structure of maximal supergravity

The structure extends to lower dimensions for higher-rank (Chevalley) group and allowed to predicted the UV behaviour maximal supergravity till seven loops [Green,Russo,Vanhove]

Determining the coupling from $D^8\mathcal{R}^4$ onward will give a direct proof of the UV behaviour $\mathcal{N} = 8$ supergravity in four dimensions

So far all higher-loop divergences in $\mathcal{N} = 8$ (and $\mathcal{N} = 4$) supergravity are given by single zeta values

Could we find of some special organizing principle putting in relation modularity and special zeta values ?

# Part II

## Tree amplitudes and MZV

$$\zeta(s_1, \cdots, s_r) = \sum_{n_r > \cdots > n_1 > 0} \prod_{i=1}^{r} \frac{1}{n_i^{s_i}}$$

# Tree-level expansion

The closed string amplitudes are given by integrals on the moduli space $\mathcal{M}_{0,n-3}$ of the Riemann sphere minus three points (mapped to the complex plane)

$$A_{\text{tree}} = \int_{\mathbb{C}^{n-3}} \exp\left( \sum_{1\leqslant i<j\leqslant n} \alpha' k_i \cdot k_j \, G_{\text{tree}}(z_i - z_j) \right) \prod_{i=2}^{n-1} d^2 z_i$$

The tree-level propagator is

$$G_{\text{tree}}(z) = \log z + \log \bar{z}$$

They are Selberg integrals which $\alpha'$ expansion is known to lead to multiple zeta values (MZV) [Terasoma, Brown, Schloterer, Stieberger, etc.]

# Tree-level expansion

The four-graviton amplitude has the expansion
$\sigma_n = \alpha'^n (s^n + t^n + u^n)$

$$A_{4g} = \frac{\mathcal{R}^4}{\sigma_3} \exp\left( - \sum_{n \geqslant 0} \frac{2\zeta(2n+1)}{2n+1} \sigma_{2n+1} \right)$$

# Tree-level expansion

The four-graviton amplitude has the expansion
$\sigma_n = \alpha'^n (s^n + t^n + u^n)$

$$A_{4g} = \mathcal{R}^4 \Big( \frac{3}{\sigma_3} + 2\zeta(3) + \zeta(5)\,\sigma_2 + \frac{2\zeta(3)^2}{3}\,\sigma_3 + \frac{\zeta(7)}{2}\,\sigma_2^2 + \frac{2}{3}\zeta(5)\zeta(3)\sigma_2\sigma_3 + \dots \Big)$$

- The expansion in polynomial only in odd zeta values
- Higher-point amplitude exhibits similar properties since only Brown's single valued multiple zetas arise in the $\alpha'$ expansion [Schloterer, Stieberger, . . . ]

# Brown's single valued multiple zetas

$$\zeta(k_1, \ldots, k_r) = \sum_{n_r > \cdots > n_1 > 0} \prod_{i=1}^{r} \frac{1}{n_i^{k_i}}$$

They are the value at 1 of the multiple polylogarithms (MPL) of

$$\mathrm{Li}_{k_1,\ldots,k_r}(z) = \sum_{n_r > \cdots > n_1 > 0} \frac{z^{n_r}}{\prod_{i=1}^{r} n_i^{k_i}}$$

This function has monodromies around $z = 0$ and $z = 1$, e.g.

