1 Motion in quadrupole fringe field

When the pole face is perpendicular to the quadrupole symmetry axis, our first order analysis of the quadrupole fringe field shows that the forces acting are

$$[F_x, F_y] = B_1 e v_s [-x, +y] W(z) \ . \tag{1}$$

Here W models the fringe field fall off. In the case of an exit face, $W(0) = 1$ and $W(L) = 0$, where L is the fringe length. When $B_1 > 0$, the quadrupole is horizontally focusing. Let x' and y' be the particle divergences. The equations of motion are

$$(d/dt)[x', y'] \ + K^2 W(t)[+x(t), -y(t)] \ = [0, 0] \tag{2}$$

Here t is synonymous with the longitudinal coordinate z. We shall now find an almost exact solution for x, y by successive approximations.

2 1st Approximation

The fringe may be written $W(t) = \bar{W} + W_{\mathrm{ac}}(t)$ where $\bar{W}$ is the average value, and W_{ac} is an alternating part. The first step is to replace $W(s)$ by $\bar{W}$ leading to

$$(d/dt)[x', y'] \ + k^2[+x(t), -y(t)] \ = [0, 0] \ , \tag{3}$$

where $k^2 = K^2 \bar{W}$. The equation has the well-known solution

$$[x(t), x'(t)] = \mathbf{T}_0 [x_0, x_0'] \quad \text{with} \quad \mathbf{T}_0 = \begin{bmatrix} C(t) & S(t) \\ C'(t) & S'(t) \end{bmatrix} \ . \tag{4}$$

Here C, S are the principal functions having the properties:

$$C(0) = 1, \quad S(0) = 0, \quad C'(0) = 0, \quad S'(0) = 1, \quad CS' - SC' = 1 \ . \tag{5}$$

There is an analogous solution fo y, y'.

3 2nd Approximation

We now restore the alternating part of $W(s)$

$$(d/dt)x' \ + k^2 x(t) \ = \ -K^2 W_{\mathrm{ac}}(t) x(t) \ . \tag{6}$$

We shall treat this as if it were an inhomogeneous equation and solvable by the method of Greens functions. First we note some properties of the Green's function $G(u,v)$.

$$G(u,v) = S(u)C(v) - C(u)S(v)\,, \qquad G(u,u) = 0\,. \tag{7}$$

$$\frac{d}{du}G(u,v) = -S(v)C'(u)\,, + C(v)S'(u) \qquad \frac{d}{du}G(u,v)|_{v=u} = 1\,. \tag{8}$$

For our particular horizontal equation, there is the property

$$C'' = -k^2 C\,, \qquad S'' = -k^2 S\,, \qquad \frac{d^2}{du^2}G(u,v) = -k^2 G(u,v)\,. \tag{9}$$

The solution is

$$x(t) = x_0 C + x_0' S - K^2 \int_0^t G(t,u) W_{\rm ac}(u) x(u) du \tag{10}$$

$$x'(t) = x_0 C' + x_0' S' - K^2 \int_0^t G'(t,u) W_{\rm ac}(u) x(u) du \tag{11}$$

$$\begin{aligned} x''(t) &= -k^2(x_0 C + x_0' S) + K^2 \left[k^2 \int_0^t G(t,u) W_{\rm ac}(u) x(u) du - W_{\rm ac}(t) x(t) \right] \\ &= -K^2 W(t) x(t) + K^2 k^2 \int_0^t G(t,u) W_{\rm ac}(u) x(u) du\,. \end{aligned} \tag{12}$$

This may also be written in a matrix form: $[x, x'] = \mathbf{T}[x_0, x_0']$ with $\mathbf{T} = \mathbf{T}_0 + \Delta\mathbf{T}$ and

$$\Delta\mathbf{T} = -K^2 \begin{bmatrix} \int_0^t G(t,u) C W_{\rm ac} du & \int_0^t G(t,u) S W_{\rm ac} du \\ \int_0^t G'(t,u) C W_{\rm ac} du & \int_0^t G'(t,u) S W_{\rm ac} du \end{bmatrix}\,. \tag{13}$$

Here we have eplicitly substituted the solution (4) into the right hand of (10, 11).

