1 Motion in quadrupole body field

There are two main types of combined function magnet: the sectoral and the rectangular. In a sectoral magnet, the majority of the bending is accomplished by the dipole component; and it is appropriate to adopt cylindrical polar coordinates reflecting the geometry of the sector, and to treat the gradient focusing as a perturbation. (See for example TRI-DN-* and its references.) The polar coordinates automatically take care of the rotation (between the entry and the exit faces) of the external cartesian coordinate systems employed for drift spaces.

A rectangular combined function magnet is essentialy a displaced quadrupole, and if the displacement is large enough we may consider a half quadrupole with a magnetic mirror plane. Because of the rectangular geometry, the motion is treated in cartesian coordinates. If the magnet achieves a large bending, then the displacements from the magnetic centre are large as are the entry and exit divergencies. Indeed, some of the longitudinal momentum is rotated into the transverse, and visa versa, at entry and exit, respectively. Thus a correct treatment demands consideration of large amplitude oscillations. Because of the symmetry of a quadrupole, quadratic terms are zero; and our corrections to the usual "linear motion" are cubic terms in the entrance coordinates. We shall study bending in the horizontal plane, in which case the vertical motion is still of small amplitude.

We adopt a cartesian coordinate system $[x, y, Z]$ with x, y horizontal and vertical, respectively, and Z aligned with the symmetry axis of the quadrupole. In a drift region, the particle speed is v_s. The particle coordinates are $\mathbf{x} = [x(t), y(t), v_s t + z(t)]$. The body quadrupole field is $\mathbf{B} = [B_1 y, B_1 x, 0]$. The particle charge and rest mass are e and m_0. The equation of motion is $\gamma m_0 (d^2\mathbf{x}/dt^2) = e(d\mathbf{x}/dt) \wedge \mathbf{B}$. We prefer to write this in terms of divergences and introduce $(d/dt)[x, z] = v_s(d/ds)[x, z] = v_s[x', z']$, where $s = v_s t$. Note, because we consider large divergence x', the angle formed with the reference axis, Z, is $\tan\theta = x'/(1 + z')$. The equations of motion become:

$$(dx'/ds) + k^2 x(1 + z') = 0 \quad (1)$$

$$(dy'/ds) - k^2 y(1 + z') = 0 \quad (2)$$

$$(dz'/ds) + k^2(yy' - xx') = 0 \; . \quad (3)$$

The initial conditions at $s = 0$ are

$$\mathbf{x}_0 = [x_0, x'_0, z_0, z'_0, y_0, y'_0] \; . \quad (4)$$

The last equation (3) can be integrated immediately. In the case of small vertical motions $y^2 \ll x^2$, this becomes

$$z'(t) = z'_0 + (k^2/2)[x^2(t) - x_0^2] \; . \quad (5)$$

For motion confined to the horizontal plane the divergences satisfy the relation

$$(1 + z')^2 + (x')^2 = 1 \quad \text{or} \quad x' = \sin\theta, \; (1 + z') = \cos\theta \; . \quad (6)$$

We shall now find an approximate solution for x, y, z by successive approximations.

1.1 1st Approximation

The first step is to treat the terms in $z'(t)$ as a small perturbation to a dominant motion given by

$$(d/ds)[x', y'] + k^2[+x, -y] = [0, 0] \,. \tag{7}$$

The equation has the well-known solution

$$[x(s), x'(s)] = \mathbf{T}_0[x_0, x_0'] \quad \text{with} \quad \mathbf{T}_0 = \begin{bmatrix} C & S \\ C' & S' \end{bmatrix} \,. \tag{8}$$

Here $C(s), S(s)$ are the principal functions having the properties:

$$C(0) = 1, \quad S(0) = 0, \quad C'(0) = 0, \quad S'(0) = 1, \quad CS' - SC' = 1 \,. \tag{9}$$

There is an analogous solution for y, y'. Explicitly, the principal functions are $C_x(s) = \cos ks$, $S_x(s) = (1/k)\sin ks$ and $C_y(s) = \cosh ks$, $S_y(s) = (1/k)\sinh ks$.

