Introduction to Nonequilibrium Processes

ANNECY, April 20-24 2015

Outline of the lectures

1. Review of statistical physics: Equilibrium versus Non-equilibrium. Dynamics, Detailed Balance and Time-reversal.

2. Out of Equilibrium: Large Deviations, Generalized Detailed Balance and the Gallavotti-Cohen theorem.

3. Work Identities: the Jarzynski and Crooks identities.

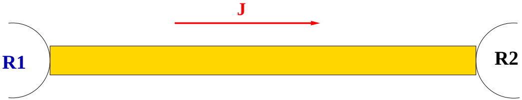
4. The Asymmetric Exclusion Process: Exact Results

5. A unifying framework: the Macroscopic Fluctuation Theory.

THE EXCLUSION PROCESS

Total Current transported through an Open System

A paradigm of a non-equilibrium system



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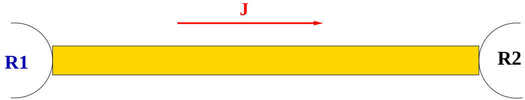
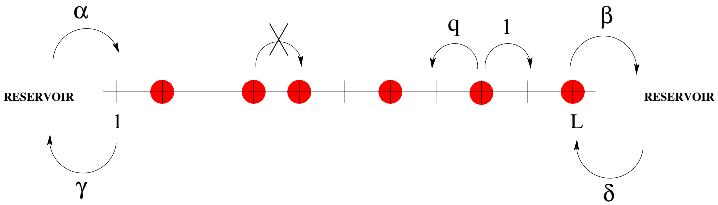
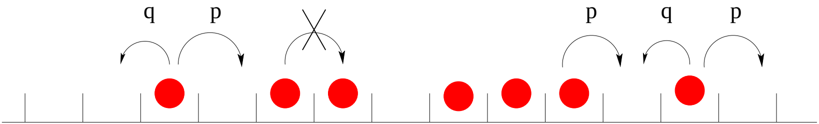


The asymmetric exclusion model with open boundaries



Classical Transport in 1d: ASEP



Asymmetric Exclusion Process. A paradigm for non-equilibrium Statistical Mechanics.

- EXCLUSION: Hard core-interaction; at most 1 particle per site.
- ASYMMETRIC: External driving; breaks detailed-balance
- PROCESS: Stochastic Markovian dynamics; no Hamiltonian.

The probability $P_t(\mathcal{C})$ to find the system in the microscopic configuration $\mathcal{C}$ at time t satisfies

$$\frac{dP_t(\mathcal{C})}{dt} = MP_t(\mathcal{C})$$

where the Markov Matrix M encodes the transitions rates amongst configurations.

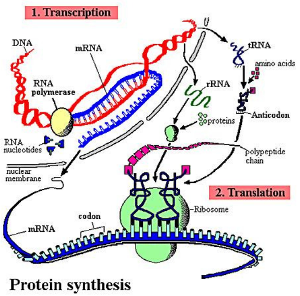
ORIGINS

- Interacting Brownian Processes (Spitzer, Harris, Liggett).
- Driven diffusive systems (Katz, Lebowitz and Spohn).
- Transport of Macromolecules through thin vessels.
  Motion of RNA templates.
- Hopping conductivity in solid electrolytes.
- Directed Polymers in random media. Reptation models.
- Interface dynamics. KPZ equation

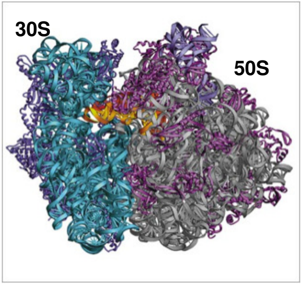
APPLICATIONS

- Traffic flow.
- Sequence matching.
- Brownian motors.

The central dogma of molecular biology



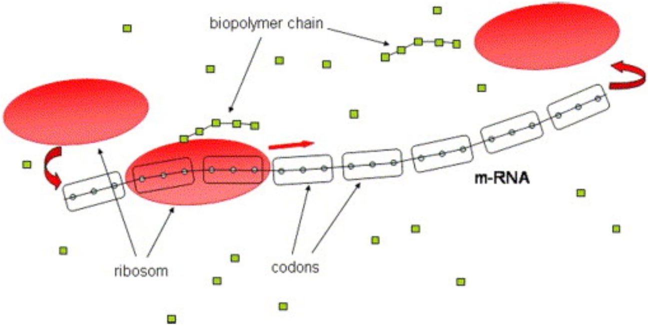
(a)



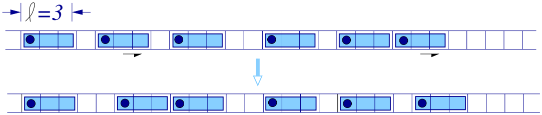
(b)

(c)

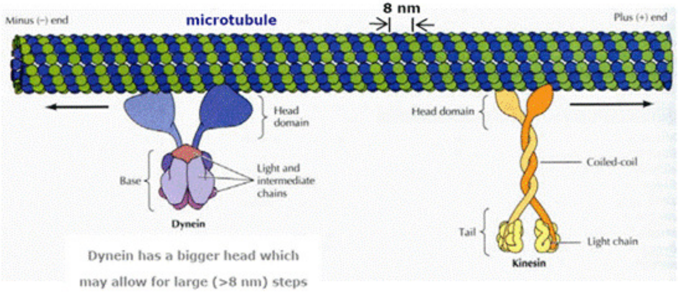
An Elementary Model for Protein Synthesis

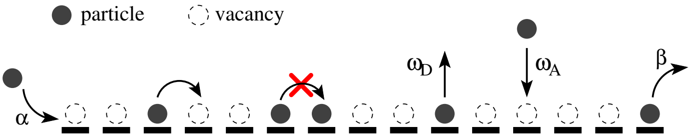


C. T. MacDonald, J. H. Gibbs and A.C. Pipkin, Kinetics of biopolymerization on nucleic acid templates, *Biopolymers* (1968).



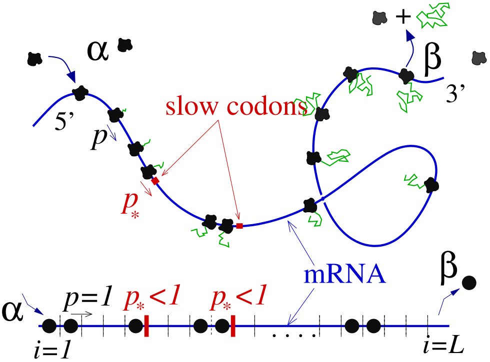
Molecular Motors and Langmuir dynamics





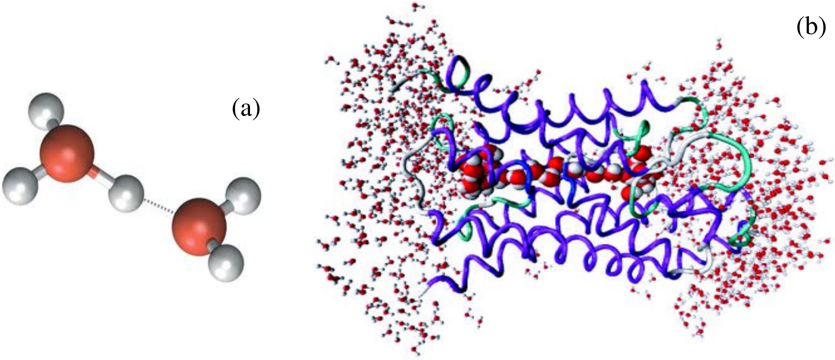
See the works of E. Frey, A. Parmeggiani and their collaborators.

Localized defects



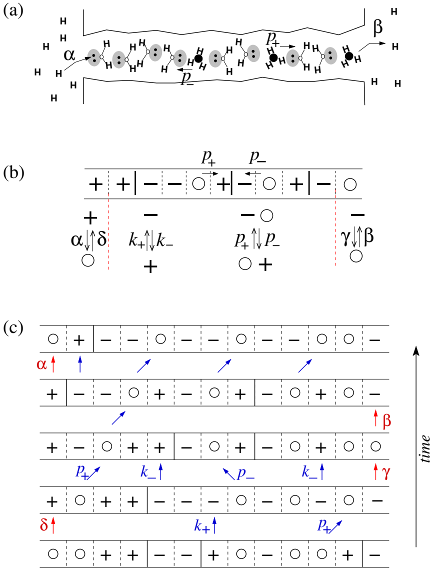
See the discussion of Lebowitz-Janowsky model.

The Grotthuss Mechanism for proton transfer



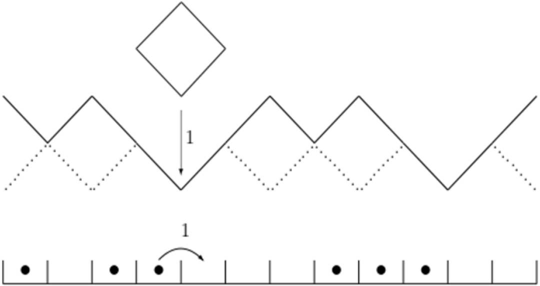
A proton hops along an oxygen backbone of a line of water molecules transiently converting each water molecule it visits into $H_3\,O^+$.

The Grotthuss Mechanism as a 3-species ASEP



A thorough study by Tom Chou and collaborators.