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ -\mathrm{Li}_1(z) & 2i\pi & & \\ -\mathrm{Li}_2(z) & 2i\pi \log z & (2i\pi)^2 & \\ \vdots & \ldots & & \end{pmatrix}$$

# Brown's single valued multiple zetas

By cancelling the monodromies at $z = 0$ and $z = 1$ one can defined a single valued function on $\mathbb{C}\backslash\{0, 1\}$

$$sv(\mathrm{Li}_{k_1,\ldots,k_r}(z))$$

Francis Brown defined the single valued MZV as their value at $z = 1$, e.g. for $n \in \mathbb{N}$

$$\zeta_{sv}(2n) = 0$$

$$\zeta_{sv}(2n+1) = 2\zeta(2n+1)$$

# Zagier single valued polylogarithms

$D_{a,b}(x)$ on $\mathbb{C}\backslash[1,+\infty[$

$$D_{a,b}(x) = (-1)^{a-1}\sum_{k=a}^{a+b-1}\binom{k-1}{a-1}\frac{(-2\log|x|)^{a+b-1-k}}{(a+b-1-k)!}\mathrm{Li}_k(x)$$
$$+(-1)^{b-1}\sum_{k=b}^{a+b-1}\binom{k-1}{b-1}\frac{(-2\log|x|)^{a+b-1-k}}{(a+b-1-k)!}(\mathrm{Li}_k(x))^*$$

e.g.

$$D_{2,3}(x) = -2\,(\ln|x|)^2\,\mathrm{Li}_2(x) + 4\ln|x|\,\mathrm{Li}_3(x) - 2\ln|x|\,(\mathrm{Li}_3(x))^* - 3\,\mathrm{Li}_4(x) + 3\,(\mathrm{Li}_4(x))^*$$

# Zagier single valued polylogarithms

$D_{a,a}(x)$ single valued on $\mathbb{C}\backslash\{0,1\}$

$$D_{a,a}(x) = 2\mathfrak{Re}\Bigg((-1)^{a-1}\sum_{k=0}^{a-1}\binom{k+a-1}{a-1} \times \frac{(-2\log|x|)^{a-1-k}}{(a-1-k)!}\mathrm{Li}_{a+k}(x)\Bigg)$$

e.g.

$$D_{1,1}(z) = \mathrm{Li}_1(z) + (\mathrm{Li}_1(z))^* = G_{\mathrm{tree}}(1-z)$$

$$D_{2,2}(z) = 2\mathfrak{Re}\Big(\log|z|\,\mathrm{Li}_2(z) - \mathrm{Li}_3(z)\Big)$$

# Zagier single valued polylogarithms

Their value at $z = 1$ gives Brown's single valued zeta

$$D_{a,b}(1) = 0 \qquad a + b \in 2\,\mathbb{N} - 1$$

$$D_{a,b}(1) \in \zeta(a + b - 1) \times \mathbb{Z} \qquad a + b \in 2\,\mathbb{N}$$

# Brown's single valued multiple zetas

The dimension $d_w^{sv}$ of the subspace of weight $w$ in the ring over $\mathbb{Q}$ of single-valued multiple-zeta values is smaller [Brown]

At weight 11 a basis of MZVs has dimension 9 is

$$\zeta(3,5,3),\ \zeta(3,5)\zeta(3),\ \zeta(3)^2\zeta(5),\ \zeta(11),$$

$$\zeta(2)\zeta(3)^3,\ \zeta(2)^4\zeta(3),\ \zeta(2)^3\zeta(5),\ \zeta(2)^2\zeta(7),\ \zeta(2)\zeta(9)\,.$$

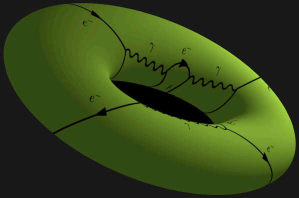
Since $\zeta_{sv}(2)=0$ and $\zeta_{sv}(3,5)=-10\zeta_{sv}(3)\zeta_{sv}(5)$
the basis at weight 11 has dimension 3 [Brown; Schnetz]

$$\zeta_{sv}(3,5,3),\ \zeta_{sv}(3)^2\zeta_{sv}(5),\ \zeta_{sv}(11)$$

with $\zeta_{sv}(3,5,3)=2\zeta(3,5,3)-2\zeta(3)\zeta(3,5)-10\zeta(3)^2\zeta(5)$

# Part III

# Loop amplitudes and elliptic polylogarithms



$$\mathrm{Li}_{s_1,\dots,s_r}(q,\bar{q},\zeta) = \sum_{n_i,m_i} \mathrm{Li}_{s_1,\dots,s_r}(q^{n_1}\bar{q}^{m_i}\zeta,\cdots)$$