1.2 2nd Approximation

We now restore the perturbation $[W_x, W_y]$:

$$\begin{aligned} (dx'/ds) + k^2 x &= -k^2 x[z_0' + (k^2/2)(x^2 - x_0^2)] \equiv W_x(s) && (10) \\ (dy'/ds) - k^2 y &= +k^2 y[z_0' + (k^2/2)(x^2 - x_0^2)] \equiv W_y(s) \,. && (11) \end{aligned}$$

We treat this as if it were an inhomogeneous equation (with a known right hand side) and solvable by the method of Greens functions. After substituting (8), the perturbations are

$$\begin{aligned} W_x &= -k^2(C_x x_0 + S_x x_0')z'(s) && (12) \\ W_y &= +k^2(C_y y_0 + S_y y_0')z'(s) && (13) \\ z'(s) &= z_0' + (k^2/2)[(C_x x_0 + S_x x_0')^2 - x_0^2)] && (14) \\ z(s) &= z_0 + \int_0^s z'(u)du \,. && (15) \end{aligned}$$

The solution is:

$$x(s) = C_x x_0 + S_x x_0' + \int_0^s G_x(s,u) W_x(u) du \tag{16}$$

$$x'(s) = C_x' x_0 + S_x' x_0' + \int_0^s G_x'(s,u) W_x(u) du \tag{17}$$

$$y(s) = C_y y_0 + S_y y_0' + \int_0^s G_y(s,u) W_y(u) du \tag{18}$$

$$y'(s) = C_y' x_0 + S_y' x_0' + \int_0^s G_y'(s,u) W_y(u) du \ . \tag{19}$$

The Green's function $G(u,v)$ has the properties:

$$G(u,v) = S(u)C(v) - C(u)S(v) \, , \qquad G(u,u) = 0 \ . \tag{20}$$

$$\frac{d}{du} G(u,v) = -S(v)C'(u) \, , + C(v)S'(u) \qquad \frac{d}{du} G(u,v)|_{v=u} = 1 \ . \tag{21}$$

Our working is not yet complete, in the next section we show the need for introducing a third approximation. Explicitly we write

2 Transfer matrix for small oscillations

The above exercise will find any number of trajectories; and we may single out one of them as *the* reference trajectory through a rectangular combined-function magnet for a particular momentum. Then we may ask what is the small amplitude motion about the reference. The equations (*) constitute a set of nonlinear mappings which take the old coordinates (positions and momenta) into new coordinates at later times. Symbolically $x(s) = T_x(s, \mathbf{x}_0)$ and $x'(s) = T_{x'}(s, \mathbf{x}_0) = (\partial/\partial s) T_x$; and similarly for the other coordinates. For brevity we adopt the notation $\mathbf{x}(s) = [x_1, x_2, x_3, x_4, x_5, x_6] = [x, x', z, z', y, y']$, so the mappings may be written $x_i = T_i(s, \mathbf{x}_0)$

If we treat the particular set of initial conditions $\mathbf{x}_0$ as defining a reference trajectory, then the transport of small deviations is given by the matrix of partial derivatives $T_{ij} = \partial T_i(s, \mathbf{x}_0)/\partial x_{j,0}$. The eigenvalues of this matrix give the oscillation frequencies for small amplitude motion about the reference trajectory.

We now give some consideration to the properties of the matrix T_{ij}. The fact that we have made the motion large amplitude in x but not in y, such that $y = y(x_0, y_0)$ but $x \neq x(x_0, y_0)$ implies the 6-D matrix cannot be symplectic. But even if we consider the sub-space $[x, x', z, z']$ the corresponding transfer matrix is not symplectic and nor should it be; because these are *not* canonical coordinates. Nevertheless, the underlying equations are conservative because only magnetic fields are present; and so the determinant of the transfer matrix, $\mathrm{Det}[T_{ij}]$, should be unity. In fact, our expressions for the elements T_{ij} are only approximate, and $\mathrm{Det}[T_{ij}]$ will deviate from unity. The deviation is a measure of the error, which we should seek to minimize. The determinant of the sub-space $[x, x', z, z']$ is

$$\mathrm{Det}[T_{ij}] = 1 + k^4[x_0^2 s^2/2 \; + x_0 x_0' s^3 (2/3) + \ldots] \, . \tag{22}$$

Typically the focal length of the magnet is greater than its physical length (L), and likewise the displacements are smaller than the quadrupole length; that is $kL < 1$ and $x_0 < L$. Consequently $|k^2 x_0 s| < 1$ and the error in (22) is a diminishing power series. The error would be appreciably reduced if the quadratic and cubic error terms (s^2, s^3) could be eliminated.

