Current disribution generating a given magnetic field law

Magnetic dipole field generated with a flat distribution of conductors

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Introduction

In the frame of a project aiming to design a fixed field accelrating gradient synchrotron, the main magnetic elements should produce a non-linear magnetic dipole field :

$$B(r) = B_0\left(\frac{r}{r_0}\right)^k F\left(\theta - \tan(\xi)\ln\left(\frac{r}{r_0}\right)\right)$$

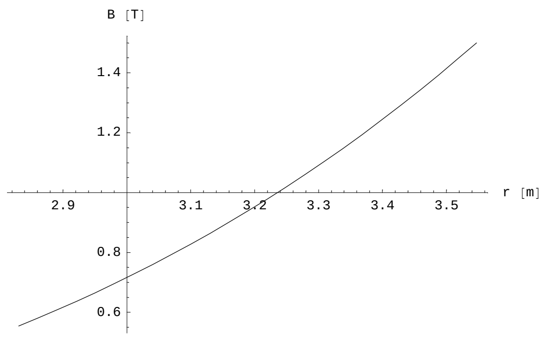
where ξ is a spiral angle,if needed.

Description

Description of the system is strictly 2D. The length where the field matches the field law is called the "good field regon".

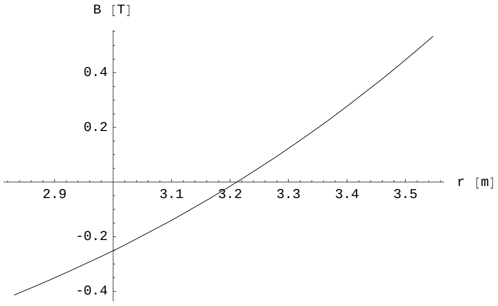
The case studied here considers $F\left(\theta - \xi\ln\left(\frac{r}{r_0}\right)\right)$=1. Hence the current intensities only depend on r (figures from RACCAM project) :

```
Plot[1.5(r/3.54688)^4.415,{r,2.83078,3.54688},AxesLabel->{"r [m]","B [T]"}];
```



A technical solution would suppose to produce this field law with the help of a main uniform magnetic so that the current intensities are minimized.

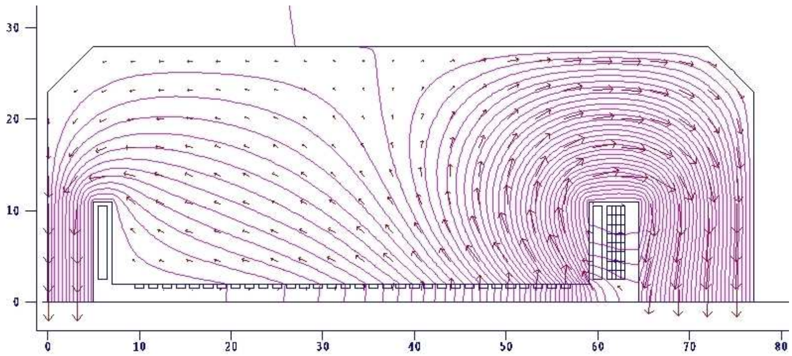
```
Plot[1.5 (r / 3.54688)^4.415 - 0.9674458, {r, 2.83078, 3.54688}, AxesLabel → {"r [m]", "B [T]"}];
```

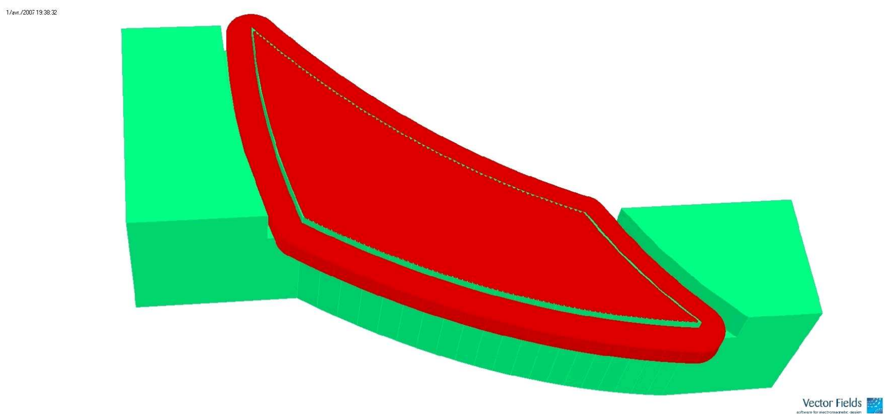


Introduction

Example of geomeries

2d finite element simulation of a dipole from the spiral injection ring KURRI-ADS with POISSON(dimensions in cm)





