1 Introduction

We shall consider fringe field effects for magnets with rotated pole faces; that is to say the entry and exit faces are not perpendicular to the reference trajectory through the magnet. Our derivation differs from those offered in standard texts [1, 2, 3] in being entirely algebraic. We begin with the simpler dipole case before addressing the quadrupole.

2 Dipole magnet

We consider a vertically oriented magnetic field $B_y = B_0$ and take coordinates which are locally perpendicular. In the interior body of the magnet where the vertical field is completely established, we take cylindrical polars $[r, y, \psi]$ with radius and angle (r, ψ) taken in the horizontal plane. The reference orbit is $[\rho, 0, \psi]$ with $\rho = \beta\gamma m_0 c/(eB_0)$ where $\beta\gamma m_0 c$ is the azimuthal momentum. At the entry/exit faces we take cartesian coordinates $[x, y, z]$ which are locally aligned with $[r, y, \theta]$ respectively. In the absence of a fringe field, the exterior region is a drift space and the motion is entirely rectilinear. The reference trajectory in the drift is $[x, y, z] = [0, 0, z]$.

2.1 Linear fringe field

Consider now a fringe field that extends from the body region and into the drift. The longitudinal component of the fringe field is launched from and perpendicular to the pole face (e.g. the plane $z = 0$) and will take its symmetry planes from that face. Let the pole be rotated by an angle ϕ about y and take cartesian coordinates $[q, y, p]$ where q and p are parallel and perpendicular, respectively, to the face and the origin is coincident with the system $[x, y, z]$.

The situation is now as follows: we know the magnetic field components in terms of coordinates $[q, y, p]$, but we wish to find the equations of motion in terms of $[x, y, z]$ as an expansion about the reference trajectory in the drift.

2.1.1 Interior body field

Just inside the interior region, the potential and body field are:

$$\Phi = -B_0 y \qquad \text{and} \qquad [B_x, B_y, B_z] = -\nabla\Phi = [0, B_0, 0]\,. \tag{1}$$

2.1.2 Exterior fringe field

Because there is complete rotational symmetry about y for the body field, so it follows that there is no influence upon the fringe by the pole-face angle ϕ. Let $W(bp)$ model the azimuthal variation of the fringe of length L and $b > 0$. For an exit fringe, $W(0) = 1$, $W(bL) = 0$ and $W'(0) = 0$, $W'(bL) = 0$. For example if $W(s) = \cos^2(s)$ then $bL = \pi/2$. Similarly for an entrance fringe, $W(0) = 1$ and $W(-bL) = 0$. The prime notation is $W'(s) = dW(s)/ds$. Just outside the magnet, the potential and fringe field are:

$$\Phi_f = -B_{0f} y W(bp) \qquad \text{and} \qquad [B_q, B_y, B_p] = -\nabla\Phi_f = B_{0f}[0, W(bp), by\, W'(bp)]\ . \tag{2}$$

Notice that the longitudinal component B_p has the same symmetry (dipole) in the $q - y$ plane as the original potential function. Matching the interior and exterior field and potential along the boundary $p = 0$, leads to the identification $B_0 = B_{0f}$.

2.1.3 Transformation of field and coordinates

The two coordinate systems are related by $[q, y, p] = \mathbf{T}[x, y, z]$ and $[z, y, z] = \mathbf{T}^{-1}[q, y, p]$, where the $\mathbf{T}^{-1}$ denotes the inverse of the matrix

$$\mathbf{T} = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix}\ . \tag{3}$$

The next step is to transform the fringe field components in $[q, y, p]$ into components directed along $[x, y, z]$

$$[B_x, B_y, B_z] = \mathbf{T}^{-1}[B_q, B_y, B_p] = B_0[by \sin\phi\, W'(bp), W(bp), by\cos\phi\, W'(bp)] \tag{4}$$

The next step is to re-write $[q, y, p]$ in terms of $[x, y, z]$ using $[q, y, p] = \mathbf{T}[x, y, z]$, leading to

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = B_0 \begin{bmatrix} by\sin\phi\, W'[b(z\cos\phi + x\sin\phi)] \\ W[b(z\cos\phi + x\sin\phi)] \\ by\cos\phi\, W'[b(z\cos\phi + x\sin\phi)] \end{bmatrix}\ . \tag{5}$$