2.1 3rd Approximation

In large part the error in the determinant arises because the longitudinal motion $[z, z']$ is a lower order approximation than the transverse $[x, x']$. This should be apparent from the fact that we have substituted the unperturbed motion (equation 8) into the expression (5) for z'; whereas x is the integral over the perturbation. The order of approximation is also manifest in the spatial frequencies present; whereas z' contains quadratic powers of C, S, the displacement x contains up to cubic powers of C, S.

The third approximation is to substitute the perturbed x, x', equations (16,17), into z', expression (5). The resulting expression for z' contains powers of x_0, x_0' up to six, which is an unncessarily high order compared with our expressions for x, x'. Consequently, we may truncate terms without compromising the accuracy of the determinant. In practise, we initially retain all terms, evaluate the determinant in a power series in s and then *a posteori* eliminate all terms in $(x_0)^n, (x_0')^m$ which do not contribute to the leading term in s^l of the determinant. The leading term is $l = 4$ and we eliminate all $(x_0)^n$ and $(x_0')^n$ with $n > 3$, but retain $(x_0)^2 (x_0')^2$. The determinant is

$$\mathrm{Det}[T_{ij}] = 1 \; + (1/12) k^4 s^4 (z_0')^2 \; + (5/36) k^6 s^5 (x_0 x_0' z_0') \; + \ldots \tag{23}$$

The identity (6) involving z' and x' implies that $z_0' \approx -(x_0')^2/2$ and so the error in the determinant is of order $(k s x_0')^4/12$. For example, if the total bending produced by the quadrupole is 60°, then the entry and exit angles are each ±30° and the error (compared with unity) is $(kL)^4/192$ because $x' = \sin\theta = \pm 1/2$.

There is an important subtlty about the transfer matrix for small oscillations $[x, x']$, it is that $[x, x']$ are not perpendicular to the reference trajectory unless $z' = 0$. In this respect, our treatment differs from that given by Carey[**?**] who takes a variable (but locally cartesian) coordinate system that follows a curved reference trajectory. Further Carey concentrates on the fringe field effects, not the body field. In order to find the transverse motion (X, X') perpendicular to the reference trajectory, one must perform a rotation.

$$
\begin{aligned}
X &= x\cos\theta - z\sin\theta = x(1+z') - zx' &\quad (24)\\
X' &= x'\cos\theta - z'\sin\theta = x'(1+z') - z'x'\ . &\quad (25)
\end{aligned}
$$

3 Reference Trajectory

Here we record explicit results for the trajectory.

$$
\begin{aligned}
x(s) &= \cos ks\left[x_0 - \frac{x_0}{64}[k^2x_0^2 - 3(x_0')^2] - \frac{sx_0'}{16}[k^2x_0^2 - 3(x_0')^2 - 8z_0']\right]\\
&+ \frac{1}{k}\sin ks\left[x_0' - \frac{x_0'}{64}[5k^2x_0^2 + 9(x_0')^2 + 32z_0'] + \frac{sx_0k^2}{16}[k^2x_0^2 - 3(x_0')^2 - 8z_0']\right]\\
&+ \cos 3ks\frac{x_0}{64}[k^2x_0^2 - 3(x_0')^2] + \frac{1}{k}\sin 3ks\frac{x_0'}{64}[3k^2x_0^2 - (x_0')^2]\ . \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
x'(s) &= \cos ks\left[x_0' + \frac{3x_0'}{64}[-3k^2x_0^2 + (x_0')^2] + \frac{sx_0k^2}{16}[k^2x_0^2 - 3(x_0')^2 - 8z_0']\right]\\
&+ k\sin ks\left[x_0 + \frac{x_0}{64}[5k^2x_0^2 - 15(x_0')^2 - 32z_0'] + \frac{sx_0'}{16}[k^2x_0^2 - 3(x_0')^2 - 8z_0']\right]\\
&+ \cos 3ks\frac{3x_0'}{64}[3k^2x_0^2 - (x_0')^2] - k\sin 3ks\frac{3x_0}{64}[k^2x_0^2 - 3(x_0')^2]\ . \qquad (27)
\end{aligned}
$$