The Kardar-Parisi-Zhang equation in 1d

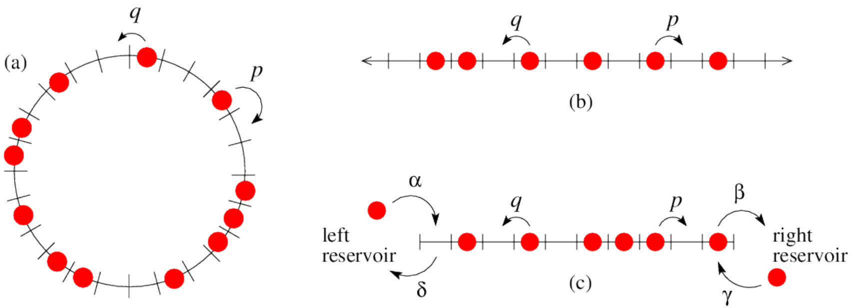


The height of an interface $h(x, t)$ satisfies the generic KPZ equation

$$\frac{\partial h}{\partial t} = \nu \frac{\partial^2 h}{\partial x^2} + \frac{\lambda}{2} \left(\frac{\partial h}{\partial x} \right)^2 + \xi(x, t)$$

The ASEP is a discrete version of the KPZ equation in one-dimension.

Various Boundary Conditions for the ASEP



The pure ASEP can be studied on a periodic chain (a), on the infinite lattice (b) or on a finite lattice connected to two reservoirs (c).

Steady state properties of ASEP

Anomalous diffusion in SEP

Consider the Symmetric Exclusion Process on an infinite one-dimensional line with a finite density ρ of particles.

Suppose that we tag and observe a particle that was initially located at site 0 and monitor its position X_t with time.

On the average $\langle X_t \rangle = 0$ but how large are its fluctuations?

- If the particles were non-interacting (no exclusion constraint), each particle would diffuse normally $\langle X_t^2 \rangle = Dt$.

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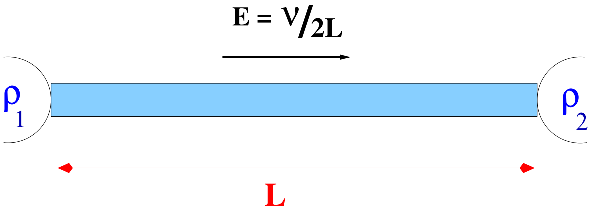
- If the particles were non-interacting (no exclusion constraint), each particle would diffuse normally $\langle X_t^2 \rangle = Dt$.
- Because of the exclusion condition, a particle displays an anomalous diffusive behaviour:

$$\langle X_t^2 \rangle = 2\frac{1-\rho}{\rho}\sqrt{\frac{Dt}{\pi}}$$

T.E. Harris, *J. Appl. Prob.* (1965).
F. Spitzer, *Adv. Math.* (1970).

The Hydrodynamic Limit: Diffusive case



Starting from the microscopic level, define local density $\rho(x,t)$ and current $j(x,t)$ with macroscopic space-time variables $x = i/L, t = s/L^2$ (diffusive scaling) and with weak asymmetry $p - q = \nu/L$.
The typical evolution of the system is given by the hydrodynamic behaviour:

$$\partial_t \rho = \frac{1}{2}\nabla^2 \rho - \nu \nabla \sigma(\rho) \quad \text{with} \quad \sigma(\rho) = \rho(1-\rho)$$

(Lebowitz, Spohn, Varadhan)

This is a Burgers type equation.

Physicist's derivation of the continuous limit

We define the binary variable $\tau_i = 0, 1$ if site i is empty or occupied. The average value $\langle \tau_i(t) \rangle$ satisfies the following equation:

$$\frac{d\langle \tau_i \rangle}{dt} = p[\langle \tau_{i-1}(1-\tau_i)\rangle - \langle \tau_i(1-\tau_{i+1})\rangle] + q[\langle \tau_{i+1}(1-\tau_i)\rangle - \langle \tau_i(1-\tau_{i-1})\rangle]$$

$$= p\langle \tau_{i-1}\rangle + q\langle \tau_{i+1}\rangle - (p+q)\langle \tau_i\rangle + (p-q)\langle \tau_i(\tau_{i+1} - \tau_{i-1})\rangle$$

For $p \neq q$: 1-point averages couple to 2-points averages etc... A hierarchy of differential equations is generated (*cf* BBGKY).

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- Take $L \to \infty$ and define the continuous space variable $x = \frac{i}{L}$.
- Define a smooth local density by $\langle \tau_i(t) \rangle = \rho(x, t)$.
- Rescale Asymmetry rates: $p = \frac{1+\nu}{2L}$ and $q = \frac{1-\nu}{2L}$
- Mean-field assumption: write the 2-points averages as products of 1-point averages.

Shocks at the microscopic scale

Applying this procedure to the previous equation leads to, after a diffusive rescaling of time $t \to t/L^2$:

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2} - \nu \frac{\partial \rho(1-\rho)}{\partial x}$$

This is known as the Burgers equation with viscosity.

Had we kept a finite asymmetry: $p - q = \mathcal{O}(1)$, the same procedure (with ballistic time-rescaling) leads to the inviscid limit of Burgers equation:

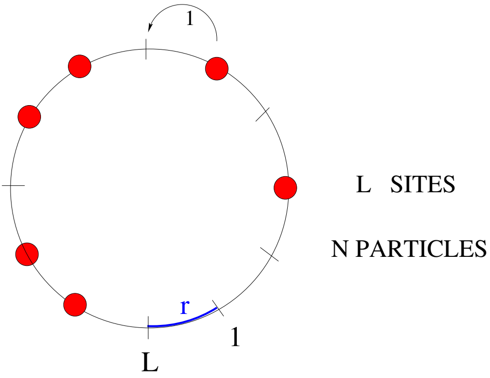
$$\frac{\partial \rho}{\partial t} = \frac{\mathbf{1}}{\mathbf{2L}}\frac{\partial^2 \rho}{\partial x^2} - \nu \frac{\partial \rho(1-\rho)}{\partial x}$$

This equation is well-known to generate shocks.

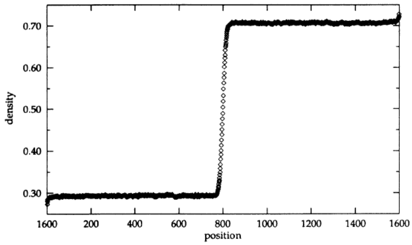
Are these shocks an artifact of the hydrodynamic limit or do they genuinely exist at the microscopic level?

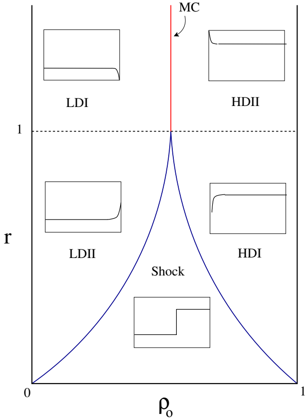
The Lebowitz-Janowsky model

The TASEP on a ring with an inhomogeneous bond with jump rate $r < 1$.



Phase diagram of the Lebowitz-Janowsky model





No exact solution of the Lebowitz-Janowsky model is available.
However, the physics of the system can be understood by a Mean-Field analysis that compares reasonably well with numerical simulations.

Mean-Field analysis of the blockage model

Through a 'normal' bond $(i, i+1)$ the current is $J_{i,i+1} = \langle \tau_i(1-\tau_{i+1}) \rangle$.
In the stationary state, this current is uniform $J_{i,i+1} = J$.

Far from the blockage and from the shock region,the density uniform (cf simulations). Thus, using Mean-Field assumption we have

$$J = \rho_{low}(1-\rho_{low}) = \rho_{high}(1-\rho_{high})$$

Two possible solutions:

- Uniform density everywhere: $\rho_{low} = \rho_{high} = \rho_0$
- Shock: $\rho_{low} = 1 - \rho_{high}$

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To find the values of the density plateaux, we apply the same analysis right at the defective bond:

$$r\rho_L(1 - \rho_1) = r\rho_{high}(1 - \rho_{low}) = J$$

Comparing the equations, we obtain

$$\rho_{low} = \frac{r}{1+r} \qquad \rho_{high} = \frac{1}{1+r} \qquad \text{and} \qquad J = \frac{r}{(1+r)^2}$$

Condition for the existence of the shock

We use the conservation of the number of particles. If we call $1 \leq S \leq L$ the position of the shock, we have $N = S\rho_{low} + (L - S)\rho_{high}$ i.e.,

$$\rho_0 = \frac{s\,r + (1-s)}{r+1} \qquad \text{with} \qquad 0 \leq \; s = \frac{S}{L} \; \leq 1$$

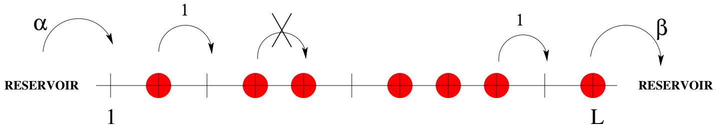
This defines the phase boundary between the uniform and the shock phases

$$\left|\rho_0 - \frac{1}{2}\right| \leq \frac{1-r}{2(r+1)}$$

- A shock will always appear for $\rho_0 = 1/2$ as soon as $r < 1$.
- We do not know if these results are exact.
- Using an improved mean field analysis, the form of the shock can be calculated. The results are not identical to simulations.

Another example of Mean-Field calculations

The mean field analysis can be applied to the TASEP on a finite lattice with open boundaries.



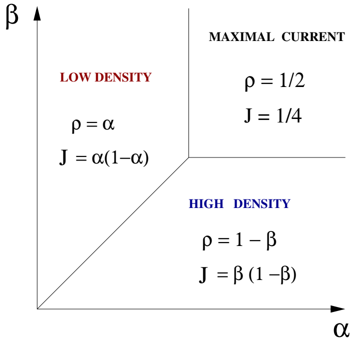
The uniform current is given by $J = \alpha(1 - \langle\tau_1\rangle) = \langle\tau_i(1 - \tau_{i+1})\rangle = \beta\langle\tau_L\rangle$
Through mean-field this leads to the harmonic recursion

$$\rho_{i+1} = 1 - \frac{J}{\rho_i}$$

with boundary conditions $\rho_1 = 1 - \frac{J}{\alpha}$ and $\rho_L = \frac{J}{\beta}$.