3.1 Errors

Ideally the equation of motion (2) is identically zero. The departure from zero is a measure of how inaccurate is our approximate solution. We substitute (10, 11, 12) into the differential equation (2) and obtain the error

$$\varepsilon(t) = -K^4 W_{\rm ac}(t) \int_0^t G(t,u) W_{\rm ac}(u) x(u) du\,. \tag{14}$$

The basic equation of motion is non-dissipative, and is therefore conservative. Thus, despite the varying coefficient $W(t)$, the determinant of the transfer matrix $\mathbf{T}$ must remain identically equal to unity, as may be confirmed by pure numerical integration of the equations of motion. We form the determinant of the matrix $\mathbf{T} = (\mathbf{T}_0 + \Delta\mathbf{T})$. In order to simplify this determinant, we substitute for $G(t,u)$ and $G'(t,u)$ from (7) and (8); giving cancellation of the K^2 terms, leading to

$$\begin{aligned} \mathrm{Det}[\mathbf{T}] &= 1 + K^4 \int_0^t G(t,u) C(u) W_{\mathrm{ac}}(u) du \int_0^t G'(t,u) S(u) W_{\mathrm{ac}}(u) du \\ &- K^4 \int_0^t G(t,u) S(u) W_{\mathrm{ac}}(u) du \int_0^t G'(t,u) C(u) W_{\mathrm{ac}}(u) du. \end{aligned} \tag{15}$$

We substitute again and find

$$\mathrm{Det}[\mathbf{T}] = 1 + K^4 \left[\int_0^t C^2 W_{\mathrm{ac}}(u) du\right] \left[\int_0^t S^2 W_{\mathrm{ac}}(u) du\right] - K^4 \left[\int_0^t C(u) S(u) W_{\mathrm{ac}}(u) du\right]^2. \tag{16}$$

Clearly, the determinant deviates from unity. The next step shall be to modify the transfer matrix so as to regulate the error ε and the determinant.

4 3rd Approximation

The matrix elements T_{11}, T_{12} relate to $x(t)$, while elements T_{21}, T_{22} relate to $x'(t)$. Now $T_{21} = T'_{11}$ and $T_{22} = T'_{12}$. The act of taking derivatives typically amplifies the effect of errors, and it is to be expected that the relative errors in $x'(t)$ are greater (by far) than those in $x(t)$; and this is confirmed by comparison with direct numerical integrations. Consequently, we take the matrix:

$$\mathbf{T} = \begin{bmatrix} C - K^2 \int_0^t G(t,u) C W_{\mathrm{ac}} du & S - K^2 \int_0^t G(t,u) S W_{\mathrm{ac}} du \\ F_{21}(t) & F_{22}(t) \end{bmatrix}, \tag{17}$$

where the functions F_{21}, F_{22} are to be determined. We substitute $[x, x'] = \mathbf{T}[x_0, x'_0]$ into the equation of motion (2) and find the error term:

$$\varepsilon = K^2 W(t) x(t) - K^4 W(t) \int_0^t G(t,u) W_{\mathrm{ac}}(u) x(u) du + [x_0 F'_{21} + x'_0 F'_{22}], \tag{18}$$

where $x(t) = x_0 C + x'_0 S$. ε must be zero independent of the coefficient x_0, x'_0, leading to differential equations for F_{21}, F_{22}, with solution:

$$F_{21}(t) = F_{21}(0) - K^2 \int_0^t C W du + K^4 \int_0^t \left[\int_0^v G[v,u] C[u] W_{ac}(u) du\right] W(v) dv \tag{19}$$

$$F_{22}(t) \;=\; F_{22}(0) - K^2\int_0^t S\,W\,du + K^4\int_0^t\left[\int_0^v G[v,u]S[u]W_{ac}(u)du\right]W(v)dv\,. \tag{20}$$

The constants $F_{21}(0), F_{22}(0)$ are chosen to make the determinant equal unity at $t=0$ and $t=L$. The determinant is

$$\mathrm{Det}\mathbf{T} = [C(t)F_{22}(t) - S(t)F_{21}(t)] - K^2\int_0^t [C(u)F_{22}(t) - S(u)F_{21}(t)]G(t,u)W_{\rm ac}(u)du\,. \tag{21}$$

At $t=0$, the determinant is $F_{22}(0)=1$. Hence $F_{21}(0)$ is chosen to make $\mathrm{Det}\mathbf{T}=1$ at $t=L$; the symbolic solution is too lengthy to record here.