$$
\begin{aligned}
z(s) &= \cos 2ks\left[\frac{x_0x_0'}{336}[23(x_0')^2 - 7(12(1-z_0') + (z_0')^2)] + \frac{s}{672}[-84(x_0')^2z_0' + k^2x_0^2(69(x_0')^2 + 7z_0'(12 - z_0'))]\right.\\
&+ \left.\frac{s^2k^2}{384}x_0x_0'z_0'[8z_0' - 9k^2x_0^2]\right]\\
&+ \cos 4ks\left[\frac{x_0x_0'}{21504}[184(x_0')^2 - 63k^2x_0^2(4 - z_0')]\right] + k\sin 4ks\left[-\frac{69}{3584}x_0^2(x_0')^2 + \frac{3s}{512}k^2x_0^3x_0'z_0'\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{1}{k}\sin 2ks\left[\frac{1}{192}[k^2x_0^2(24-12z_0'+(z_0')^2)+12(x_0')^2(-2+3z_0')]\right.\\
&+ \quad \left.\frac{sk^2}{5376}x_0x_0'[63k^2x_0^2(-4+z_0')+8(69(x_0')^2+28z_0'(6-z_0')]-\frac{s^2k^4}{96}x_0^2(z_0')^2\right]\\
&+ \quad z_0+\frac{3k^2}{1024}x_0^3x_0'(4-z_0')+\frac{x_0x_0'}{2688}[-207(x_0')^2+56(12(1-z_0')+(z_0')^2)]\\
&+ \quad s\left[z_0'+\frac{(x_0')^2}{4}(1-z_0')-\frac{k^2x_0^2}{896}[224+23(x_0')^2]\right]+\frac{s^2k^2}{48}x_0x_0'(z_0')^2+\frac{s^3k^4}{144}x_0^2(z_0')^2\ . \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
z'(s) \quad &= \quad \cos 2ks\left[\frac{1}{224}[k^2x_0^2(56+23(x_0')^2)-56(x_0')^2(1-z_0')]\right.\\
&- \quad \left.\frac{k^2x_0x_0'}{2688}\left[63k^2x_0^2(4+z_0')-8[69(x_0')^2+14z_0'(12-z_0')]\right]-\frac{s^2k^4}{48}x_0^2(z_0')^2\right]\\
&+ \quad \cos 4ks\left[-\frac{69}{896}k^2x_0^2(x_0')^2+\frac{3s}{128}k^4x_0^3x_0'z_0'\right]-k\sin 4ks\left[\frac{x_0x_0'}{10752}[-504k^2x_0^2+368(x_0')^2+63k^2x_0^2z_0']\right]\\
&+ \quad k\sin 2ks\left[\frac{1}{5376}[-8x_0x_0'[23(x_0')^2-168(2-z_0')-63k^2x_0^3x_0'(4-z_0')]\right.\\
&+ \quad \left.\frac{s}{112}[28(x_0')^2z_0'-k^2x_0^2(23(x_0')^2+28z_0')]+\frac{s^2k^2}{192}kx_0x_0'z_0'(9k^2x_0^2-8z_0')\right]\\
&+ \quad z_0'+\left[\frac{(x_0')^2}{4}(1-z_0')-\frac{k^2x_0^2}{896}[224+23(x_0')^2]\right]+\frac{s}{24}k^2x_0x_0'(z_0')^2+\frac{s^2}{48}k^4x_0^2(Z_0')^2\ . \qquad (29)
\end{aligned}
$$