Gravitational waves

Theory, detection principles and VIRGO

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Context

- Gravity and General Relativity
- Linearized gravity
- Gravitational waves
- Generation of gravitational waves
- Scientific goals of a detection

The full calculations can be found, for example, in :

"General Relativity", M.P. Hobson, G. Efstathiou and N. Lasenby Cambridge University Press
"General Relativity", R.M. Wald, The University of Chicago Press

A very nice general public introductory book :

"A Journey into Gravity and Space-time", J. A. Wheeler, Scientific American Library

How does it work ?

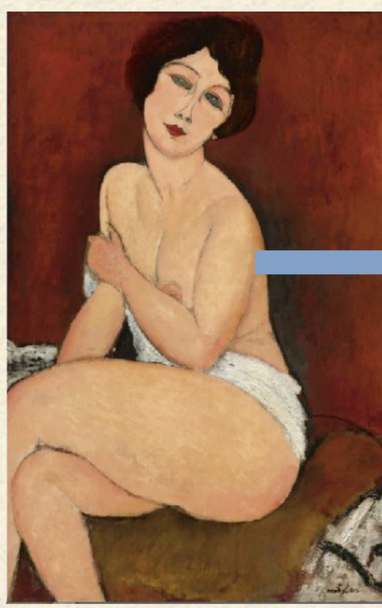
- One of the main ingredients : gravity

- Contributes to the birth, life and death
  - of stars
  - of groups of stars (galaxies and galaxy clusters)
  - and of the universe

How does gravity work ?

- Newton :

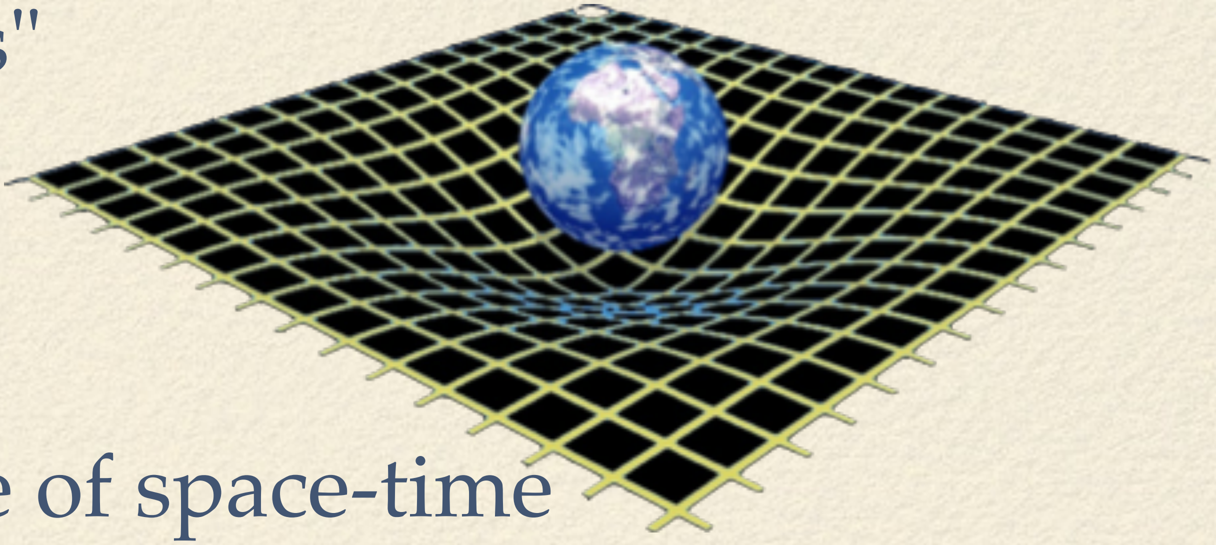
$$\vec{F} = G \cdot m_1 m_2 \cdot \frac{1}{r^2} \cdot \vec{u}$$