5 Example of $\cos^2(bt)$ fringe field

Evidently, the utility of this approach is limited by the need to find closed form expressions for the double integrals; and this limits $W(t)$ to simple functions. We shall study the case that $W(t)=\cos^2(bt)$ for an exit field, with $b=\pi/(2L)$ and $b\neq k$. This is more realistic than a linear decay, but still short of "ideal" because real fringe fields tend to have an initial rapid fall off, but a lingering tail. For this case $\bar{W}=1/2$, $k=K/\sqrt{2}$ and $W_{\rm ac}=(1/2)\cos(2bt)$.

5.1 Horizontal motion

For the horizontal motion $C(t)=\cos(kt)$ and $S(t)=\sin(kt)/k$. The partial transfer matrix is $\Delta\mathbf{T}=$

$$\frac{K^2}{4b(b^2-k^2)}\begin{bmatrix} \sin bt[k\cos bt\sin kt - b\sin bt\cos kt] & \cos bt[b\cos bt\sin kt - k\sin bt\cos kt] \\ (-2b^2+k^2)\cos kt\sin 2bt + bk\cos^2 bt\sin kt & (-2b^2+k^2)\sin kt\sin 2bt + bk\cos^2 bt\cos kt \end{bmatrix} \tag{22}$$

The error (before introducing F_{21}, F_{22}) is $\varepsilon=$

$$\frac{K^4\cos 2bt}{8b(b^2-k^2)}\left[x_0\sin bt(-b\cos kt\sin bt + k\cos bt\sin kt) + (x_0'/k)\cos bt(-k\cos kt\sin bt + b\cos bt\sin kt)\right]\,. \tag{23}$$

The determinant (before introducing F_{21}, F_{22}) is

$$1+\frac{K^4}{64(b^2-k^2)}\left[\frac{(k^2-5b^2)}{2b^2(b^2-k^2)}+\frac{\cos 4bt}{2b^2}+\frac{\cos 2(b-k)t}{k(b-k)}-\frac{\cos 2(b+k)t}{k(b+k)}\right]\,. \tag{24}$$

5.1.1 Corrected matrix elements

The next step is to find the matrix coefficients F_{21}, F_{22}:

$$F'_{21} + K^2 \cos^2 bt \left[\cos kt + \frac{K^2}{4b(b^2-k^2)} \sin bt(-b\cos kt \sin bt + k\cos bt \sin kt)\right] = 0\,, \tag{25}$$

$$F'_{22} + \frac{K^2}{k} \cos^2 bt \left[\sin kt + \frac{K^2}{4b(b^2-k^2)} \cos bt(-k\cos kt \sin bt + b\cos bt \sin kt)\right] = 0\,. \tag{26}$$

The integrals can be performed in closed form, but are rather lengthy.

5.2 Vertical motion

Analogously, for the vertical plane For the horizontal motion $C(t) = \cosh(kt)$ and $S(t) = \sinh(kt)/k$. The driving term for the inhomogeneous equation is $+K^2 W_{\mathrm{ac}} y(t)$. $d^2/dt^2 G(t,u) = +k^2 G(t,u)$. The top row of the partial transfer matrix is $\Delta[T_{11}, T_{12}] =$

$$\frac{K^2}{b(b^2+k^2)} \Big[\ 2\sin bt[k\cos bt \sinh kt + b\sin bt \cosh kt],\quad 2\cos bt[-b\cos bt \sinh kt + k\sin bt \cosh kt]\ \Big]\,. \tag{27}$$

The lower row is $\Delta[T_{21}, T_{22}] = \Delta[T'_{11}, T'_{12}]$.