The TASEP Phase diagram

A precise analysis of the mean-field equations can be carried out in the $L \to \infty$ limit (Derrida, Domany, Mukamel 1992).



This phase diagram is the correct one. However, predicted density profiles and correlations are not obtained corrected by the mean-field approximation.

ASEP: a Markov Process

Any exact study requires to analyze the Master Equation:

$$\frac{\mathbf{dP_t}}{\mathbf{dt}} = \mathbf{M.P_t}$$

Non-diagonal entries of M are positive and $M(\mathcal{C},\mathcal{C}) = -\sum_{\mathcal{C}'\neq\mathcal{C}} M(\mathcal{C},\mathcal{C}')$
$\rightarrow$ the sums of the terms in each vertical column of M vanish:

$$(\mathbf{1},\mathbf{1},\ldots,\mathbf{1})\mathbf{M} = \mathbf{0}$$

- Complex Eigenvalues: $M\psi = E\psi$ with $\Re(E) \leq 0$ (Perron-Frobenius)
- **Ground State** $E = 0$ corresponds to the **stationary state** *(unique)*.
- Excited States $\rightarrow$ relaxation times.

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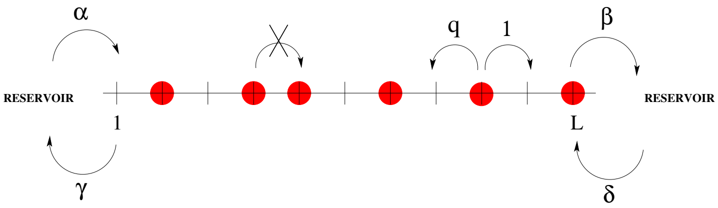
- Transport properties; statistics of the total current?

The Matrix Ansatz (DEHP,1993)



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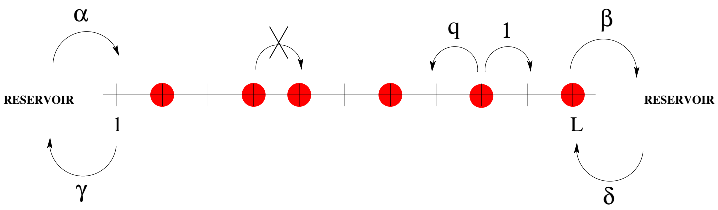

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where $\tau_i = 1$ (or 0) if the site i is occupied (or empty).
The normalization constant $Z_L = \langle W| (D+E)^L |V\rangle$.

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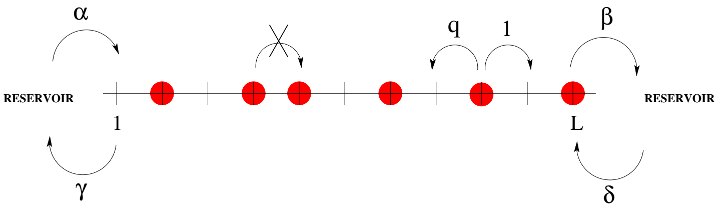

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The operators D and E, the vectors $\langle W|$ and $|V\rangle$ satisfy

$$D\,E - qE\,D = (1-q)(D + E)$$

$$(\beta\, D - \delta\, E)|V\rangle = |V\rangle \quad \text{and} \quad \langle W|(\alpha\, E - \gamma\, D) = \langle W|$$

Representations of the quadratic algebra

The algebra encodes combinatorial recursion relations between systems of different sizes.

Generically, the representations are infinite dimensional (q-deformed oscillators).

Infinite dimensional Representation:

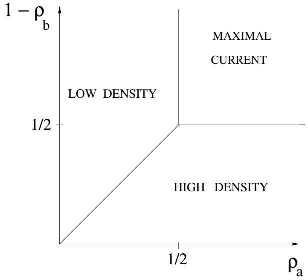
$D = 1 + d$ where $d = q$-deformed right-shift.

$E = 1 + e$ where $e = q$-deformed left-shift.

$$D = \begin{pmatrix} 1 & \sqrt{1-q} & 0 & 0 & \dots \\ 0 & 1 & \sqrt{1-q^2} & 0 & \dots \\ 0 & 0 & 1 & \sqrt{1-q^3} & \dots \\ & & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad E = D^\dagger$$

The matrix Ansatz allows one to calculate Stationary State Properties (currents, correlations, fluctuations) and to derive the Phase Diagram in the infinite size limit (DEHP,1993).

The Phase Diagram of the open ASEP



$\rho_a = \frac{1}{a_+ + 1}$: effective left reservoir density.

$\rho_b = \frac{b_+}{b_+ + 1}$: effective right reservoir density.

$$a_\pm = \frac{(1 - q - \alpha + \gamma) \pm \sqrt{(1 - q - \alpha + \gamma)^2 + 4\alpha\gamma}}{2\alpha}$$

$$b_\pm = \frac{(1 - q - \beta + \delta) \pm \sqrt{(1 - q - \beta + \delta)^2 + 4\beta\delta}}{2\beta}$$

The TASEP algebra

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$$\begin{aligned} D\,E &= D + E \\ D\,|\beta\rangle &= \frac{1}{\beta}\,|\beta\rangle \\ \langle\alpha|\,E &= \frac{1}{\alpha}\langle\alpha| \end{aligned}$$

Average Stationary Current:

$$J = \langle \tau_i(1-\tau_{i+1})\rangle = \frac{\langle\alpha|C^{i-1}\,D\,E\,C^{L-i-1}|\beta\rangle}{\langle\alpha|C^L|\beta\rangle} = \frac{\langle\alpha|C^{L-1}|\beta\rangle}{\langle\alpha|C^L|\beta\rangle} = \frac{Z_{L-1}}{Z_L}$$

Equal-time Steady State Correlations

More generally, the Matrix Ansatz gives access to all equal time correlations in the steady-state.

Density Profile:

$$\rho_i = \langle \tau_i \rangle = \frac{\langle \alpha | C^{i-1}\, D\, C^{L-i} | \beta \rangle}{\langle \alpha | C^L | \beta \rangle}$$

Multi-body correlations:

$$\langle \tau_{i_1} \tau_{i_2} \ldots \tau_{i_k} \rangle = \frac{\langle \alpha | C^{i_1-1}\, D\, C^{i_2-i_1-1}\, D\, \ldots D\, C^{L-i_k} | \beta \rangle}{\langle \alpha | C^L | \beta \rangle}$$

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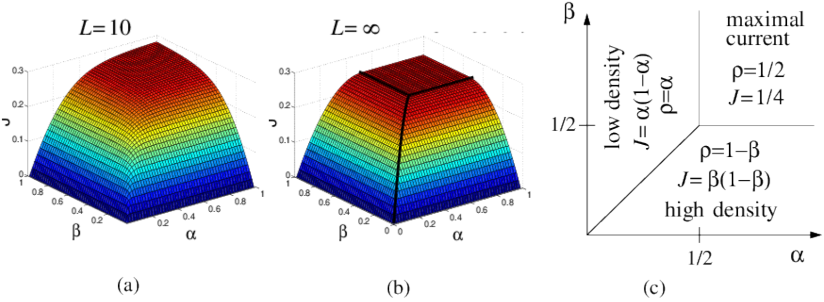
The expressions look formal but it is possible to derive explicit formulae: either by using purely combinatorial/algebraic techniques or via a specific representation (e.g., C can be chosen as a discrete Laplacian).

$$\langle \alpha | C^L | \beta \rangle = \sum_{p=1}^{L} \frac{p\,(2L-1-p)!}{L!\,(L-p)!} \frac{\beta^{-p-1} - \alpha^{-p-1}}{\beta^{-1} - \alpha^{-1}}$$

A very large body of knowledge has been developed around this Matrix Ansatz: see the review of R. Blythe and M. R. Evans.

Time-dependent Properties

The Matrix Ansatz allows us to calculate steady state properties in particular equal-time correlations, as for example the average current through the system in the long time limit.



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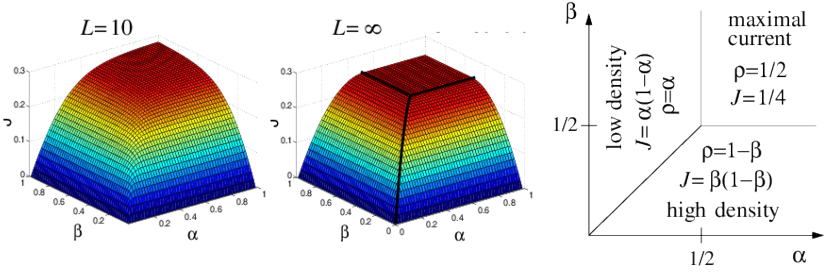


(a) (b) (c)

How do we access to time-dependent properties?

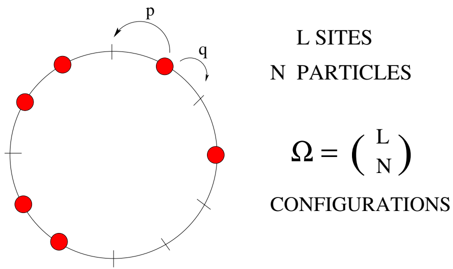
- How does the system relax to its stationary state?
- What do the fluctuations of the current look like? What about its probability distribution?