2.1.4 Expansion about reference trajectory

The next step is to make an expansion of this field about the reference trajectory $[x, y, z] = [0, 0, z]$. It is clear that a linear expansion is to first order in (x, y); but to what order in the longitudinal coordinate z? Actually, we prefer not to make an explicit expansion in powers of z, but rather retain the functional form $W(bz)$, etc. For the dipole magnet, the powers of b coincide with the powers of x, y, z and so b and its powers are a handy way of keeping track of which terms to retain. Thus, to first order in b the result is

$$\frac{1}{B_0}\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} W(bZ) + \begin{bmatrix} y \sin\phi \\ x \sin\phi \\ y \cos\phi \end{bmatrix} bW'(bZ) \,. \tag{6}$$

Here, for brevity, we have introduced $Z = z\cos\phi$. Note, we do not have a true quadrupolar field ($\Phi = r^2 \sin 2\theta$) here. The angular multipole in the (q, y) plane is still a dipole; but the act of rotating and displacing has introduced additional terms.

2.1.5 Forces

The next step is to find the forces acting on a moving particle with charge e and position and velocity vectors $\mathbf{X} = [x(t), y(t), v_s t]$ and $\mathbf{U} = d\mathbf{X}/dt = [\dot{x}(t), \dot{y}(t), v_s]$ where $\dot{z} = v_s$. The force is $\mathbf{F} = e\,\mathbf{U} \wedge \mathbf{B}_f$. We neglect the terms in $\dot{x}$ and $\dot{y}$ as small compared with v_s and obtain the components

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = eB_0 v_s \begin{bmatrix} -bx\sin\phi\, W'(bZ) \;-W(bZ) \\ +by\sin\phi\, W'(bZ) \\ 0 \end{bmatrix} \,. \tag{7}$$

Clearly we have focusing/defocusing terms in y and x (respectively) proportional to $b\sin\phi\, W'$ and a dipole bending term in F_x which is independent of the poleface rotation ϕ. The bending persists when there is no poleface rotation, but the focusing disappears when $\phi = 0$.

Notice that for an exit face, the field is decreasing and W' is negative. Consequently, if $\phi > 0$ then the edge is vertically focusing and horizontally defocusing. Because the dipole, $-W$, is negative, this implies the particle trajectory is bent toward the centre of curvature of the arc-shaped reference trajectory in the interior body-field region; which is as expected.

2.1.6 Range of integration

In a moment we shall integrate over the forces to find the change in momenta. Note that while the fringe range is $p = [0, L]$, it is $Z = [-x_{\text{in}} \sin\phi,\; L - x_{\text{out}} \sin\phi]$ with x_{in} and x_{out} constrained to line in the planes $p = 0$ and $p = L$, respectively. However, the difference between integrating Z over the p range versus the Z range is a small second order effect. Indeed, it is often zero because, for example, $W[b(L - x\sin\phi)] \approx W(bL) + W'(bL)(-bx\sin\phi) = 0$ where $W(bL) = 0$ and $W'(bL) = 0$. Corresponding to $Z = [0, L]$, the range of $z = [0, L/\cos\phi]$.

2.1.7 Deflections

It might seem as if the $\tan\phi$ terms appearing in the text book derivations [1, 2, 3] are missing from our equations. However, they emerge when we calculate the deflections from crossing the fringe. Let $(\dot{x}, \dot{y}) = (x', y')v_s$. Then the change in divergence is given by

$$\begin{aligned} dy'/dz &= [by \sin\phi\, W'(bZ)]/\rho && (8)\\ \Delta y' &= \int_0^{L/\cos\phi} dz[by \sin\phi\, W'(bZ)]/\rho && (9)\\ &= \int_0^L \frac{dZ}{\cos\phi}[by \sin\phi\, W'(bZ)]/\rho\ . && (10) \end{aligned}$$

In the limit of a narrow fringe, the vertical position y_0 does not change, and so can be removed from the integral.

$$\Delta y' = (by_0/\rho)\tan\phi \int_0^L W'(bZ)dZ = (y_0/\rho)\tan\phi[W(bL) - W(0)] = -(y_0/\rho)\tan\phi\ . \qquad (11)$$

Similarly in the limit of a thin fringe, the horizontal deflection becomes:

$$\begin{aligned} \Delta x' &= +(x_0/\rho)\tan\phi - \int_0^L W(bZ)dZ/\cos\phi && (12)\\ &= +(x_0/\rho)\tan\phi - (L/\rho)\bar{W}\sec\phi\ , && (13) \end{aligned}$$

where $\bar{W}$ is the average value of W. For example, if $W(s) = \cos^2(s)$ then $\bar{W} = 1/2$.