The error (before introducing F_{21}, F_{22}) is $\varepsilon =$

$$\frac{K^4 \cos 2bt}{8b(b^2+k^2)} \left[-y_0 \sin bt(b\cosh kt \sin bt + k\cos bt \sinh kt) + (y'_0/k)\cos bt(-k\cosh kt \sin bt + b\cos bt \sinh kt)\right]\,. \tag{28}$$

The determinant (before introducing F_{21}, F_{22}) is

$$1 + \frac{K^4}{64(b^2+k^2)^2}\left[\frac{-(k^2+5b^2)}{2b^2} + \frac{(b^2+k^2)\cos 4bt}{2b^2} + \frac{2}{k}(k\cos 2bt \cosh 2kt + b\sin 2bt \sinh 2kt)\right]\,. \tag{29}$$

5.2.1 Corrected Matrix Elements

The next step is to find the matrix coefficients F_{21}, F_{22}:

$$F'_{21} - K^2 \cos^2 bt \left[\cosh kt + \frac{K^2}{4b(b^2+k^2)} \sin bt(b\cosh kt \sin bt + k\cos bt \sinh kt)\right] = 0\,, \tag{30}$$

$$F'_{22} - \frac{K^2}{k}\cos^2 bt\left[\sinh kt + \frac{K^2}{4b(b^2+k^2)}\cos bt(k\cosh kt\sin bt - b\cos bt\sinh kt)\right] = 0\,. \tag{31}$$

The integrals can be performed in closed form, but are rather lengthy.

6 Numerical example

For the $\cos^2(bt)$ fringe field we now show a numerical example for the parameters $k = \pi/3$, $b = 5\pi/3$ and $L = 3/10$. We consider the particle trajectory with $x_0 = 0.2$ and $x'_0 = 20°$.

6.1 Horizontal and vertical motion

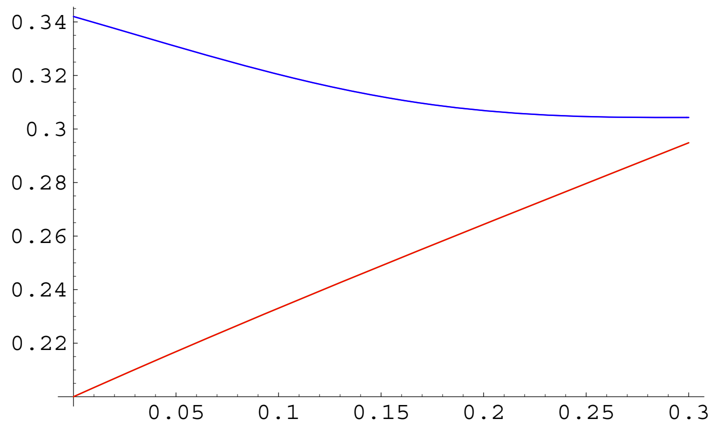

Figure 1: Trajectory x (red) and x' (blue)

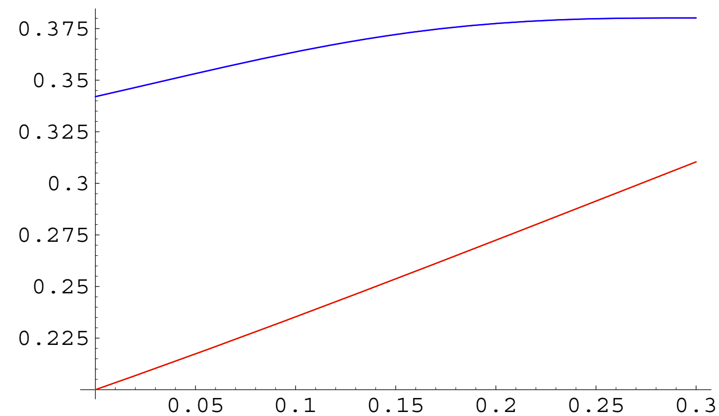

Figure 2: Trajectory y (red) and y' (blue)

In a falling fringe field, the particle divergences x', y' tend toward constant values.

6.2 Errors in horizontal motion

Here we compare numerical evaluation of our analytic expressions against direct numerical integration of the equations of motion for x, x'.