BETHE ANSATZ for ASEP:

A crash-course

The Periodic ASEP

We consider the asymmetric exclusion process on a homogeneous ring: jumps in the positive (trigonometric) direction occur with rate p, jumps in the negative direction occur with rate q.



By rescaling time we can always make $p \to 1$ and $q \to x = \frac{q}{p}$. We shall perform this rescaling at the end of our calculations.

The Eigenvalue Problem for the Markov Matrix

A configuration of the system at time t can be specified by the position of the N particles on the ring of size L:

$$1 \leq x_1 < \ldots < x_N \leq L \, .$$

With this representation, the eigenvalue equation becomes:

$$\begin{aligned} & E\psi(x_1, \ldots, x_N) = \\ & p \sum_i{}' \left[\psi(x_1, \ldots, x_i - 1, \ldots, x_N) - \psi(x_1, \ldots, x_i, \ldots x_N)\right] \\ + \quad & q \sum_i{}' \left[\psi(x_1, \ldots, x_i + 1, \ldots, x_N) - \psi(x_1, \ldots, x_i, \ldots x_N)\right] \end{aligned}$$

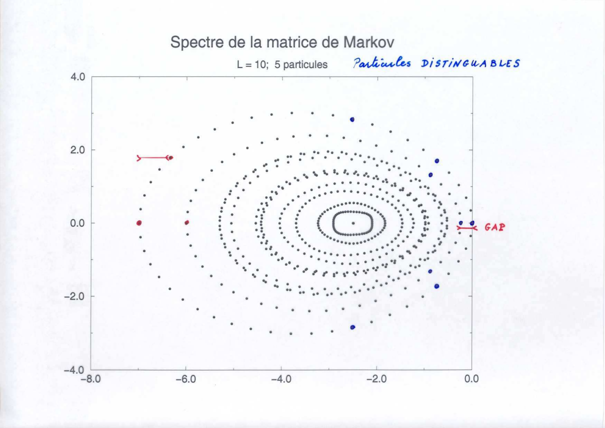
where the sum are restricted over the indices i such that $x_{i-1} < x_i - 1$ and over the indices j such that $x_j + 1 < x_{j+1}$: These conditions ensure that the corresponding jumps are allowed.

This equation looks like a discrete Laplacian but with special boundary conditions.

Spectrum

Complex Eigenvalues $M\psi = E\psi$ with $\Re(E) \leq 0$ (Perron-Frobenius)

- Ground State $E = 0$ corresponds to the stationary state.
- Excited States $\rightarrow$ relaxation times.



ASEP: An Integrable Spin Chain

MAPPING TO A NON-HERMITIAN SPIN CHAIN

$$M = \sum_{l=1}^{L} \left(q\mathbf{S}_l^+\mathbf{S}_{l+1}^- + p\mathbf{S}_l^-\mathbf{S}_{l+1}^+ + \frac{p+q}{4}\mathbf{S}_l^z\mathbf{S}_{l+1}^z - \frac{p+q}{4} \right)$$

Complex Eigenvalues $M\psi = E\psi$:

- Ground State : $E = 0$, $P = \Omega^{-1}$ (non-degenerate).
- Excited States : $\Re(E) < 0$ (Perron-Frobenius).

Excitations correspond to relaxation times.

TASEP : $p = 1, q = 0$

The single particle case

For $N = 1$, the eigenvalue equation reads

$$E\psi(x) = p\psi(x-1) + q\psi(x+1) - (p+q)\psi(x)\,,$$

with $1 \leq x \leq L$ and where periodicity is assumed: $\psi(x+L) = \psi(x)\,.$
This is a linear recursion of order 2. Thus

$$\psi(x) = Az_+^x + Bz_-^x\,,$$

where $r = z_\pm$ are the two roots of the characteristic equation

$$qr^2 - (E+p+q)r + p = 0\,.$$

Because of the periodicity condition at least one of the two characteristic values is a L-th root of unity (Since $z_+z_- = p/q$, only one of them can be a root of unity when $p \neq q$).

The general solution is

$$\psi(x) = Az^x \qquad \text{with} \qquad z^L = 1$$

This is a *plane wave* with momentum $2k\pi/L$ and with eigenvalue

$$E = \frac{p}{z} + qz - (p+q)$$

The two particles case

When $N = 2$, the exclusion condition begins to play a role and the general eigenvalue equation has to be be split into two different cases.

- Generic case: x_1 and x_2 are separated by at least one empty site

$$
\begin{aligned}
E\psi(x_1, x_2) &= p\left[\psi(x_1 - 1, x_2) + \psi(x_1, x_2 - 1)\right] \\
&+ q\left[\psi(x_1 + 1, x_2) + \psi(x_1, x_2 + 1)\right] \\
&- 2(p + q)\psi(x_1, x_2)
\end{aligned}
$$

- Adjacency case: Here $x_2 = x_1 + 1$, some jumps are forbidden and the eigenvalue equation becomes

$$E\psi(x_1, x_1 + 1) = p\psi(x_1 - 1, x_1 + 1) + q\psi(x_1, x_1 + 2) - (p + q)\psi(x_1, x_1 + 1)$$

This equation differs from the generic equation for $x_2 = x_1 + 1$: There are missing terms. Equivalently, one can impose the generic equation everywhere and add the *cancellation boundary condition:*

$$p\psi(x_1, x_1) + q\psi(x_1 + 1, x_1 + 1) - (p + q)\psi(x_1, x_1 + 1) = 0$$

Bethe Wave Function for N=2

In the generic case, particles jump totally independently: the solution of the generic equation can be written as as a product of plane waves

$$\psi(x_1, x_2) = A z_1^{x_1} z_2^{x_2}$$

with the eigenvalue

$$E = p\left(\frac{1}{z_1} + \frac{1}{z_2}\right) + q\,(z_1 + z_2) - 2(p+q)$$

However, the cancellation condition will not be satisfied in general.

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However, the cancellation condition will not be satisfied in general. • **Crucial Observation:** The eigenvalue E is invariant by the permutation $z_1 \leftrightarrow z_2$: there are **two** plane waves $A z_1^{x_1} z_2^{x_2}$ and $B z_2^{x_1} z_1^{x_2}$ with the **same** eigenvalue E.

One should try a linear combination of plane-waves of the form:

$$\psi(x_1, x_2) = A_{12} z_1^{x_1} z_2^{x_2} + A_{21} z_2^{x_1} z_1^{x_2}$$

where the amplitudes A_{12} and A_{21} are yet arbitrary but can be chosen to fulfill the adjacency cancellation condition: **Bethe Ansatz** (Bethe, 1931)

Calculation of the Amplitudes Ratio

The adjacency cancellation condition will be fulfilled if the amplitudes satisfy

$$(p + qz_1z_2)\,(A_{12} + A_{21}) = (p + q)(A_{12}z_2 + A_{21}z_1)$$

Equivalently

$$\frac{A_{21}}{A_{12}} = -\frac{qz_1z_2 - (p+q)z_2 + p}{qz_1z_2 - (p+q)z_1 + p}$$

The eigen-equation is now satisfied in all the cases.

We must now impose the boundary conditions (here periodicity): this will **quantify** the Bethe roots z_1 and z_2.

Periodicity condition. The Bethe Equations

We now implement the periodicity condition that takes into account the fact that the system is defined on a ring. This constraint can be written as follows for $1 \leq x_1 < x_2 \leq L$:

$$\psi(x_1, x_2) = \psi(x_2, x_1 + L)$$

i.e., $$A_{12} z_1^{x_1} z_2^{x_2} + A_{21} z_2^{x_1} z_1^{x_2} = A_{12} z_1^{x_2} {z_2}^{x_1+L} + A_{21} z_2^{x_2} {z_1}^{x_1+L}$$

This leads to a new relation between the amplitudes:

$$\frac{A_{21}}{A_{12}} = z_2^L = \frac{1}{z_1^L}$$

Using the known value of the amplitudes-ratio, we deduce

$$z_1^L = -\frac{q z_1 z_2 - (p+q) z_1 + p}{q z_1 z_2 - (p+q) z_2 + p}$$

$$z_2^L = -\frac{q z_1 z_2 - (p+q) z_2 + p}{q z_1 z_2 - (p+q) z_1 + p}$$

These are the Bethe Ansatz Equations for $N = 2$.

N=3 (and larger)

For a system containing three particles, located at $x_1 \leq x_2 \leq x_3$, the generic equation can be written from as above. But now, the special adjacency cases are more complicated.

• **Two-Body collisions:** *Two particles are next to each other and the third one is 'far apart'.* This reduces to $N = 2$ (with a spectator). There are now two equations that correspond to the two cases $x_1 = x \leq x_2 = x + 1 \ll x_3$ and $x_1 \ll x_2 = x \leq x_3 = x + 1$:

$$p\psi(x, x, x_3) + q\psi(x + 1, x + 1, x_3) - (p + q)\psi(x, x + 1, x_3) = 0$$
$$p\psi(x_1, x, x) + q\psi(x_1, x + 1, x + 1) - (p + q)\psi(x_1, x, x + 1) = 0$$

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$$p\psi(x_1, x, x) + q\psi(x_1, x + 1, x + 1) - (p + q)\psi(x_1, x, x + 1) = 0$$

• **Triple collision:** the three particles are adjacent, with $x_1 = x$, $x_2 = x + 1$ and $x_3 = x + 2$. The cancellation condition becomes

$$\begin{aligned} p \quad & [\psi(x, x, x + 2) + \psi(x, x + 1, x + 1)] + \\ q \quad & [\psi(x + 1, x + 1, x + 2) + \psi(x, x + 2, x + 2)] \\ - \quad & (p + q)\psi(x, x + 1, x + 2) - (p + q)\psi(x, x + 1, x + 2) = 0 \end{aligned}$$

Not a new constraint, just a linear combination of the Two-Body collisions.