Mention the reversal of the B_p component at entrance ad exit (because it is directed away from the poleface).

2.2 Nonlinear fringe field

The results above are correct to lowest order. For the moment, consider the case $\phi = 0$. Notice that all we did was to take the simple two-dimensional potential $\Phi(x, y)$ and multiply by a varying longitudinal factor to generate a three-dimensional potential. However, this potential does not satisfy Laplace's equation for free space; to compensate for the curvature introduced in direction z, we must introduce additional curvature in the $r = (x, y)$ directions. Hence to obtain the correct second order terms, we must consider a higher-order potential function.

In two dimensions, the multipolar expansion for a potential in the (x, y)-plane is $\Phi = \sum_n A_n r^n e^{jn\theta}$ where $r = \sqrt{x^2 + y^2}$ and $y/x = \tan\theta$. For three dimensions, the potential is given by the Fourier-Bessel series $\Phi = \sum_k e^{ikz} \sum_n A_{k,n} I_n(kr) e^{jn\theta}$ where

I_n are the hyperbolic Bessel functions. We may use the expansion of $I_1(kr) \approx (kr/2) + (kr/2)^3/2 + \ldots$ as a guide to which terms must be added. Hence the trial potential function for the fringe:

$$\Phi_f = -G(p)y \ + H(p)y(q^2 + y^2) \ . \tag{14}$$

Substitution into Laplace's equation gives the approximate condition $H(p) = G''/8$ and the error in the Laplacian is reduced by the multiplicative factor $\varepsilon = -(q^q + y^2)G^{iv}/(8G^{ii}) \approx (q^2 + y^2)b^2/8$. For the error to be small, the transverse displacements should be smaller than the length of the fringe field (that is $\varepsilon \ll 1$).

2.2.1 Potential, field and coordinate transformations

Starting from the fringe-field potential

$$\Phi_f = B_0 y[-W(bp) \ + \ (q^2 + y^2)(b^2/8)W''(bp)] \ , \tag{15}$$

we repeat the previous steps. First we calculate $\mathbf{B}_f = [B_q, B_y, B_p] = -\nabla\Phi_f$. Next we transform the components according to $[B_x, B_y, B_z] = \mathbf{T}^{-1}\mathbf{B}_f(q, y, p)$. Then we re-write the coordinate dependence according to $[q, y, p] = \mathbf{T}[x, y, z]$. Finally we make an expansion about the reference trajectory $[x, y, z] = [0, 0, z]$; only this time we consider large amplitude motion and take quadratic terms in x, y, z. The *additional* nonlinear terms up to b^2 are

$$\Delta \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = B_0 \begin{bmatrix} y[3x - 5x\cos 2\phi + 2Z\sin\phi] \\ -3y^2 + x^2(4\sin^2\phi - \cos^2\phi) + 2xZ\sin\phi - Z^2\tan^2\phi \\ 2y[(5x\cos^2\phi - z\sin\phi)\tan\phi] \end{bmatrix} \frac{b^2}{8} W''(bZ) \ . \tag{16}$$

2.2.2 Forces and deflections

We then form the vector cross product of the velocity and field to find the components of the force $\mathbf{F} = [F_x, F_y, F_z]$. The *additional* nonlinear forces beyond those given above are $\Delta F_x = -ev_s\Delta B_y$, $\Delta F_y = +ev_s\Delta B_x$ and $\Delta F_z = 0$. By integration of these forces, we find the *additional* deflections arising from the nonlinear terms (assuming the fringe is sufficiently thin that there is no appreciable change in x or y.) The boundary conditions $W'(0) = 0$ and $W'(bL) = 0$ will eliminate many of the terms.

$$\Delta y' \ = \ +(1/4)(y/\rho)\tan\phi \tag{17}$$

$$\Delta x' \ = \ -(1/4)(x/\rho)\tan\phi \ + (1/4)(L/\rho)\bar{W}\sec\phi\tan^2\phi \ . \tag{18}$$

If we consider also that there are large angles (x', y') then there are additional terms appearing in the forces:

$$\Delta \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = eB_0 v_s \begin{bmatrix} (y\cos\phi)y' \\ (y\cos\phi)x' \\ (xx' - yy')\sin\phi \end{bmatrix} bW'(bZ) + eB_0 v_s \begin{bmatrix} 0 \\ 0 \\ x' \end{bmatrix} W(bZ)\ . \tag{19}$$

The resulting additional deflections are

$$\Delta x' = -y'\,(y/\rho) \tag{20}$$

$$\Delta y' = +x'\,(y/\rho) \tag{21}$$

$$\Delta z' = [(yy' - xx')/\rho]\tan\phi + (L/\rho)\bar{W}x'\sec\phi\ . \tag{22}$$

Treatments of the linear effects of the dipole fringe field are widespread, as noted above, and the second order matrix elements are derived in Reference [2]. The equivalent effects for a quadrupole magnet are less widely known, particularly for magnets with rotated entry/exit faces.