6.2.1 Before correction

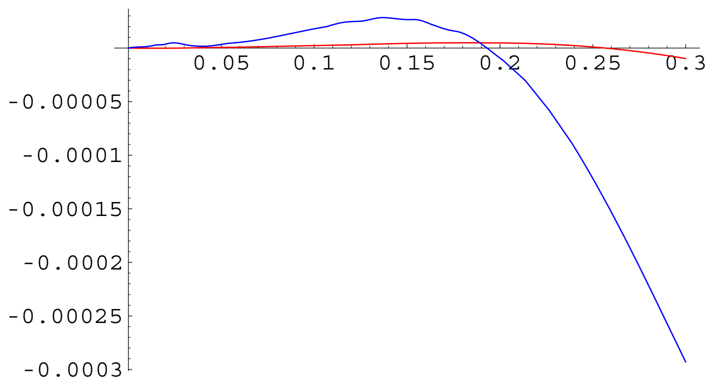

Figure 3: Relative fractional error in x (red) and x' (blue)

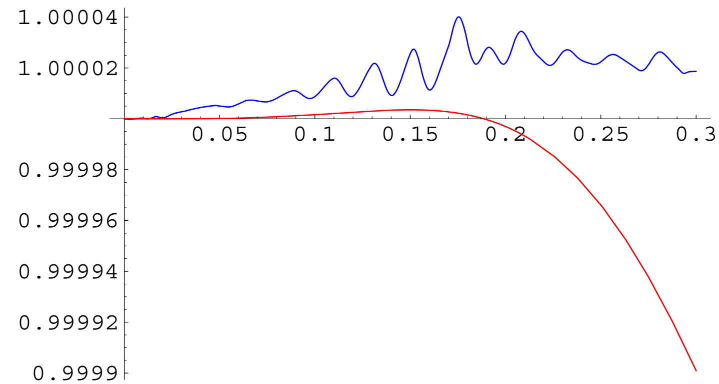

Figure 4: Determinants via numerical (blue) and Greens function (red)

6.2.2 After introducing F_{21}, F_{22} correction

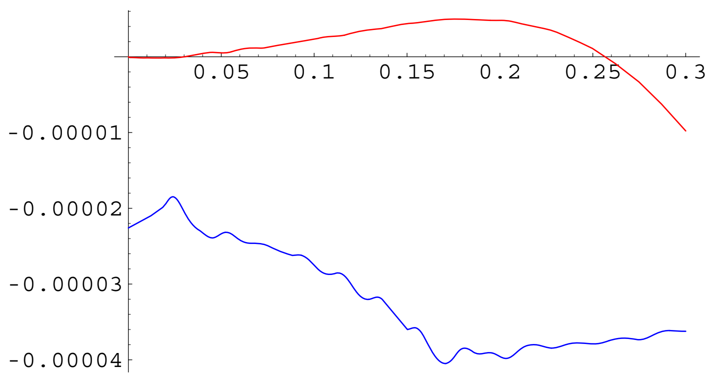

Figure 5: Relative fractional error in x (red) and x' (blue)

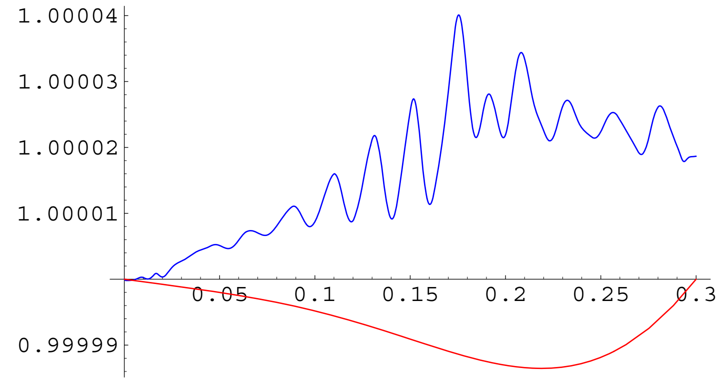

Figure 6: Determinants via numerical (blue) and Greens function (red)

From the figures 3-6 it is clear that there is an order of magnitude reduction in the errors after introducing the F_{21}, F_{22} matrix elements.

6.3 Errors in vertical motion

Here we compare numerical evaluation of our analytic expressions against direct numerical integration of the equations of motion for y, y'.