Bethe Ansatz for N=3

The fact that 3-body collisions 'factorise' into 2-body collisions is the *crucial property* at the very heart of the Bethe Ansatz.

The plane wave $\psi(x_1, x_2, x_3) = A z_1^{x_1} z_2^{x_2} z_3^{x_3}$ is a solution of the generic equation with the eigenvalue

$$E = p\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right) + q\,(z_1 + z_2 + z_3) - 3(p+q)$$

However, collision conditions are not satisfied.

Note that E is invariant (degenerate) by permuting z_1, z_2, z_3.

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- **TRY the Bethe Wave function:**

$$\begin{aligned}\psi(x_1, x_2, x_3) &= A_{123}\, z_1^{x_1} z_2^{x_2} z_3^{x_3} + A_{132}\, z_1^{x_1} z_3^{x_2} z_2^{x_3} + A_{213}\, z_2^{x_1} z_1^{x_2} z_3^{x_3} \\ &\quad + A_{231}\, z_2^{x_1} z_3^{x_2} z_1^{x_3} + A_{312}\, z_3^{x_1} z_1^{x_2} z_2^{x_3} + A_{321}\, z_3^{x_1} z_2^{x_2} z_1^{x_3}\end{aligned}$$

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- Fix all amplitude ratios by the 2-collision conditions.
- Quantize the Bethe roots z_1, z_2 and z_3 via the periodicity condition

$$\psi(x_1, x_2, x_3) = \psi(x_2, x_3, x_1 + L)$$

(This yields the Bethe equations).

The general N case

For general values of N, one can have k-body collisions with k=2,3...N. However, all multi-body collisions 'factorize' into 2-body collisions. *ASEP can be diagonalized by Bethe Ansatz.*

- **Bethe Wave function:**

$$\psi(x_1, x_2, \ldots, x_N) = \sum_{\sigma \in S_N} A_\sigma \, z_{\sigma(1)}^{x_1} z_{\sigma(2)}^{x_2} \cdots z_{\sigma(N)}^{x_N}$$

- Eigenvalue: $E = p \sum_{i=1}^N \frac{1}{z_i} + q \sum_{i=1}^N z_i - N(p+q)$

- Periodicity Condition (for $1 \leq x_1 < x_2 < \ldots < x_N \leq L$):

$$\psi(x_1, x_2, \ldots, x_N) = \psi(x_2, x_3, \ldots, x_N, x_1 + L)$$

The Bethe Ansatz Equations

$$z_i^L = (-1)^{N-1} \prod_{j \neq i} \frac{q z_i z_j - (p+q) z_i + p}{q z_i z_j - (p+q) z_j + p}$$

for $i = 1, \ldots N$.

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- The Bethe equations are a system of N algebraic equations of order L whereas the characteristic polynomial of the Markov Matrix is of order 2^L.

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$$\psi(x_1 + 1, x_2 + 1, \ldots, x_N + 1) = (z_1 \ldots z_N)\ \psi(x_1, x_2, \ldots, x_N)$$

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• In the symmetric case ($p = q = 1$), the Bethe equations are identical to those derived by H. Bethe for the Heisenberg XXX chain, in 1931.

• For the TASEP case ($p = 1$ and $q = 0$), the wave function has the structure of a determinant:

$$\psi(x_1, \ldots, x_N) = \det\left(\frac{z_i^{x_j}}{(1 - z_i)^j}\right)$$

By expanding this determinant the generic form for the Bethe wave function is recovered. *It can also be shown directly that this determinant satisfies the eigenvalue equation and all the collision conditions.*

Bethe Equations for TASEP

For TASEP, the Bethe equations take a simpler form.
Making the change of variable $\zeta_i = \frac{2}{z_i} - 1$, these equations become

$$\mathbf{(1-\zeta_i)^N(1+\zeta_i)^{L-N} = -2^L \prod_{j=1}^{N} \frac{\zeta_j - 1}{\zeta_j + 1}} \quad \text{for} \quad i = 1, \dots N$$

Note that the r.h.s. is a constant independent of i: There is an effective DECOUPLING.

The corresponding eigenvalue is

$$\mathbf{E = \frac{1}{2}(-N + \sum_j \zeta_j)}$$

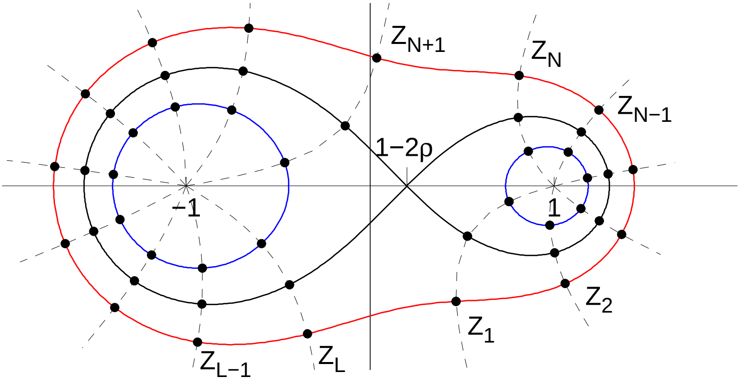
For a fixed value of the r.h.s. the roots lie on curves that satisfy

$$|1-\zeta|^{\rho}\, |1+\zeta|^{1-\rho} = const$$

where $\rho = N/L$ is the density.

Labelling the roots of the TASEP Bethe Equations

The loci of the roots (for $q = 0$) are remarkable curves: **The Cassini Ovals**



Procedure for solving the TASEP Bethe Equations

- For any given value of Y, *SOLVE* $(1 - z_i)^N (1 + z_i)^{L-N} = Y$. The roots are located on Cassini Ovals
- *CHOOSE* N *roots* $z_{c(1)}, \dots z_{c(N)}$ amongst the L available roots, with a *choice set* $c : \{c(1), \dots, c(N)\} \subset \{1, \dots, L\}$.
- *SOLVE* the self-consistent equation $\mathbf{A_c}(\mathbf{Y}) = \mathbf{Y}$ where

$$A_c(Y) = -2^L \prod_{j=1}^{N} \frac{z_{c(j)} - 1}{z_{c(j)} + 1} .$$

- *DEDUCE* from the value of Y, the $z_{c(j)}$'s and the energy corresponding to the choice set c :

$$2E_c(Y) = -N + \sum_{j=1}^{N} z_{c(j)} .$$

Calculation of the GAP

The first excited state is solution of a transcendental equation. For a density ρ:

$$E_1 = \underbrace{-2\sqrt{\rho(1-\rho)}\frac{6.509189337\ldots}{L^{3/2}}}_{\text{RELAXATION}} \pm \underbrace{\frac{2i\pi(2\rho-1)}{L}}_{\text{OSCILLATIONS}} .$$

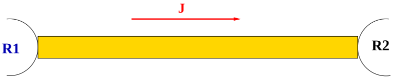
• Non-diffusive: Largest relaxation time $T \sim L^z$ with $z = 3/2$ *(D. Dhar, L.H. Gwa and H. Spohn, D. Kim)*.

• Oscillations $\rightarrow$ Traveling waves probed by dynamical correlations *(M. Barma, S. Majumdar, P. Krapivsky)*.

• Classification of higher excitations *(J. de Gier and F.H.L. Essler, 2006)*.

Application to Current Fluctuations

Large Deviations of the Total Current



Let Y_t be the total charge transported through the system (total current) between time 0 and time t.

In the stationary state: a non-vanishing mean-current $\frac{Y_t}{t} \to J$

The fluctuations of Y_t obey a Large Deviation Principle:

$$P\left(\frac{Y_t}{t} = j\right) \sim e^{-t\Phi(j)}$$

$\Phi(j)$ being the *large deviation function* of the total current.

Equivalently, we can consider the moment-generating function

$$\langle e^{\mu Y_t} \rangle \simeq e^{E(\mu)t} \qquad \text{when} \quad t \to \infty$$

Related by Legendre transform: $E(\mu) = \max_j (\mu j - \Phi(j))$

The Periodic ASEP Case

Large Deviations of the Current

Total current Y_t**,** *total distance covered by all the* N *particles, hopping on a ring of size* L*, between time 0 and time* t*.*

WHAT IS THE STATISTICS of Y_t**?**

Let $P_t(\mathcal{C}, Y)$ be the joint probability of being at time t in configuration $\mathcal{C}$ with $Y_t = Y$. The time evolution of this joint probability can be deduced from the original Markov equation, by splitting the Markov operator

$$M = M_0 + M_+ + M_-$$

into transitions for which $\Delta Y = 0$, $+1$ or -1.

$$\frac{dP_t(\mathcal{C}, Y)}{dt} = \sum_{\mathcal{C}'} M_0(\mathcal{C}, \mathcal{C}')P_t(\mathcal{C}', Y) + \sum_{\mathcal{C}'} M_+(\mathcal{C}, \mathcal{C}')P_t(\mathcal{C}', Y-1) + \sum_{\mathcal{C}'} M_-(\mathcal{C}, \mathcal{C}')P_t(\mathcal{C}', Y+1)$$

The Laplace transform of $P_t(\mathcal{C}, Y)$ with respect to Y, defined as

$$\hat{P}_t(\mathcal{C}, \mu) = \sum_Y \mathrm{e}^{\mu Y} P_t(\mathcal{C}, Y),$$

satisfies a dynamical equation governed by the deformation of the Markov Matrix M, obtained by adding a jump-counting *fugacity* μ:

$$\frac{d\hat{P}_t}{dt} = M(\mu)\hat{P}_t$$

with

$$M(\mu) = M_0 + \mathrm{e}^{\mu} M_+ + \mathrm{e}^{-\mu} M_-$$

The Matrix $M(\mu)$ is not a Markov Matrix in general (it does not conserve probability). But it is a matrix with positive off-diagonal entries and the Perron-Frobenius Theorem can still be applied: $M(\mu)$ has a unique dominant eigenvalue, denoted by $E(\mu)$, with eigenvector $F_\mu(\mathcal{C})$

$$M(\mu).F_\mu = E(\mu)F_\mu$$

When $t \to \infty$, we have

$$\hat{P}_t(\mathcal{C}, \mu) \sim \mathrm{e}^{E(\mu)t} F_\mu(\mathcal{C})$$

Cumulant generating function

From the previous result, one deduces that when $t \to \infty$:

$$\boxed{\left\langle \mathrm{e}^{\mu Y_t} \right\rangle \simeq \mathrm{e}^{E(\mu) t}}$$

The cumulant generating function $E(\mu)$ is the eigenvalue with maximal real part of the deformed operator $M(\mu)$

$$M(\mu) = M_0 + \mathrm{e}^{\mu} M_+ + \mathrm{e}^{-\mu} M_-$$

corresponding to splitting the Markov operator $M = M_0 + M_+ + M_-$ according to the increments of the total current.