# Genus-one amplitude

By unitarity the special properties of the $\alpha'$ expansion at tree-level amplitude will reappear in some way at one-loop

$$A_{1-\text{loop}}(\alpha' s_{ij}) = \int_{\mathcal{F}} \mathcal{B}_N(\alpha' s_{ij}|\tau)\frac{d^2\tau}{\tau_2^2}$$

$\mathcal{F}$ is a fundamental domain for $SL(2,\mathbb{Z})$

$$\mathcal{B}_N(s_{ij}|\tau) = \prod_{n=1}^{N}\int_{\Sigma}\frac{d^2 z_n}{\tau_2}\,\text{exp}\left(\sum_{1\leqslant i<j\leqslant N}\alpha' s_{ij}\,G_{1-\text{loop}}(z_i - z_j|\tau)\right)$$

# Genus-one amplitude

The analytic part (i.e. not on the logarithmic thresholds which start from $\alpha'^7 D^8 \mathcal{R}^4$ )

$$\mathcal{B}_1(s,t,u|\tau) = \sum_{p,q=0}^{\infty} j^{(p,q)}(\tau)\, \sigma_2^p\, \sigma_3^q$$

$j^{(p,q)}(\tau)$ are $SL(2,\mathbb{Z})$ modular functions of weight zero

$$j^{(p,q)}(\gamma\cdot\tau) = j^{(p,q)}(\tau) \qquad \gamma \in SL(2,\mathbb{Z})$$

Integrating these modular functions lead to $a_g$ in ten or lower-dimensions when multiplied by the appropriate lattice factor [Green, Vanhove, Russo; Angelantonij, Florakis, Pioline; ... ]

# The one-loop Green function

The one-loop green function satisfies

$$4\partial_z\bar\partial_{\bar z} G_{1-\text{loop}}(z|\tau) = -4\pi\delta^{(2)}(z)+\frac{4\pi}{\tau_2}; \qquad \int_\Sigma d^2z\, G_{1-\text{loop}}(z|\tau) = 0$$

solved by the modular invariant expression $z = \nu + \tau u$

$$G_{1-\text{loop}}(z|\tau) = -\ln\left|\frac{\vartheta_1(z|\tau)}{\eta(\tau)}\right|^2 - \frac{\pi}{2\tau_2}(z-\bar z)^2$$

it has the lattice sum expansion

$$G_{1-\text{loop}}(z|\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2}{\pi|m\tau+n|^2}\, e^{2i\pi(m\nu-nu)}$$

# Elliptic polylogarithm

This Green function is Zagier's a singled value elliptic 1-log
$q = e^{2\pi i\tau}$ and $\zeta = e^{2\pi i z} = q^{u} e^{2i\pi v}$

$$G_{1-\text{loop}}(z|\tau) = \sum_{n\geqslant 0} D_{1,1}(q^{n}\zeta) + \sum_{n\geqslant 1} D_{1,1}(q^{n}/\zeta) + 2\pi\tau_2\left(u^2 - u + \frac{1}{6}\right)$$

This expression is singled value in $\zeta \in \mathbb{C}^{\times}/q^{\mathbb{Z}}$ on the torus

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$$G_{1-\text{loop}}(z|\tau) = 2\Re\text{e}\Big(\sum_{n\geqslant 0}\text{Li}_1\left(q^n\zeta\right) + \sum_{n\geqslant 1}\text{Li}_1\left(q^n/\zeta\right)\Big) + 2\pi\tau_2\left(u^2 - u + \frac{1}{6}\right)$$

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# Elliptic multiple-polylogarithm