6.3.1 Before correction

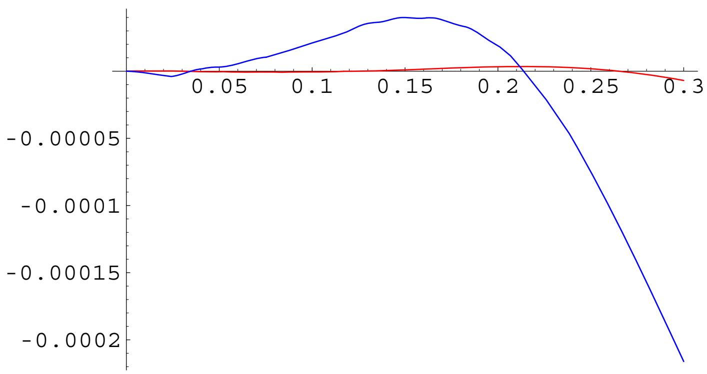

Figure 7: Relative fractional error in y (red) and y' (blue)

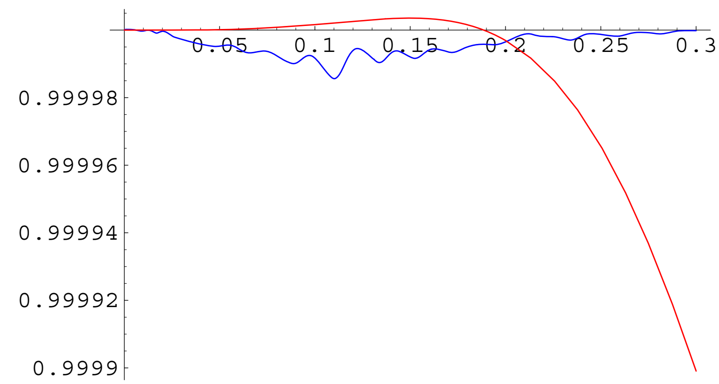

Figure 8: Determinants via numerical (blue) and Greens function (red)

6.3.2 After introducing F_{21}, F_{22} correction

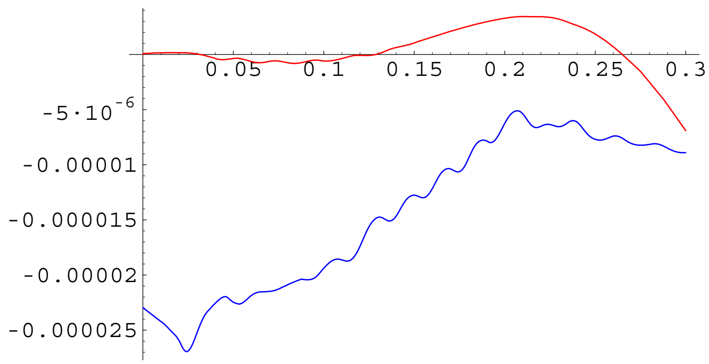

Figure 9: Relative fractional error in y (red) and y' (blue)

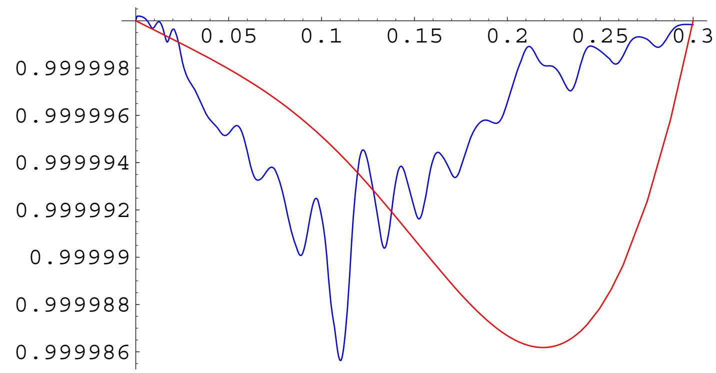

Figure 10: Determinants via numerical (blue) and Greens function (red)

From the figures 7-10 it is clear that there is an order of magnitude reduction in the errors after introducing the F_{21}, F_{22} matrix elements.