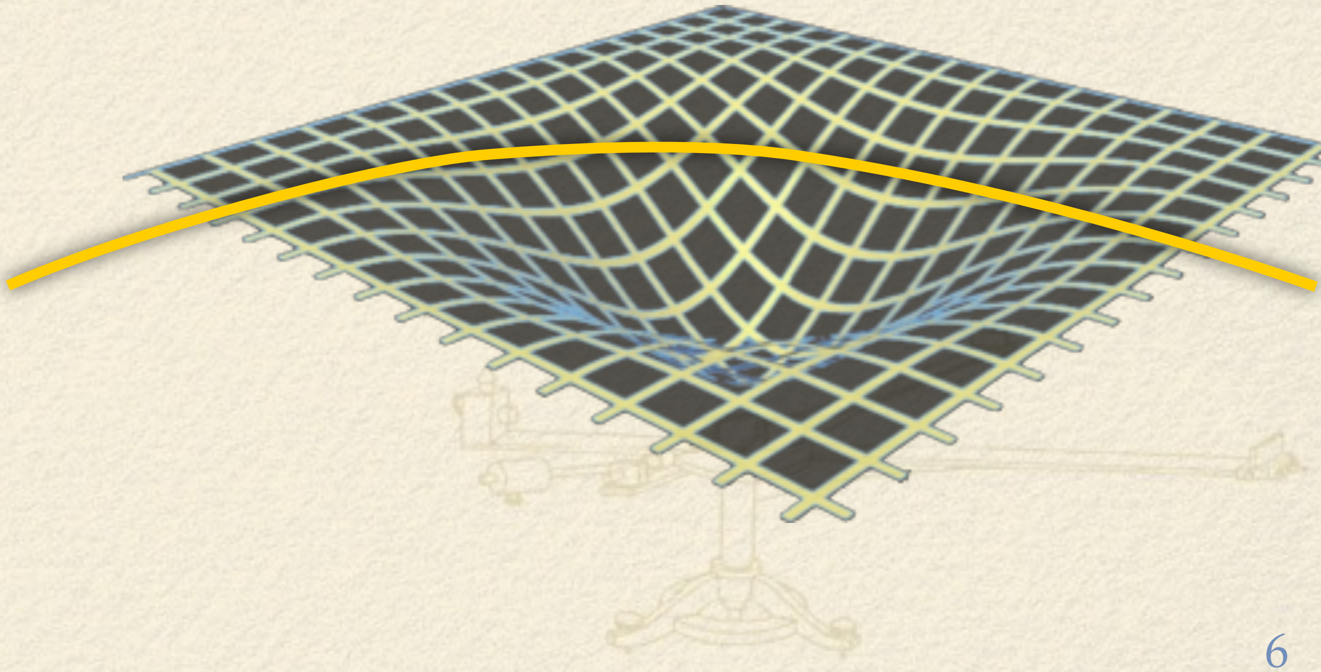


$$\vec{F} \qquad -\vec{F}$$

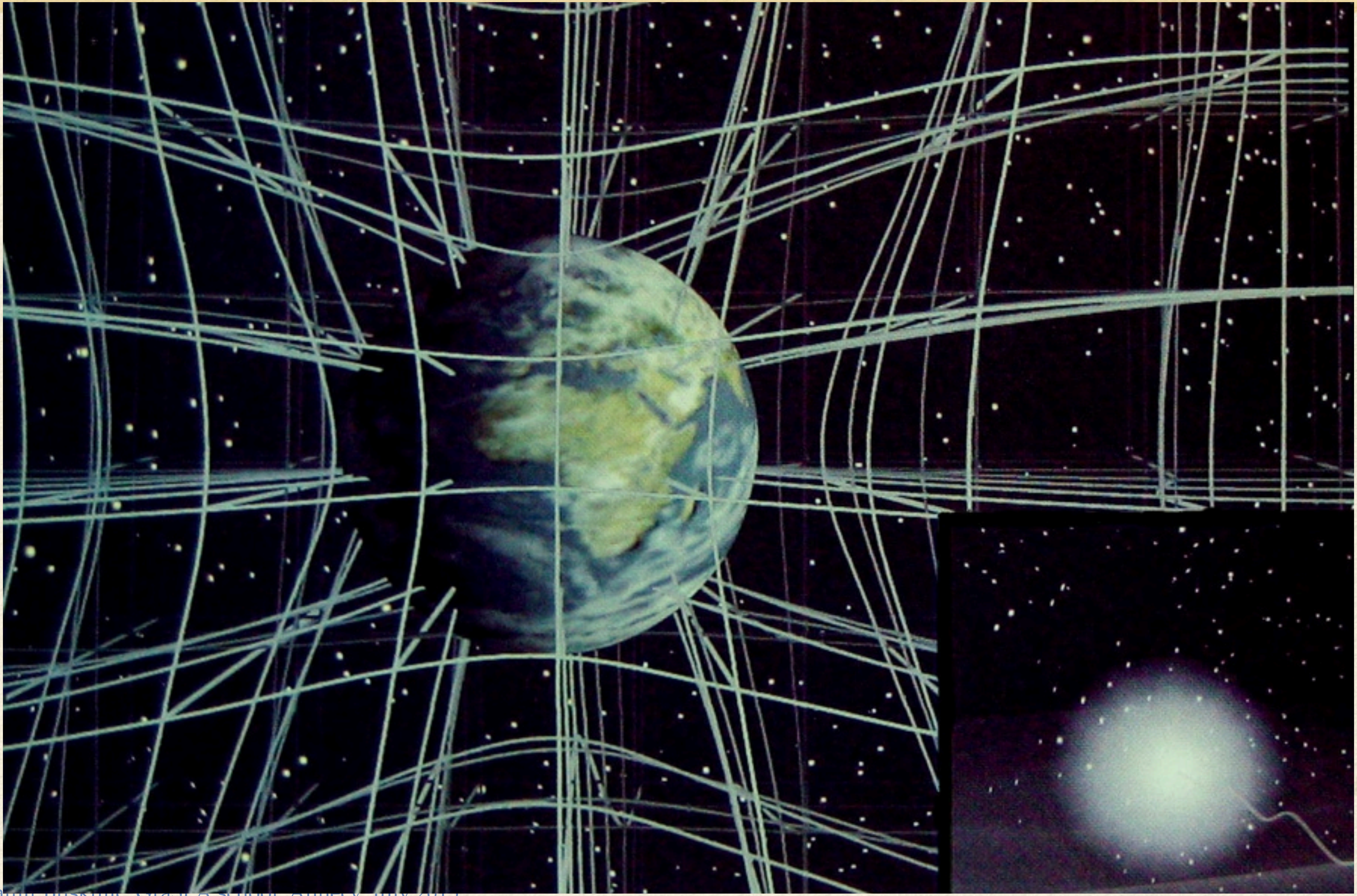
How does gravity work ?

- Experiences show that this is not a complete picture
- Einstein : «General Relativity» (GR) theory
  - A mass "bends" and "deforms" space-time
  - The trajectory of a mass is influenced by the curvature of space-time





How does gravity work ?