The large deviation function $\Phi(j)$ of the current is defined as

$$P\left(\frac{Y_t}{t} = j\right) \sim e^{-t\Phi(j)}$$

Legendre transform

The large deviation function $\Phi(j)$ is related to the cumulant generating function $E(\mu)$ by a Legendre transform:

$$\boxed{E(\mu) = \max_j \left(\mu j - \Phi(j)\right)}$$

Indeed,

$$\langle \mathrm{e}^{\mu Y_t} \rangle = \int \mathrm{e}^{\mu Y_t} P(Y_t)\, dY_t = t \int \mathrm{e}^{\mu\, t\, j} P\left(\frac{Y_t}{t} = j\right)\, dj$$

Keep the dominant exponential behaviour in the long time limit

$$\mathrm{e}^{E(\mu)t} \simeq \int \mathrm{e}^{t[\mu j - \Phi(j)]}\, dj$$

Conclude by saddle-point method.

Bethe Ansatz for current statistics

The current statistics is reduced to an eigenvalue problem, solvable by Bethe Ansatz.

The Bethe Equations are given by

$$z_i^L = (-1)^{N-1} \prod_{j=1}^{N} \frac{x\mathrm{e}^{-\mu} z_i z_j - (1+x) z_i + \mathrm{e}^{\mu}}{x\mathrm{e}^{-\mu} z_i z_j - (1+x) z_j + \mathrm{e}^{\mu}}$$

The eigenvalues of $M(\mu)$ are

$$E(\mu; z_1, z_2 \ldots z_N) = \mathrm{e}^{\mu} \sum_{i=1}^{N} \frac{1}{z_i} + x\mathrm{e}^{-\mu} \sum_{i=1}^{N} z_i - N(1+x)\,.$$

The Bethe equations do not decouple unless $x = 0$

(This TASEP case was solved by B. Derrida and J. L. Lebowitz, 1998).

TASEP CASE (Derrida Lebowitz 1998)

$E(\mu)$ is calculated by Bethe Ansatz to all orders in μ, thanks to the decoupling property of the Bethe equations.
The structure of the solution is given by a parametric representation of the cumulant generating function $E(\mu)$:

$$\mu = -\frac{1}{L}\sum_{k=1}^{\infty} \frac{[kL]!}{[kN]!\,[k(L-N)]!}\frac{B^k}{k}\ ,$$

$$E = -\sum_{k=1}^{\infty} \frac{[kL-2]!}{[kN-1]!\,[k(L-N)-1]!}\frac{B^k}{k}\ .$$

Mean Total current:

$$J = \lim_{t\to\infty}\frac{\langle Y_t\rangle}{t} = \frac{N(L-N)}{L-1}$$

Diffusion Constant:

$$D = \lim_{t\to\infty}\frac{\langle Y_t^2\rangle - \langle Y_t\rangle^2}{t} = \frac{LN(L-N)}{(L-1)(2L-1)}\frac{C_{2L}^{2N}}{\left(C_L^N\right)^2}$$

Exact formula for the large deviation function.

Functional Bethe Ansatz for the General Case

After a change of variable, $y_i = \frac{1-\mathrm{e}^{-\mu} z_i}{1-x\mathrm{e}^{-\mu} z_i}$, the Bethe equations read

$$\mathrm{e}^{L\mu} \left(\frac{1-y_i}{1-xy_i} \right)^L = -\prod_{j=1}^{N} \frac{y_i - xy_j}{xy_i - y_j} \quad \text{for} \quad i = 1 \ldots N \,.$$

Let T be auxiliary variable playing a symmetric role w.r.t. all the y_i:

$$\mathrm{e}^{L\mu} \left(\frac{1-T}{1-xT} \right)^L = -\prod_{j=1}^{N} \frac{T - xy_j}{xT - y_j} \quad \text{for} \quad i = 1 \ldots N \,.$$

$$i.e.\, P(T) = \mathrm{e}^{L\mu}(1-T)^L \prod_{j=1}^{N}(xT - y_j) + (1-xT)^L \prod_{j=1}^{N}(T - xy_j) = 0.$$

But $P(y_i) = 0$ (Bethe Eqs.). Thus, $Q(T) = \prod_{i=1}^{N}(T - y_i)$ divides $P(T)$:

$$Q(T) \;\; \text{DIVIDES} \;\; e^{L\mu}(1-T)^L Q(xT) + (1-xT)^L x^N Q(T/x).$$

Functional Bethe Ansatz

There exist two polynomials $Q(T)$ and $R(T)$ such that

$$\boxed{Q(T)R(T) = e^{L\mu}(1-T)^L Q(xT) + x^N (1-xT)^L Q(T/x)}$$

where $Q(T)$ of degree N vanishes at the Bethe roots.
Functional Bethe Ansatz (Baxter's TQ equation): Restatement of the Bethe Ansatz as a purely algebraic problem. This equation is solved perturbatively w.r.t. μ.

Knowing $Q(T)$, we obtain an expansion of $E(\mu)$. This provides the full statistics of the current and its large deviations.

Cumulants of the Current

- Mean Current: $J = (1-x)\frac{N(L-N)}{L-1} \sim (1-x)L\rho(1-\rho)$ for $L \to \infty$

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$$D \sim 4\phi L\rho(1-\rho)\int_0^\infty du \frac{u^2}{\tanh \phi u} e^{-u^2}$$

when $L \to \infty$ and $x \to 1$ with *fixed value* of $\phi = \frac{(1-x)\sqrt{L\rho(1-\rho)}}{2}$.

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when $L \to \infty$ and $x \to 1$ with *fixed value* of $\phi = \frac{(1-x)\sqrt{L\rho(1-\rho)}}{2}$.

- Third cumulant (Skewness):

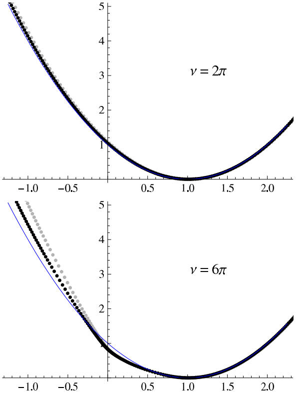
$$\frac{E_3}{\phi(\rho(1-\rho))^{3/2}L^{5/2}} \simeq -\frac{4\pi}{3\sqrt{3}} + 12\int_0^\infty dudv \frac{(u^2+v^2)e^{-u^2-v^2} - (u^2+uv+v^2)e^{-u^2-uv-v^2}}{\tanh \phi u \tanh \phi v}$$

$\to$ Non Gaussian fluctuations. TASEP limit for $\phi \to \infty$:

$$E_3 \simeq \left(\frac{3}{2} - \frac{8}{3\sqrt{3}}\right)\pi(\rho(1-\rho))^2 L^3$$

$$
\begin{aligned}
\frac{E_3}{6L^2} &= \frac{1-x}{L-1}\sum_{i>0}\sum_{j>0}\frac{C_L^{N+i}C_L^{N-i}C_L^{N+j}C_L^{N-j}}{(C_L^N)^4}(i^2+j^2)\frac{1+x^i}{1-x^i}\frac{1+x^j}{1-x^j} \\
&- \frac{1-x}{L-1}\sum_{i>0}\sum_{j>0}\frac{C_L^{N+i}C_L^{N+j}C_L^{N-i-j}}{(C_L^N)^3}\frac{i^2+ij+j^2}{2}\frac{1+x^i}{1-x^i}\frac{1+x^j}{1-x^j} \\
&- \frac{1-x}{L-1}\sum_{i>0}\sum_{j>0}\frac{C_L^{N-i}C_L^{N-j}C_L^{N+i+j}}{(C_L^N)^3}\frac{i^2+ij+j^2}{2}\frac{1+x^i}{1-x^i}\frac{1+x^j}{1-x^j} \\
&- \frac{1-x}{L-1}\sum_{i>0}\frac{C_L^{N+i}C_L^{N-i}}{(C_L^N)^2}\frac{i^2}{2}\left(\frac{1+x^i}{1-x^i}\right)^2 \\
&+ (1-x)\frac{N(L-N)}{4(L-1)(2L-1)}\frac{C_{2L}^{2N}}{(C_L^N)^2} \\
&- (1-x)\frac{N(L-N)}{6(L-1)(3L-1)}\frac{C_{3L}^{3N}}{(C_L^N)^3}
\end{aligned}
$$

Full large deviation function (weak asymmetry)



$$E\left(\frac{\mu}{L}\right) \simeq \frac{\rho(1-\rho)(\mu^2+\mu\nu)}{L} - \frac{\rho(1-\rho)\mu^2\nu}{2L^2} + \frac{1}{L^2}\psi[\rho(1-\rho)(\mu^2+\mu\nu)]$$

with $\quad \psi(z) = \sum_{k=1}^{\infty} \frac{B_{2k-2}}{k!(k-1)!} z^k$

The General Case (S. Prolhac, 2010)

The function $E(\mu)$ is again obtained in a parametric form:

$$\mu = -\sum_{k\geq 1} C_k \frac{B^k}{k} \quad \text{and} \quad E = -(1-x)\sum_{k\geq 1} D_k \frac{B^k}{k}$$

C_k and D_k are combinatorial factors enumerating some tree structures.
There exists an auxiliary function

$$W_B(z) = \sum_{k\geq 1} \phi_k(z) \frac{B^k}{k}$$

such that C_k and D_k are given by complex integrals along a small contour that encircles 0 :

$$C_k = \oint_{\mathcal{C}} \frac{dz}{2\,i\,\pi} \frac{\phi_k(z)}{z} \quad \text{and} \quad D_k = \oint_{\mathcal{C}} \frac{dz}{2\,i\,\pi} \frac{\phi_k(z)}{(z+1)^2}$$

The function $W_B(z)$ contains the full information about the statistics of the current.