3 Quadrupole Fringe Field Effects

The basic configuration is that of a rectangular quadrupole magnet described in terms of a cartesian system $[x, y, z]$ with the z-axis colinear with the zero-field centre-line of the magnet; and $x - z$ and $y - z$ being the horizontal and vertical symmetry planes. Again we consider the possibility of a rotated pole face, and take a system of coordinates $[q, y, p]$ for the fringe-field region.

3.1 Fringe field with rotated pole face

Contrary to the case of the dipole, the quadrupole magnetic field is not rotation symmetric about the y-axis; and when the pole face is not cut perpendicular to the magnet principal axis, z, it is not immediately clear how the exterior fringe field relates to the interior body field. However, some "asymptotic" properties are evident. Sufficiently deep within the interior, the body field is essentially two-dimensional; and sufficiently far from the pole face, the fringe field will be dominated by the field source (namely the four pole pieces) and the "longitudinal" component B_p is directed perpendicular and away from the rotated pole face.

3.1.1 Transition region

The issue then, is how large and how significant is the transition region between these two "asymptotic" zones where the field shapes are relatively simple. To answer that question, we have performed some 3D-modeling of a quadrupole magnet with entry/exit faces rotated about the y-axis. In our model, we took a magnet 10 units in length along its centreline, and having the radius to the hyperbolic pole tips of 3 units. The pole faces were then cut at 10° to the perpendicular. For this model, the 2-D interior body field is established within 1/2 unit of the pole face; and the exterior fringe field approaches the simple 3-D "asymptotic" form within 1 unit distance of the poleface. The finge field falls almost to zero within 5 units distance of the poleface. (This is typical of a quadrupole: the longitudinal extent of the fringe is roughly twice the radius to poletip.) The conclusion is that eliminating the details of the transition region from our analytical model will little compromise our results because the "asymptotic" forms are established within a short distance either side of the pole face.

3.2 Linear fringe field

Let B_1 (tesla/metre) be the quadrupole gradient. In the interior region, the potential and body field are:

$$\Phi = -B_1 xy \qquad \text{and} \qquad [B_x, B_y, B_z] = -\nabla\Phi = [B_1 y, B_1 x, 0]\,. \tag{23}$$

Let $W(bp)$ model the fringe-field fall off, as above. In the exterior region, the potential and fringe field are:

$$\Phi_f = -B_{1f} xy\, W(bp) \quad \text{and} \quad [B_q, B_y, B_p] = -\nabla\Phi_f = B_{1f}[yW(bp), qW(bp), qybW'(bp)]\,. \tag{24}$$

Notice how the longitudinal component B_p has the same form in the $q-y$ plane as the 2-D potential from which it is derived; the angular symmetry is quadrupolar (from the pole face geometry), but the radial ($r = \sqrt{q^2+y^2}$) dependence is quadratic.

3.2.1 Transformation of field and coordinates

The next step is to transform the fringe field components in $[q, y, p]$ into components directed along $[x, y, z]$, and then to investigate continuity of field and potential in the boundary plane $p = 0$. The components are $[B_x, B_y, B_z] = \mathbf{T}^{-1}[B_q, B_y, B_p]$ or, explicitly,

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = B_1 \begin{bmatrix} y\cos\phi\, W(bp) + bqy\sin\phi\, W'(bp) \\ qW(bp) \\ -y\sin\phi\, W(bp) + bqy\cos\phi\, W'(bp) \end{bmatrix}. \tag{25}$$

At the point $[q, y, 0]$, we find $[x, y, z] = [q\cos\phi, y, -q\sin\phi]$. Hence $\Phi = -B_1 qy\cos\phi$ while $\Phi_f = -B_{1f} qy$. The interior field has components $[B_x, B_y, B_z] = B_1[y, q\cos\phi, 0]$ while the exterior field has components $[B_x, B_y, B_z] = B_{1f}[y\cos\phi, q, -y\sin\phi]$.