Introduction

Principle

To calculate the field, we simply use the Biot and Savart law. $\{\xi,\eta\}$ are the conductor coordinates and $\{x, 0\}$ the observation coordinates in the median plane :

$$B_x(x, \xi, \eta) = \frac{\mu_0 I}{4\pi} \frac{\eta}{(x-\xi)^2+\eta^2} \qquad B_x(x, \xi, \eta) = \frac{\mu_0 I}{4\pi} \frac{x-\xi}{(x-\xi)^2+\eta^2}$$

For N conductor :

$$B_x(x, \{\xi_i, \eta_i\}) = \frac{\mu_0}{4\pi} \sum_{i=1}^{N} I_i \frac{\eta_i}{(x-\xi_i)^2+\eta_i^2} \qquad B_y(x, \{\xi_i, \eta_i\}) = \frac{\mu_0}{4\pi} \sum_{i=1}^{N} I_i \frac{x-\xi_i}{(x-\xi_i)^2+\eta_i^2}$$

The system is then reduced to a simple linear matrix equation $A_{x,y} \times I + B_y = 0$

For P observation points and N conductors, the matrices write as follow :

$$B_{x,y} = \begin{pmatrix} B_{x,y1} \\ B_{x,y2} \\ \vdots \\ B_{x,yP} \end{pmatrix} \qquad I = \begin{pmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{pmatrix}$$

$$A_y = \begin{pmatrix} \frac{x_1-\xi_1}{(x_1-\xi_1)^2+(y_1-\eta_1)^2} & \cdots & \frac{x_1-\xi_N}{(x_1-\xi_N)^2+(y_1-\eta_N)^2} \\ \vdots & \ddots & \vdots \\ \frac{x_P-\xi_1}{(x_P-\xi_1)^2+(y_P-\eta_1)^2} & \cdots & \frac{x_P-\xi_N}{(x_P-\xi_N)^2+(y_P-\eta_N)^2} \end{pmatrix} \qquad A_x = \begin{pmatrix} \frac{\eta_1}{(x_1-\xi_1)^2+(y_1-\eta_1)^2} & \cdots & \frac{\eta_N}{(x_1-\xi_N)^2+(y_1-\eta_N)^2} \\ \vdots & \ddots & \vdots \\ \frac{\eta_1}{(x_P-\xi_1)^2+(y_P-\eta_1)^2} & \cdots & \frac{\eta_N}{(x_P-\xi_N)^2+(y_P-\eta_N)^2} \end{pmatrix}$$

Matrix A dimensions are P×N : the number of observation points is often chosen greater than the number of conductors, hence the system is overdetermined.

To reduce the system size we simply multiply by the transpose of A :

$$A^T A I = A^T B$$

For any of the studied geometries, the matrix rank is always equal to the number of columns : there always should exist a unique solution.

Further remarks

Currents calcuculated are approximations : the influence of iron and the geometry of the yoke are not taken into account.

Further iterations are necessary using a finite element model.

Series of Conductors

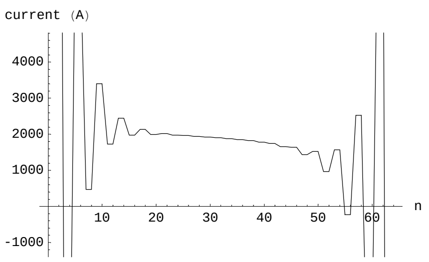
With interdistance gap

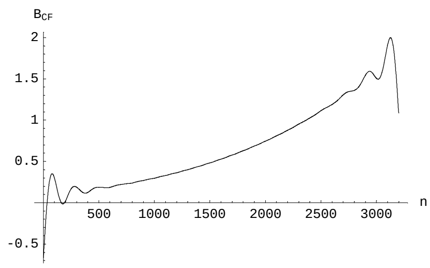
To minimize the number of conductors used, and if field precision requirements allow it, field oscillation can be left at a certain amount by leaving space between conductors.

If matrix condition number of A is sufficiently low, the system is solved directly using the normal equation $A^T A I = A^T B$.

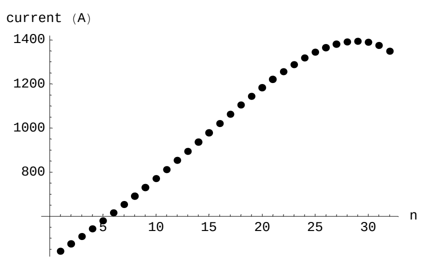
If the model extent is limited to the "good field region", extremities can strongly influence the results by producing current oscillations.