The function $W_B(z)$ is the solution of a functional Bethe equation:

$$W_B(z) = -\ln\left(1 - BF(z)e^{X[W_B](z)}\right)$$

where

$$F(z) = \frac{(1+z)^L}{z^N}$$

The operator X is a integral operator

$$X[W_B](z_1) = \oint_{\mathcal{C}} \frac{dz_2}{\imath 2\pi\, z_2} W_B(z_2) K(z_1, z_2)$$

with the kernel

$$K(z_1, z_2) = 2\sum_{k=1}^{\infty} \frac{x^k}{1-x^k} \left\{ \left(\frac{z_1}{z_2}\right)^k + \left(\frac{z_2}{z_1}\right)^k \right\}$$

Solving this Functional Bethe Ansatz equation to all orders enables us to calculate cumulant generating function. For $x = 0$, the TASEP result is readily retrieved.

The function $W_B(z)$ also contains information on the 6-vertex model associated with the ASEP.

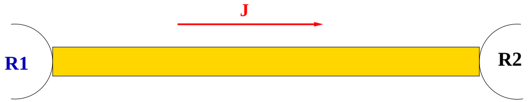
From the Physics point of view, the solution allows one to

- Classify the different universality classes (KPZ, EW).
- Study the various scaling regimes.
- Investigate the hydrodynamic behaviour.

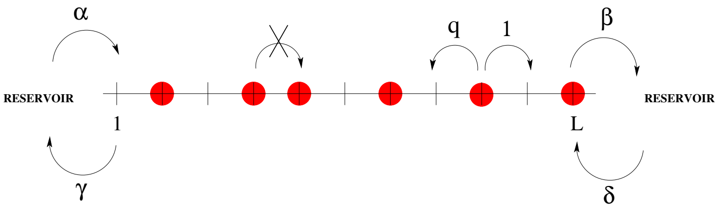
Current Fluctuations in the open ASEP

The Current in the Open System

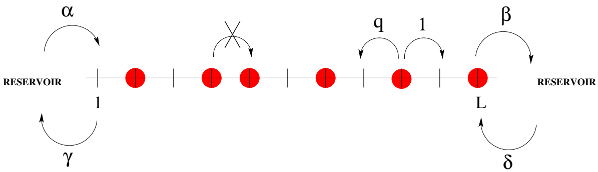
The fundamental paradigm



The asymmetric exclusion model with open boundaries



NB: the asymmetry parameter in now denoted by q.



The stationary probability of a configuration $\mathcal{C}$ is given by

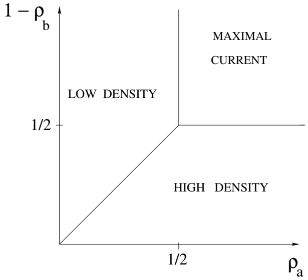
$$P(\mathcal{C}) = \frac{1}{Z_L} \langle W| \prod_{i=1}^{L} \left(\tau_i D + (1-\tau_i) E\right) |V\rangle$$

where $\tau_i = 1$ (or 0) if the site i is occupied (or empty) and the normalization constant is $Z_L = \langle W| (D+E)^L |V\rangle$

The operators D and E, the vectors $\langle W|$ and $|V\rangle$ satisfy

$$\begin{aligned} D\,E - qE\,D &= (1-q)(D+E) \\ (\beta\,D - \delta\,E)\,|V\rangle &= |V\rangle \\ \langle W|(\alpha\,E - \gamma\,D) &= \langle W| \end{aligned}$$

The Phase Diagram



$\rho_a = \frac{1}{a_+ + 1}$: effective left reservoir density.

$\rho_b = \frac{b_+}{b_+ + 1}$: effective right reservoir density.

$$a_\pm = \frac{(1 - q - \alpha + \gamma) \pm \sqrt{(1 - q - \alpha + \gamma)^2 + 4\alpha\gamma}}{2\alpha}$$

$$b_\pm = \frac{(1 - q - \beta + \delta) \pm \sqrt{(1 - q - \beta + \delta)^2 + 4\beta\delta}}{2\beta}$$

Representations of the quadratic algebra

The algebra encodes combinatorial recursion relations between systems of different sizes.

Infinite dimensional Representation:

$D = 1 + d$ where d is a q-destruction operator.

$E = 1 + e$ where e is a q-creation operator.

$$d = \begin{pmatrix} 0 & \sqrt{1-q} & 0 & 0 & \dots \\ 0 & 0 & \sqrt{1-q^2} & 0 & \dots \\ 0 & 0 & 0 & \sqrt{1-q^3} & \dots \\ & & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad e = d^\dagger$$

The matrix Ansatz allows one to calculate Stationary State Properties (currents, correlations, fluctuations) and to derive the Phase Diagram in the infinite size limit.

Total Current

The observable Y_t counts the total number of particles exchanged between the system and the left reservoir between times 0 and t.

Hence, $Y_{t+dt} = Y_t + y$ with

- $y = +1$ if a particle enters at site 1 (at rate α),
- $y = -1$ if a particle exits from 1 (at rate γ)
- $y = 0$ if no particle exchange with the left reservoir has occurred during dt.

Statistical properties of Y_t:

- Average current: $J(q, \alpha, \beta, \gamma, \delta, L) = \lim_{t\to\infty} \frac{\langle Y_t \rangle}{t}$
  It can be calculated by the steady-state matrix Ansatz $J = \frac{Z_{L-1}}{Z_L}$.
- Current fluctuations: $\Delta(q, \alpha, \beta, \gamma, \delta, L) = \lim_{t\to\infty} \frac{\langle Y_t^2 \rangle - \langle Y_t \rangle^2}{t}$
  The fluctuations of the total current. It does not depend on the stationary measure only.
- Cumulant Generating Function: $\langle e^{\mu Y_t} \rangle \simeq e^{E(\mu) t}$ for $t \to \infty$ $E(\mu)$ encodes the statistical properties of the total current.

Current Statistics: Mathematical Framework

These three mutually exclusive types of transitions lead to a splitting the Markov operator:

$$M = M_0 + M_+ + M_-$$

- M_0 corresponds to transitions that do not modify the value of Y.
- M_+ are transitions that increment Y by 1: a particle enters the system from the left reservoir.
- M_- encodes rates in which Y decreases by 1, if a particle exits the system from the left reservoir (does not happen in the simplest TASEP case).

The cumulant-generating function $E(\mu)$ when $t \to \infty$, $\langle \mathrm{e}^{\mu Y_t} \rangle \simeq \mathrm{e}^{E(\mu) t}$, is the dominant eigenvalue of the deformed matrix

$$M(\mu) = M_0 + \mathrm{e}^{\mu} M_+ + \mathrm{e}^{-\mu} M_-$$

The current statistics is again reduced to an eigenvalue problem.

Analytic Procedure

Call $F_\mu(\mathcal{C})$ of the dominant eigenvector F_μ of $M(\mu)$. We have:

$$M(\mu).F_\mu = E(\mu)F_\mu$$

This dominant eigenvector can be formally expanded w. r. t. μ:

$$F_\mu(\mathcal{C}) = P(\mathcal{C}) + \mu R_1(\mathcal{C}) + \mu^2 R_2(\mathcal{C}) \dots$$

- For $\mu = 0$: $M(\mu = 0)$ is the original Markov operator, $E(\mu = 0) = 0$ and $P(\mathcal{C})$ is the stationary weight of the configuration $\mathcal{C}$: $M.P = 0.$
- The generalized weight vector $R_k(\mathcal{C})$ satisfies an inhomogeneous linear equation: $M.R_k = \Phi_k\,(P, R_1, \dots R_{k-1})$, Φ_k being a linear functional.
- For each value of k, we show that F_μ can be represented by a matrix product Ansatz up to corrections of order μ^{k+1}.
- Knowing F_μ up to corrections of order μ^{k+1}, we calculate $E(\mu)$ to order μ^{k+1}.