Clearly these functions are not continuous across the boundary unless $\phi = 0$; and this was to be expected because we have eliminated the transition region and the field rotations that occur there. We adopt a compromise, set $B_{1f} = B_1$ and incur a scale error of order $\cos\phi$. The next step is to re-write $[q, y, p]$ in terms of $[x, y, z]$ using $[q, y, p] = \mathbf{T}[x, y, z]$, leading to $p = (z\cos\phi + x\sin\phi)$ and $q = (x\cos\phi - z\sin\phi)$.

3.2.2 Expansion about reference trajectory

The next step is to make an expansion of this field about the reference trajectory $[x, y, z] = [0, 0, z]$. Again, the question arises for a given order in x, y, what order in z? For the quadrupole, the parameter b is not such a handy tool to regularize the expansion. Again, we pursue the course *not* to expand $W(bZ)$, etc, in powers of Z. But what to do with the mixed powers of x and z arising from q^n? There are two options: (A) accept all terms $x^i y^j$ (irrespective of z^k) that satisfy $i + j = n$ for a calculation of order n; and (B) to eliminate all terms $x^i + y^j + z^k$ that satisfy $i + j + k > n$. Option (A) is less restrictive and induces small corrections to the coefficients obtained under option (B).

We shall take option (A) with n=1; to first order the result is

$$\frac{1}{B_1}\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} y\cos\phi \\ (x\cos\phi - Z\tan\phi) \\ -y\sin\phi \end{bmatrix} W(bZ) - \begin{bmatrix} y\sin\phi\tan\phi \\ x\sin\phi\tan\phi \\ y\sin\phi \end{bmatrix} bZW'(bZ). \tag{26}$$

The next step is to find the forces $\mathbf{F} = e\,\mathbf{U}\wedge\mathbf{B}_f$. As before we neglect terms in $\dot{x}$ and $\dot{y}$ as small compared with v_s, and obtain the components

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = B_1 e v_s \begin{bmatrix} -x[\cos\phi\, W(bZ) - bZ\sin\phi\tan\phi\, W'(bZ)] + Z\tan\phi\, W(bZ) \\ +y[\cos\phi\, W(bZ) - bZ\sin\phi\tan\phi\, W'(bZ)] \\ 0 \end{bmatrix}. \tag{27}$$

Clearly we have focusing/defocusing terms in x and y (respectively). Under the assumption $W \geq 0$, the body field is horizontally focusing. In addition there is a dipole term in F_x which bends in the opposite direction to the horizontal focusing, assuming $\phi > 0$. The dipole term should come as no surprise, because our geometry implies the particle travels off-axis in the exterior quadrupolar fringe field.

3.2.3 Deflections

As above, we may find (thin-lens type) approximate expressions for the deflexions produced by these fields under the condition that the fringe is sufficiently narrow that there is no change in the coordinate values within the fringe. Let us introduce the parameter

$$\rho \equiv \frac{\gamma m_0 v_s}{e(B_1 L)} , \tag{28}$$

which is the analogue of the dipole bending radius only with $B_1 \times L$ replacing B_0. The change in the divergences are:

$$\begin{aligned} \Delta y' &= +(y/\rho)\bar{W} \sec^2\phi && (29) \\ \Delta x' &= -(x/\rho)\bar{W} \sec^2\phi \ + (\bar{L}/\rho) \sec\phi \tan\phi \ . && (30) \end{aligned}$$

Here $\bar{L} = \int_0^L ZW(aZ)dZ \approx L/6$.

3.3 Nonlinear fringe field

To find the second order terms in the fields and forces, we have first to find the higher order correction to the potential function in the region exterior to the quadrupole. As above, we appeal to the Fourier-Bessel expansion for guidance on the correct form. The relevant term for a quadrupole is $I_2(kr)e^{ikz}e^{j2\theta}$ and the radial dependence is $I_2(kr) \approx (kr/2)^2/2 + (kr)^4/6 + \ldots$. Hence the trial potential function for the fringe

$$\Phi_f = -G(p)qy \ + H(p)qy(q^2 + y^2) \ . \tag{31}$$

Substitution into Laplace's equation gives the approximate condition $H(p) = G''/12$ and the error in the Laplacian is reduced by the factor $\varepsilon \approx (q^2 + y^2)b^2/12$. We do not have to re-match the potential on the boundary plane $p = 0$, because the higher order terms are small provided that $\varepsilon \ll 1$. Starting from the fringe field potential

$$\Phi_f = B_1 qy[-W(bp) \ + (q^2 + y^2)(b^2/12)W''(bp)] , \tag{32}$$

we repeat the previous steps of field transformation and coordinate substitution, and expand about the reference trajectory $[x, y, z] = [0, 0, z]$.