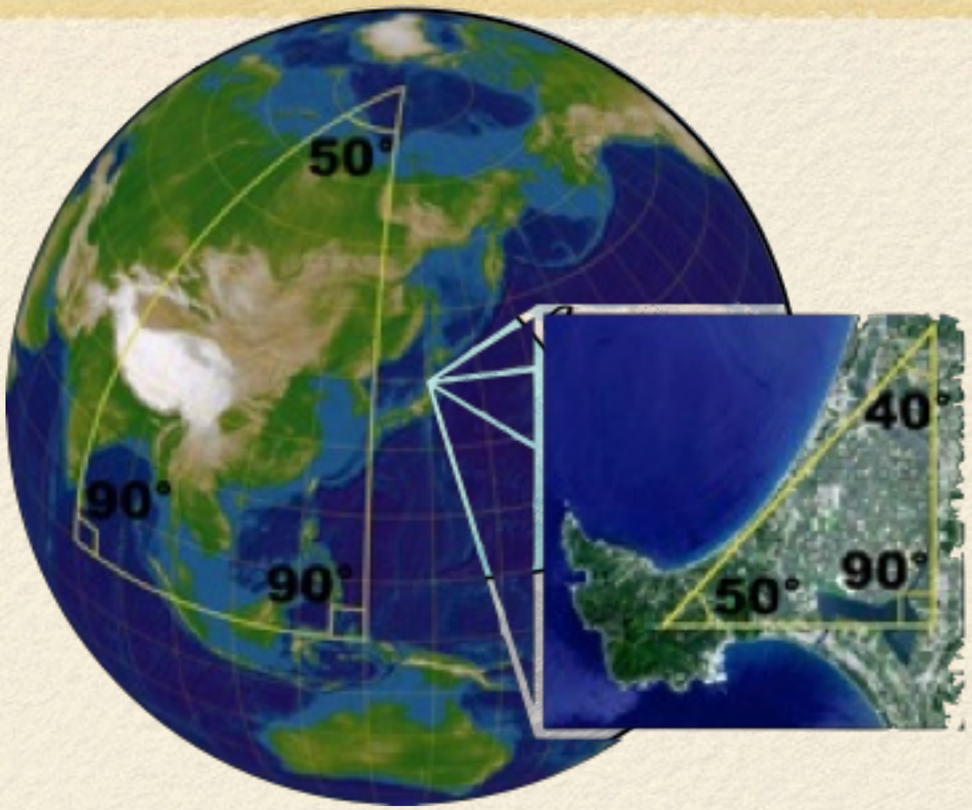
"Curved" space-time

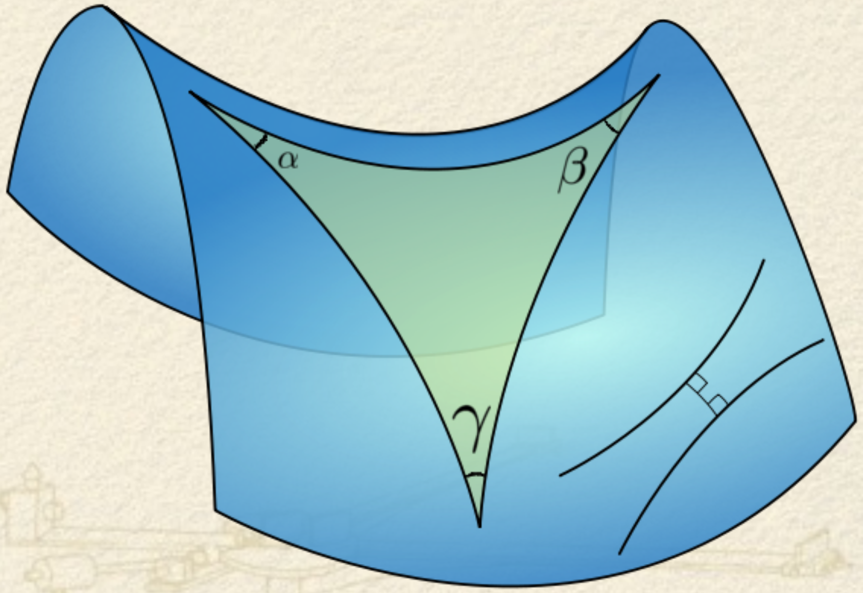
- What is a curved space ? (= "manifold")
  - examples : sphere, sadle
- Can we measure curvature ?
  - we cannot see our space from "outside"
  - but we can measure angles
  - the sum of the angles of a triangle is not always equal to π !
  - positive curvature