Non singled valued elliptic multiple polylogarithms have appeared in other different contexts in QFT and string theory

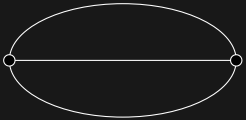
- QFT : sunset, three-loop banana graph [Bloch, Vanhove, Kerr, Weinzierl, Adams, Bodgner, . . . ]
- Open string expansion [Scholterer, Mathes, Broedel, . . . ]

They can be constructed using a elliptic generalisation of Chen iterated integral for multiple-polylogarithm (see Schloterer's talk)

They are related to the construction of Brown and Levin

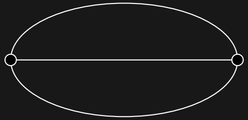
# Part IV

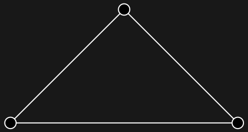
## Modular graph functions



$$= \sum_{n,m \geqslant 0} a_{n,m} q^n \bar{q}^m.$$

# Genus-one world-sheet graphs





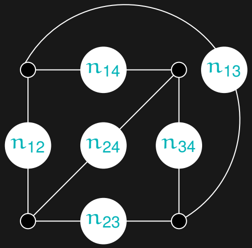
The propagator is one-loop propagator $G_{1-\text{loop}}$


$$= G(z_i - z_j|\tau)^n.$$

Recall $\zeta = e^{2i\pi z}$

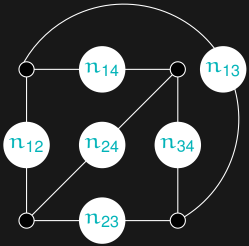
# Genus-one world-sheet graphs

$$I_\Gamma(q) \;=$$




$$I_\Gamma(q) = \prod_{k=2}^{4} \int_\Sigma \frac{d^2 z_k}{2\pi\tau_2} \prod_{1\leqslant i<j\leqslant 4} G_{1-\mathrm{loop}}(z_j - z_i|\tau)^{n_{ij}}$$

# Genus-one world-sheet graphs

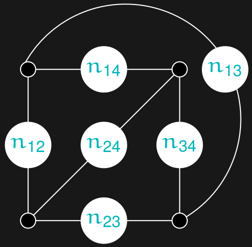
$I_\Gamma(q) =$ 


Using the lattice sum representation for the propagator

$$G_{1-\text{loop}}(z|\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2}{\pi|m\tau+n|^2}\, e^{2i\pi(m v - n u)}$$

# Genus-one world-sheet graphs

$I_\Gamma(q) \;=$

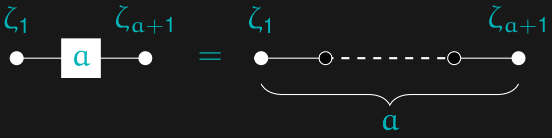


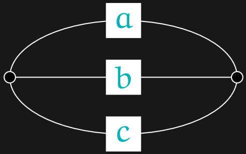

The lattice momentum space Feynman representation [Green, Vanhove; Green, Russo, Vanhove; Green, d'Hoker, Vanhove]

$$I_\Gamma(q) = \sum_{p_1,\dots,p_w\in\mathbb{Z}\tau+\mathbb{Z}}{}^{'} \prod_{\alpha=1}^{w} \frac{\tau_2}{\pi|p_\alpha|^2} \prod_{i=1}^{N} \delta\left(\sum_{\alpha=1}^{w} p_\alpha\right).$$

# Modular graph functions

They satisfy a lot of important algebraic relations




$$= C_{a,b,c}(q;\zeta_1/\zeta_1')$$

$$C_{a_1,\dots,a_\rho}(\tau) = \sum_{(m_r,n_r)\neq(0,0)} \delta\Big(\sum_r (m_r\tau+n_r)\Big) \prod_{r=1}^{\rho} \left(\frac{\tau_2}{\pi|m_r\tau+n_r|^2}\right)^{a_r}$$