3.4 Restrictive 2nd order expansion

We adopt option (B) and eliminate any cubic terms $x^i y^j z^k$ with $i + j + k > 2$. The *additional* nonlinear field is

$$\Delta \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = B_1 \begin{bmatrix} xy \sin 2\phi \\ x^2 \cos\phi \sin\phi \\ xy \cos 2\phi \end{bmatrix} b\, W'(bZ) \ . \tag{33}$$

3.4.1 Forces and deflections

The next step is to find the additional forces: $\Delta F_x = -ev_s \Delta B_y$, $\Delta F_y = +ev_s \Delta B_x$ and $\Delta F_z = 0$. By integration of these forces, we find the *additional* deflections:

$$\Delta x' = +[x^2/(L\rho)] \sin\phi \tag{34}$$

$$\Delta y' = -[2xy/(L\rho)] \sin\phi \; . \tag{35}$$

These higher order terms are of order (x/L) smaller than the quadrupole fringe focusing/defocusing terms; and become important when the amplitude of the horizontal oscillation is comparable with the length of the fringe field.

If we consider that there are large angles, then there are further terms appearing in the forces:

$$\Delta \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = eB_1 v_s \begin{bmatrix} -yy' \sin\phi \\ +yx' \sin\phi \\ (xx' - yy') \cos\phi - x' Z \tan\phi \end{bmatrix} W(bZ) \; . \tag{36}$$

The resulting additional deflections are:

$$\Delta x' = -y' \, (y/\rho) \bar{W} \tan\phi \tag{37}$$

$$\Delta y' = +x' \, (y/\rho) \bar{W} \tan\phi \tag{38}$$

$$\Delta z' = (xx' - yy') \bar{W}/\rho \; - (\bar{L}/\rho) x' \sec\phi \tan\phi \; . \tag{39}$$

3.5 Full 2nd order expansion

We take option (A) and retain all terms in z consistent with $x^i y^j$ and $i + j \leq 2$. The *additional* nonlinear field, compared with equations (26)+(33), is

$$\Delta \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \frac{B_1}{4} \begin{bmatrix} -y(x - 3x\cos 2\phi + Z\sin\phi)\tan\phi \\ x^2 \cos\phi \sin\phi + \tan\phi [y^2 - x\sin\phi(Z + 2\sin\phi) + (Z^2/3)\tan^2\phi] \\ y(Z\sin\phi - 6x\cos^2\phi)\tan^2\phi \end{bmatrix} b^2 Z W''(bZ) \tag{40}$$

3.5.1 Forces and deflections

The *additional* deflections, compared with equations (29, 30), are

$$\Delta y' = -[xy/(L\rho)][(1/2) + \sec^2\phi] \sin\phi \tag{41}$$

$$\begin{aligned}\Delta x' &= +[x^2/(2L\rho)][(1/2)+\sec^2\phi]\sin\phi - [y^2/(4L\rho)]\sec\phi\tan\phi \\ &+ [x/(2\rho)]\bar{W}\tan^2\phi - [\bar{L}/(2\rho)]\sec\phi\tan^3\phi \ .\end{aligned} \tag{42}$$

If we consider again the large angles, then there are further terms beyond those in equation (36) appearing in the forces:

$$\Delta \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = eB_1 v_s \begin{bmatrix} +y' \\ -x' \\ 0 \end{bmatrix} y\frac{b^2Z^2}{4}W''(bZ)\sin\phi\tan^2\phi \ . \tag{43}$$

The resulting additional deflections, compared with equations (29, 30) are:

$$\Delta x' = -y'(y/\rho)\bar{W}\tan\phi[1-(\tan^2\phi)/2] \tag{44}$$

$$\Delta y' = +x'(y/\rho)\bar{W}\tan\phi[1-(\tan^2\phi)/2] \tag{45}$$

$$\Delta z' = (xx'-yy')\bar{W}/\rho - (\bar{L}/\rho)x'\sec\phi\tan\phi \ . \tag{46}$$

References

[1] John Livingood: Principles of Cyclic Accelerators, chapter4, published by D. Van Nostrand Co., Inc., 1961.

[2] David Carey: The Optics of Charged Particle Beams (chapter 8), Volume 6 of the Accelerators and Storage Rings series; published by Harwodd Academic Publishers, 1987.

[3] Helmut Wiedemann: Particle Accelerator Physics: basic principles and linear beam dynamics, chapter 5, Published by Springer Verlag, 1993.