$$\sum \text{angles} = \alpha + \beta + \gamma > \pi$$

  - negative curvature

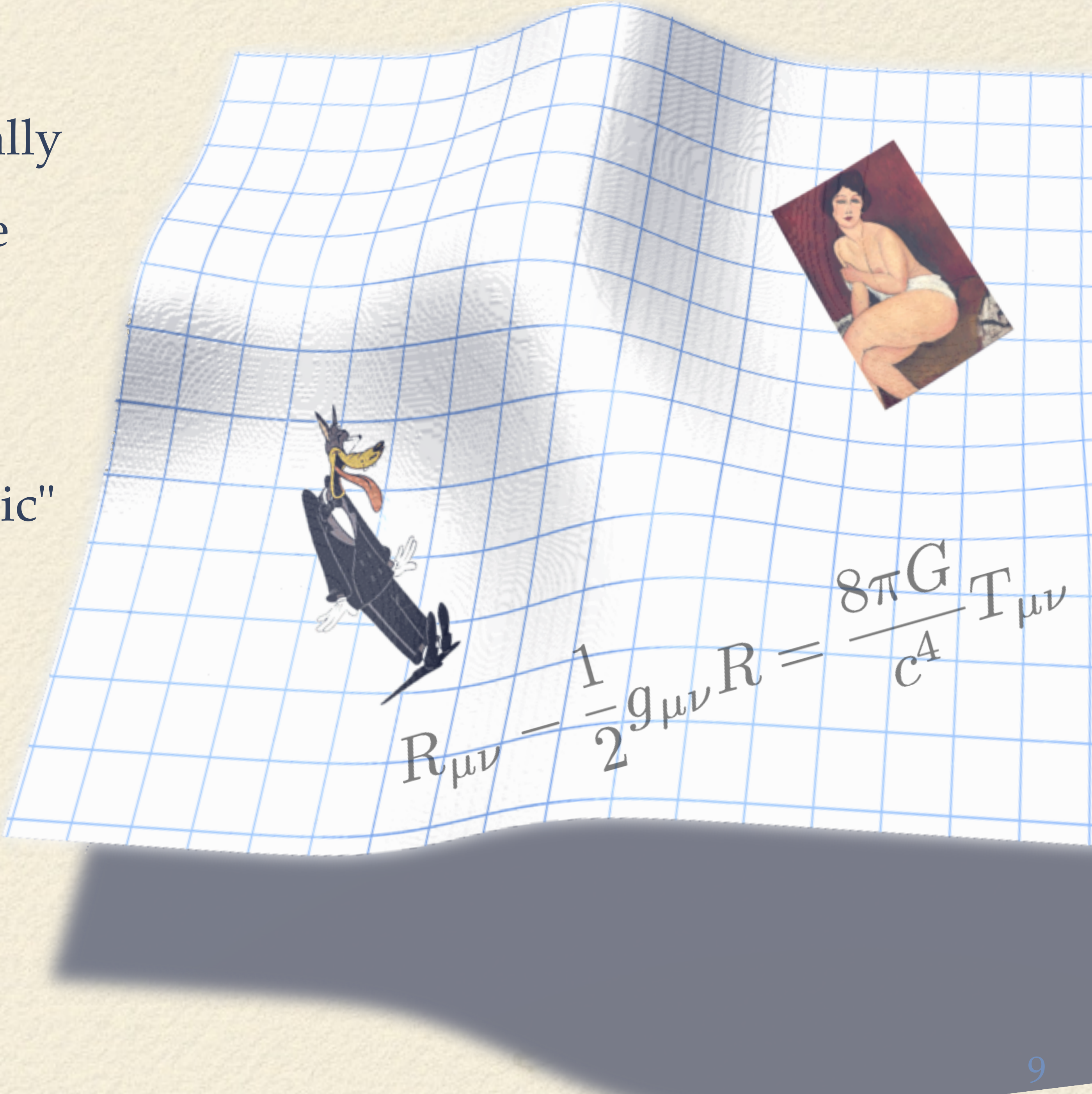
$$\sum \text{angles} = \alpha + \beta + \gamma < \pi$$