# Modular graph functions

$$C_{1,1,1} = E_3 + \zeta(3); \qquad C_{2,2,1} = \frac{2}{5} E_5 + \frac{\zeta(5)}{30}$$

$$40 C_{2,1,1,1} = 300\, C_{3,1,1} + 120\, E_2 E_3 - 276\, E_5 + 7\, \zeta(5)$$

$$C_{1,1,1,1,1} = 60 C_{3,1,1} + 10\, E_2 E_3 - 48\, E_5 + 10\, \zeta(3) E_2 + 16\, \zeta(5)$$

The loop order is not respected : one , two,three, four loops

The relations are between graphs with the same number of propagators

# Modular graph function and MZV I

The lattice sum displays a clear parallel with the MZV sum

These functions have a mixed $q$ and $\bar{q}$ expansion

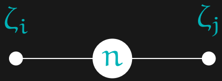
$$F(q) = \sum_{n\geqslant 0, m\geqslant 0} c_{n,m} q^n \bar{q}^m$$

The constant term $c_{0,0}$ is a Laurent polynomial in $y = \pi\tau_2$

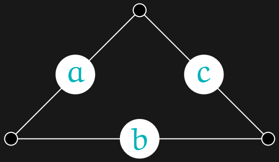
$$C_{2,1,1}\Big|_{0,0} = \frac{2y^4}{14175} + \frac{\zeta(3)y}{45} + \frac{5\zeta(5)}{12y} - \frac{\zeta(3)^2}{4y^2} + \frac{9\zeta(7)}{16y^3}$$

The $C_{a,b,c}$ satisfy differential equations
$(\Delta - \lambda)C_{a,b,c} = P(E_a, \zeta(a))$ and only contain $\zeta_{sv}$

# Modular graph function and MZV




$$= G(z_i - z_j|\tau)^n .$$




$$= D_{a,b,c}(q) .$$

$$D_{3,1,1}(q)\Big|_{0,0} = \frac{2y^5}{22275} + \frac{y^2\zeta(3)}{45} + \frac{11\zeta(5)}{60} + \frac{105\zeta(7)}{32y^2} - \frac{3\zeta(3)\zeta(5)}{2y^3} + \frac{81\zeta(9)}{64y^4}$$

# Modular graph function and MZV

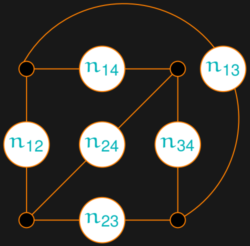
Zerbini computed that

$$\frac{D_{1,1,5}(q)}{4^7}\bigg|_{0,0} = \frac{62\,y^7}{10945935} + \frac{\zeta_{sv}(3)}{243}y^4 + \frac{119}{648}\zeta_{sv}(5)y^2 + \frac{11}{54}\zeta_{sv}(3)^2 y + \frac{21}{32}\zeta_{sv}(7) + \frac{23}{}\frac{\zeta_{sv}(3)\zeta_{sv}(5)}{6y} + \frac{7115\zeta_{sv}(9) - 1800\zeta_{sv}(3)^2}{576y^2} + \frac{1245\zeta_{sv}(3)\zeta_{sv}(7) - 150\zeta_{sv}(5)^2}{64y^3} + \frac{288\zeta_{sv}(3,5,3) - 4080\zeta_{sv}(5)\zeta_{sv}(3)^2 - 9573\zeta_{sv}(11)}{256y^4} + \frac{2475\zeta_{sv}(5)\zeta_{sv}(7) + 1125\zeta_{sv}(9)\zeta_{sv}(3)}{128y^5} - \frac{1575}{64}\frac{\zeta_{sv}(13)}{y^6}$$