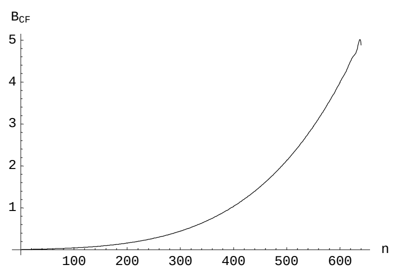
If the number of observation points is equal to the number of conductors, the resulting intenisties are the same for any conductor, with opposite sign at opposite positions with respect to the median plane.





The model length is limited to the "good field region". The number of oservation points is increased to a large value.





The model length is threee times "good field region".The number of oservation points is minimized and precision is far better.

These two series of curves that te model more depending on the geometry of the system than on the number of observation points.

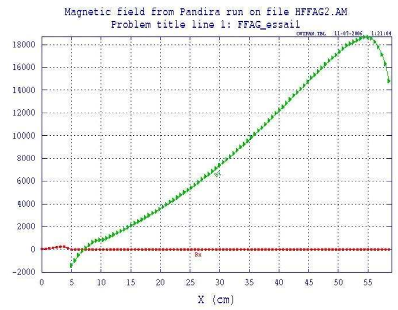
Here the matrix condition number is lower than 1000.

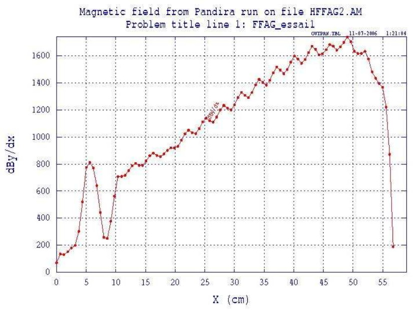
Series of conductors

Comparison to finite elements simulation

A simple 2D finite element model with POISSON of the KURRI-ADS spiral dipole.

Calculated current intensities are directly applied to the real geometry in 2D.





Maximum and minimum field are close to those expected with the theoretical field law

Series of stripe conductors

To minimize field oscillations, conductors interdistance is reduced to zero by increasing conductors width.

Geometry parameters of conductors are included : width and thickness.

Thickness is used in the final extent of field oscillations smoothing : a conducting sheet instead of conductors with variable thickness.

The matrix contains integrals for each stripe with $p_j < \xi < q_j$ at each observation point $x_i = \int_{p_j}^{q_j} f_i(\xi)\, d\xi$ (P observation points and N stripes).

```
x =.; p =.; q =.; ξ =.; η =.; h =.; g =.; p = Table[{p1, p2, p3, p4}];
q = Table[{q1, q2, q3, q4}]; f = Table[{f1[ξ, η], f2[ξ, η], f3[ξ, η], f4[ξ, η]}];
Table[ Integrate[ f[[j]], {ξ, p[[i]], q[[i]] }, {η, h[ξ], h[ξ] + e} ], {j, 1, 4},
  {i, 1, 4}] // MatrixForm
```

$$\begin{pmatrix} \int_{p1}^{q1}\int_{h[\xi]}^{e+h[\xi]} f1[\xi,\eta]\,d\eta\,d\xi & \int_{p2}^{q2}\int_{h[\xi]}^{e+h[\xi]} f1[\xi,\eta]\,d\eta\,d\xi & \int_{p3}^{q3}\int_{h[\xi]}^{e+h[\xi]} f1[\xi,\eta]\,d\eta\,d\xi & \int_{p4}^{q4}\int_{h[\xi]}^{e+h[\xi]} f1[\xi,\eta]\,d\eta\,d\xi \\ \int_{p1}^{q1}\int_{h[\xi]}^{e+h[\xi]} f2[\xi,\eta]\,d\eta\,d\xi & \int_{p2}^{q2}\int_{h[\xi]}^{e+h[\xi]} f2[\xi,\eta]\,d\eta\,d\xi & \int_{p3}^{q3}\int_{h[\xi]}^{e+h[\xi]} f2[\xi,\eta]\,d\eta\,d\xi & \int_{p4}^{q4}\int_{h[\xi]}^{e+h[\xi]} f2[\xi,\eta]\,d\eta\,d\xi \\ \int_{p1}^{q1}\int_{h[\xi]}^{e+h[\xi]} f3[\xi,\eta]\,d\eta\,d\xi & \int_{p2}^{q2}\int_{h[\xi]}^{e+h[\xi]} f3[\xi,\eta]\,d\eta\,d\xi & \int_{p3}^{q3}\int_{h[\xi]}^{e+h[\xi]} f3[\xi,\eta]\,d\eta\,d\xi & \int_{p4}^{q4}\int_{h[\xi]}^{e+h[\xi]} f3[\xi,\eta]\,d\eta\,d\xi \\ \int_{p1}^{q1}\int_{h[\xi]}^{e+h[\xi]} f4[\xi,\eta]\,d\eta\,d\xi & \int_{p2}^{q2}\int_{h[\xi]}^{e+h[\xi]} f4[\xi,\eta]\,d\eta\,d\xi & \int_{p3}^{q3}\int_{h[\xi]}^{e+h[\xi]} f4[\xi,\eta]\,d\eta\,d\xi & \int_{p4}^{q4}\int_{h[\xi]}^{e+h[\xi]} f4[\xi,\eta]\,d\eta\,d\xi \end{pmatrix}$$