Generalized Matrix Ansatz

One can prove that the dominant eigenvector of the deformed matrix $M(\mu)$ is given by the following matrix product representation:

$$F_\mu(\mathcal{C}) = \frac{1}{Z_L^{(k)}} \langle W_k | \prod_{i=1}^{L} (\tau_i D_k + (1-\tau_i) E_k) | V_k \rangle + \mathcal{O}\left(\mu^{k+1}\right)$$

The matrices D_k and E_k are constructed recursively (knowing D_1 and E_1)

$$\begin{aligned} D_{k+1} &= (1 \otimes 1 + d \otimes e) \otimes D_k + (1 \otimes d + d \otimes 1) \otimes E_k \\ E_{k+1} &= (1 \otimes 1 + e \otimes d) \otimes E_k + (e \otimes 1 + 1 \otimes e) \otimes D_k \end{aligned}$$

The boundary vectors $\langle W_k|$ and $|V_k\rangle$ are also obtained recursively:

$$|V_k\rangle = |\beta\rangle|\tilde{V}\rangle|V_{k-1}\rangle \quad \text{and} \quad \langle W_k| = \langle W^\mu|\langle \tilde{W}^\mu|\langle W_{k-1}|$$

$$[\beta(1-d) - \delta(1-e)]\,|\tilde{V}\rangle = 0$$

$$\langle W^\mu|[\alpha(1+\mathrm{e}^{\mu}\, e) - \gamma(1+\mathrm{e}^{-\mu}\, d)] = (1-q)\langle W^\mu|$$

$$\langle \tilde{W}^\mu|[\alpha(1-\mathrm{e}^{\mu}\, e) - \gamma(1-\mathrm{e}^{-\mu}\, d)] = 0$$

Structure of the solution I

For arbitrary values of q and $(\alpha, \beta, \gamma, \delta)$, and for any system size L the parametric representation of $E(\mu)$ is given by

$$\mu = -\sum_{k=1}^{\infty} C_k(q; \alpha, \beta, \gamma, \delta, L) \frac{B^k}{2k}$$

$$E = -\sum_{k=1}^{\infty} D_k(q; \alpha, \beta, \gamma, \delta, L) \frac{B^k}{2k}$$

The coefficients C_k and D_k are given by contour integrals in the complex plane:

$$C_k = \oint_{\mathcal{C}} \frac{dz}{2\,i\,\pi} \frac{\phi_k(z)}{z} \quad \text{and} \quad D_k = \oint_{\mathcal{C}} \frac{dz}{2\,i\,\pi} \frac{\phi_k(z)}{(z+1)^2}$$

There exists an auxiliary function

$$W_B(z) = \sum_{k \geq 1} \phi_k(z) \frac{B^k}{k}$$

that contains the full information about the statistics of the current.

Structure of the solution II

This auxiliary function $W_B(z)$ solves a functional Bethe equation:

$$W_B(z) = -\ln\left(1 - BF(z)e^{X[W_B](z)}\right)$$

- The operator X is a integral operator

$$X[W_B](z_1) = \oint_{\mathcal{C}} \frac{dz_2}{\imath 2\pi\, z_2} W_B(z_2) K\left(\frac{z_1}{z_2}\right)$$

with kernel $\quad K(z) = 2\sum_{k=1}^{\infty} \frac{q^k}{1-q^k}\left\{z^k + z^{-k}\right\}$

- The function $F(z)$ is given by

$$F(z) = \frac{(1+z)^L(1+z^{-1})^L(z^2)_\infty(z^{-2})_\infty}{(a_+z)_\infty(a_+z^{-1})_\infty(a_-z)_\infty(a_-z^{-1})_\infty(b_+z)_\infty(b_+z^{-1})_\infty(b_-z)_\infty(b_-z^{-1})_\infty}$$

where $(x)_\infty = \prod_{k=0}^{\infty}(1-q^k x)$ and $a_\pm$, $b_\pm$ depend on the boundary rates.

- The complex contour $\mathcal{C}$ encircles 0, $q^k a_+, q^k a_-, q^k b_+, q^k b_-$ for $k \geq 0$.

Discussion

- These results are of *combinatorial nature*: *valid for arbitrary values of the parameters and for any system sizes with no restrictions.*
- *Average-Current:*

$$J = \lim_{t\to\infty} \frac{\langle Y_t \rangle}{t} = (1-q)\frac{D_1}{C_1} = (1-q)\frac{\oint_\Gamma \frac{dz}{2\,i\,\pi}\frac{F(z)}{z}}{\oint_\Gamma \frac{dz}{2\,i\,\pi}\frac{F(z)}{(z+1)^2}}$$

(cf. T. Sasamoto, 1999.)

- *Diffusion Constant:*

$$\Delta = \lim_{t\to\infty} \frac{\langle Y_t^2 \rangle - \langle Y_t \rangle^2}{t} = (1-q)\frac{D_1 C_2 - D_2 C_1}{2 C_1^3}$$

where C_2 and D_2 are obtained using

$$\phi_1(z) = \frac{F(z)}{2} \quad \text{and} \quad \phi_2(z) = \frac{F(z)}{2}\left(F(z) + \oint_\Gamma \frac{dz_2 F(z_2) K(z/z_2)}{2\imath\pi z_2}\right)$$

(cf. the TASEP case: B. Derrida, M. R. Evans, K. M., 1995)

Asymptotic behaviour

- Maximal Current Phase:

$$\mu = -\frac{L^{-1/2}}{2\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{(2k)!}{k!k^{(k+3/2)}} B^k$$

$$\mathcal{E} - \frac{1-q}{4}\mu = -\frac{(1-q)L^{-3/2}}{16\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{(2k)!}{k!k^{(k+5/2)}} B^k$$

- Low Density (and High Density) Phases:
  Dominant singularity at a_+: $\phi_k(z) \sim F^k(z)$. By Lagrange Inversion:

$$E(\mu) = (1-q)(1-\rho_a)\frac{\mathrm{e}^{\mu} - 1}{\mathrm{e}^{\mu} + (1-\rho_a)/\rho_a}$$

(cf de Gier and Essler, 2011).

Current Large Deviation Function:

$$\Phi(j) = (1-q)\left\{\rho_a - r + r(1-r)\ln\left(\frac{1-\rho_a}{\rho_a}\frac{r}{1-r}\right)\right\}$$

where the current j is parametrized as $j = (1-q)r(1-r)$.
Matches the predictions of Macroscopic Fluctuation Theory, as observed by T. Bodineau and B. Derrida.

The TASEP case

Here $q = \gamma = \delta = 0$ and (α, β) are arbitrary.
The parametric representation of $E(\mu)$ is

$$\mu = -\sum_{k=1}^{\infty} C_k(\alpha, \beta) \frac{B^k}{2k}$$

$$E = -\sum_{k=1}^{\infty} D_k(\alpha, \beta) \frac{B^k}{2k}$$

with

$$C_k(\alpha, \beta) = \oint_{\{0,a,b\}} \frac{dz}{2i\pi} \frac{F(z)^k}{z} \quad \text{and} \quad D_k(\alpha, \beta) = \oint_{\{0,a,b\}} \frac{dz}{2i\pi} \frac{F(z)^k}{(1+z)^2}$$

where

$$F(z) = \frac{-(1+z)^{2L}(1-z^2)^2}{z^L(1-az)(z-a)(1-bz)(z-b)}, \qquad a = \frac{1-\alpha}{\alpha}, \qquad b = \frac{1-\beta}{\beta}$$

A special case of TASEP

In the case $\alpha = \beta = 1$, a parametric representation of the cumulant generating function $E(\mu)$:

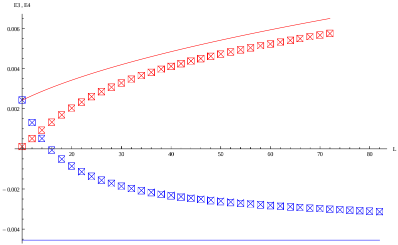
$$\mu = -\sum_{k=1}^{\infty} \frac{(2k)!}{k!} \frac{[2k(L+1)]!}{[k(L+1)]!\,[k(L+2)]!} \frac{B^k}{2k} \,,$$

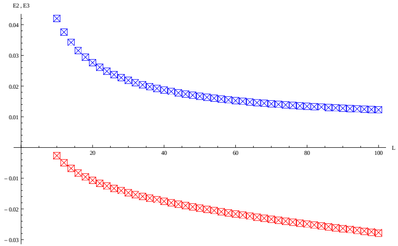
$$E = -\sum_{k=1}^{\infty} \frac{(2k)!}{k!} \frac{[2k(L+1)-2]!}{[k(L+1)-1]!\,[k(L+2)-1]!} \frac{B^k}{2k} \,.$$

First cumulants of the current

- Mean Value : $J = \frac{L+2}{2(2L+1)}$
- Variance : $\Delta = \frac{3}{2} \frac{(4L+1)![L!(L+2)!]^2}{[(2L+1)!]^3(2L+3)!}$
- Skewness :
  $E_3 = 12 \frac{[(L+1)!]^2[(L+2)!]^4}{(2L+1)[(2L+2)!]^3} \left\{ 9 \frac{(L+1)!(L+2)!(4L+2)!(4L+4)!}{(2L+1)![(2L+2)!]^2[(2L+4)!]^2} - 20 \frac{(6L+4)!}{(3L+2)!(3L+6)!} \right\}$
  For large systems: $E_3 \to \frac{2187-1280\sqrt{3}}{10368} \pi \sim -0.0090978...$

Numerical results (DMRG)





Left: **Max. Current** ($q = 0.5$, $a_+ = b_+ = 0.65$, $a_- = b_- = 0.6$), Third and Fourth cumulant.

Right: **High Density** ($q = 0.5$, $a_+ = 0.28$, $b_+ = 1.15$, $a_- = -0.48$ and $b_- = -0.27$), Second and Third cumulant.

M. Gorissen, A. Lazarescu, K. M. and C. Vanderzande, PRL **109** 170601 (2012)