# Modular graph function as svEMZ

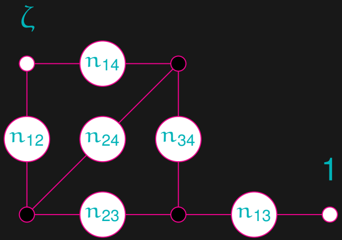
We showed in [Green, D'Hoker, Gurdogan, Vanhove] that the modular graph functions $I_\Gamma(q)$ are the value at $\zeta = 1$ of a single value elliptic multiple polylogarithms

$I_\Gamma(q) =$



# Modular graph function as svEMZ

One can open any world-sheet graph

$I_\Gamma(q;\zeta) =$ 


$$I_\Gamma(q;\zeta) = \prod_{k=2}^{4} \int_\Sigma \frac{d^2 \log \zeta_k}{4\pi^2\tau_2} \prod_{1\leqslant i<j\leqslant 4} D_{1,1}(q;\zeta_j/\zeta_i)^{n_{ij}} \times$$

$$\times \left( \frac{D_{1,1}(q;\zeta_1\zeta/\zeta_3)}{D_{1,1}(q;\zeta_1/\zeta_3)} \right)^{n_{13}}$$

# Modular graph function as svEMZ

$$I_\Gamma(q;\zeta) = \prod_{k=2}^{4} \int_\Sigma \frac{d^2 \log \zeta_k}{4\pi^2\tau_2} \prod_{1\leqslant i<j\leqslant 4} D_{1,1}(q;\zeta_j/\zeta_i)^{n_{ij}} \times \times \left(\frac{D_{1,1}(q;\zeta_1\zeta/\zeta_3)}{D_{1,1}(q;\zeta_1/\zeta_3)}\right)^{n_{13}}$$

the integral is single valued in $\zeta$ and evaluates to at $\zeta = 1$

$$I_\Gamma(q;1) = I_\Gamma(q)$$

# Eichler integrals, period polynomials

The modular graph functions relations can be understood properties of the period polynomials arising from Eichler integral

$$C_{1,1,1} = E_3 + \zeta(3)$$

Both $E_3(q)$ and $C_{1,1,1}(q)$ are obtained from Eichler integrals of *holomorphic* Eisenstein series

$$E_3(q) = \frac{2\Re\mathrm{e}\left(2 + 4\pi\tau_2 \frac{d}{d\log q}\right)\widetilde{G}_5(q)}{(4\pi\tau_2)^2}$$

$$C_{1,1,1}(q) = \frac{2\Re\mathrm{e}\left(2 + 4\pi\tau_2 \frac{d}{d\log q}\right)\left(\widetilde{G}_5(q) + \frac{1}{2}\pi^3\zeta(3)(\log q)^2\right)}{(4\pi\tau_2)^2},$$

# Eichler integrals, period polynomials

The modular graph functions relations can be understood properties of the period polynomials arising from Eichler integral

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Both $E_3(q)$ and $C_{1,1,1}(q)$ are obtained from Eichler integrals of *holomorphic* Eisenstein series

$$\widetilde{G}_5(q) = \zeta(-5)\,\frac{(\log q)^6}{5!} + \zeta(5) + 2\sum_{n=1}^{\infty} \mathrm{Li}_5\left(q^n\right)$$

$$\left(\frac{d}{d\log q}\right)^5 \widetilde{G}_5(q) = \frac{120}{(2i\pi)^6} \sum_{(m,n)\neq(0,0)} \frac{1}{(m\tau+n)^6}$$

# Outlook

- The relations between modular graph functions leads to interesting relations to Eichler integrals and period polynomials
- String theory provide nice avenue for studying the new modular functions produced by string and show how MZV arise from non-trivial modular forms/functions
- The modular graph relations are very non obvious relations between the lattice sums. A systematic understanding of these relations is needed for non-BPS coupling in string theory
- Space-time supersymmetry needs very similar functions. We hope this will help understanding the non-BPS couplings and allow to use the method of [Green,Russo,Vanhove] to address UV question of maximal supergravity in four dimensions