Field components :

$$\rightarrow \text{vertical} : B_{y,i}(\xi) = \int \left(-\text{ArcTan}\left[\frac{g}{x-\xi}\right] + \text{ArcTan}\left[\frac{h[\xi]}{x-\xi}\right]\right) d\xi$$

$$\rightarrow \text{horizontal} : B_{x,i}(\xi) = \frac{1}{2}\int (\text{Log}[g^2 + (x-\xi)^2] - \text{Log}[(x-\xi)^2 + h[\xi]^2])\, d\xi$$

Thickness of conductors can vary with the intensity if current

Width of the conductors can vary in order to allow sharp variaton of the field where needed (for great values of k).

Series of stripe conductors

Geometry dependence

This calculation strongly depends on the geometry of the system : matrix condition number can easily reach very large values showing that the linear system is ill-conditionned.

$A^T.A$ can be more ill-conditionned than A.

More sofisticated technics are used in these cases : least square solutions and other matrix norms minimization have been tested (QR decomposition, pseudoinverse, Cholesky decomposition, all I found in the *Mathematica* help)

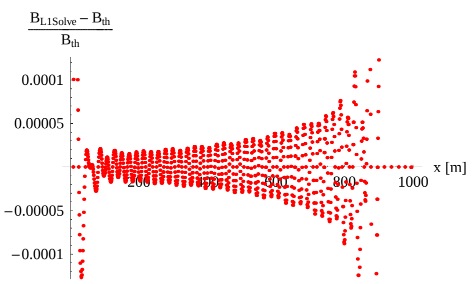
Least square solutions are included in the function LinearSolve.

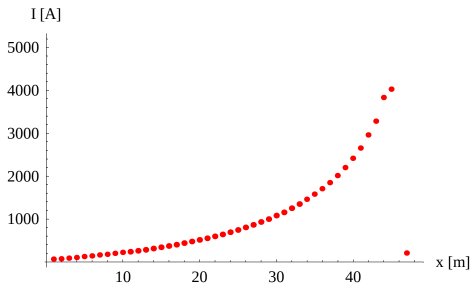
Here I merged the two methods using 1-norm and ∞-norm with the proposed algorithms. It seems these kind of technics preserve important properties of the system.

The technics aims to minimize 1-norm : $\text{Min} \; \| A x - b \|_1 = \text{Min} \sum_i (A x - b)_i$

Example

Results obtained with the 1-norm minimization give the best accuracy (∞-norm is close to this accuracy)





Series of stripe conductors

Singular case : thickness very small compared to width

Initial formulas :

$$B_{x,i}(\xi) = \frac{1}{2} \int (\text{Log}[g^2 + (x-\xi)^2] - \text{Log}[(x-\xi)^2 + h[\xi]^2]) \, d\xi$$

$$B_{x,i}(\xi) = \frac{1}{2} \int (\text{Log}[g^2 + (x-\xi)^2] - \text{Log}[(x-\xi)^2 + h[\xi]^2]) \, d\xi$$

From them, it can be deduced that:

→ The thickness deasappears from the horizontal component and can be easily integrated:

$$B_{x,i}(\xi) = \frac{1}{2} \left(-2 \, g \, \text{ArcTan}\left[\frac{x-\xi}{g}\right] - (x-\xi) \, \text{Log}\left[\frac{g^2 + (x-\xi)^2}{(x-\xi)^2}\right] \right)$$

→ The vertical component has linear dependence on the thickness

$$B_{y,i}(\xi) = \int -\frac{(x-\xi) \, h[\xi]}{g^2 + (x-\xi)^2} \, d\xi$$

Following this, two situations appear :

→ if h[ξ] has polynomial expression, the integration is analytical,

→ if h[ξ] has any expression, the integration is numerical.

Summary

We wrote a series of functions which generate the matrix A, covering the folllowing cases : conductor, stripe with variable thickness and width, sheets.

Accuracy of the calculation is sufficientdepends on the conditionning of the matrix A

Provided that the appropriate numerical method is used (partly available in Mathematica), approximate current distribution is well calculated

Aims for future calculations :
- include matrix norms minimization in the current loops mehod,
- include leakage flux from empirical formulas (J.-B. Lagrange doing this work)

⇒ 3d finite element simulations still have do be brought out : to confirm or infirm what current loops let appear and evaluate the correction of the current intensities to .