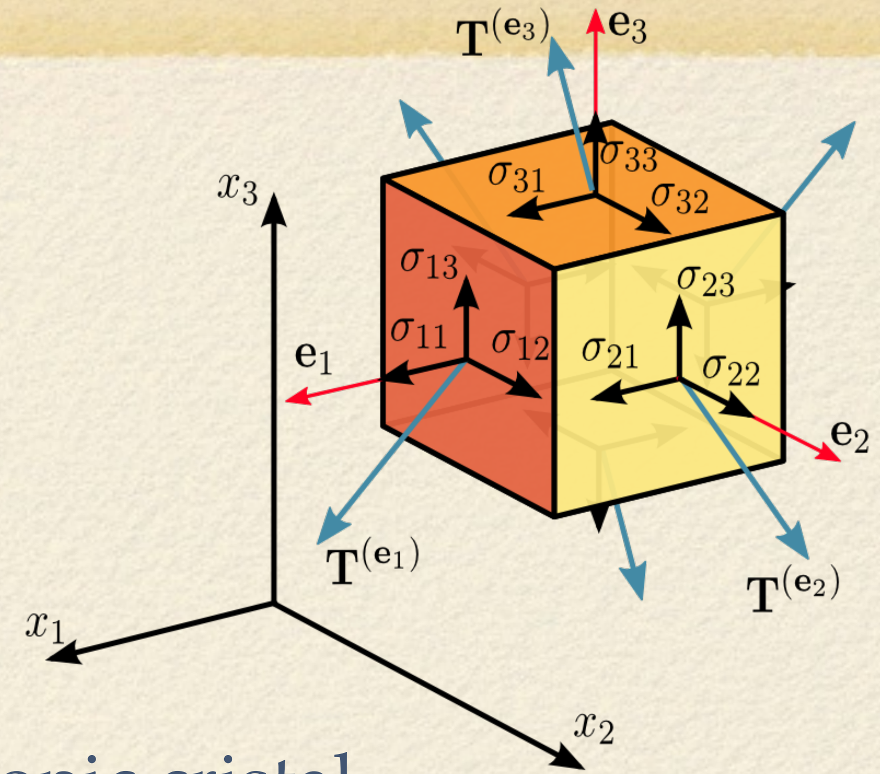
Curvature of space-time

- Newton : space is Euclidian (flat) and time is universal
  - flat space-time !
- General Relativity
  - space is curved and time is defined locally
  - one cannot go "out" to see the curvature
    - "intrinsically" curved space
    - intrinsic curvature
  - go straight (free fall) = follow a "geodesic"
  - note that the time is also curved !
  - as a first approximation, finds the results (trajectories) of newtonian mechanics



"Reminder" about tensors

- Tensor = mathematical object
- Does not depend on the coordinate system
- Extends the notion of vector
- In a specific coordinate system,
  multidimensional array
- Example : electrical conductivity of an anisotropic cristal



$$j^i = \sigma^i_j E^j$$

- Note : summation is implicit over repeated indices
  (Einstein convention)

$$\sigma^i_j E^j \equiv \sum_j \sigma^i_j E^j$$

The metric

- In space-time, measure
  - the distance between two points
  - the angle between two vectors
- Measure of the distance between two infinitesimally close events in spacetime
- Need a "metric", start from the "line element" seen in special relativity :

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \quad \text{with c=1}$$

- Which can be written

$$\boxed{ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta}$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad dx^0 = dt, \quad dx^1 = dx, \quad dx^2 = dy, \quad dx^3 = dz$$

- is the metric of a flat spacetime, the Minkowski spacetime, used in special relativity

$\eta_{\mu\nu}$

The metric

- But the space is not flat !
- The metric can be general : $g_{\mu\nu}$
- It contains all information about spacetime curvature
- It is a rank 2 tensor
- The curvature is also defined by another tensor, which depends on $g_{\mu\nu}$, the Ricci tensor $R_{\mu\nu}$
- But what generates curvature of spacetime ?

The Einstein field equations

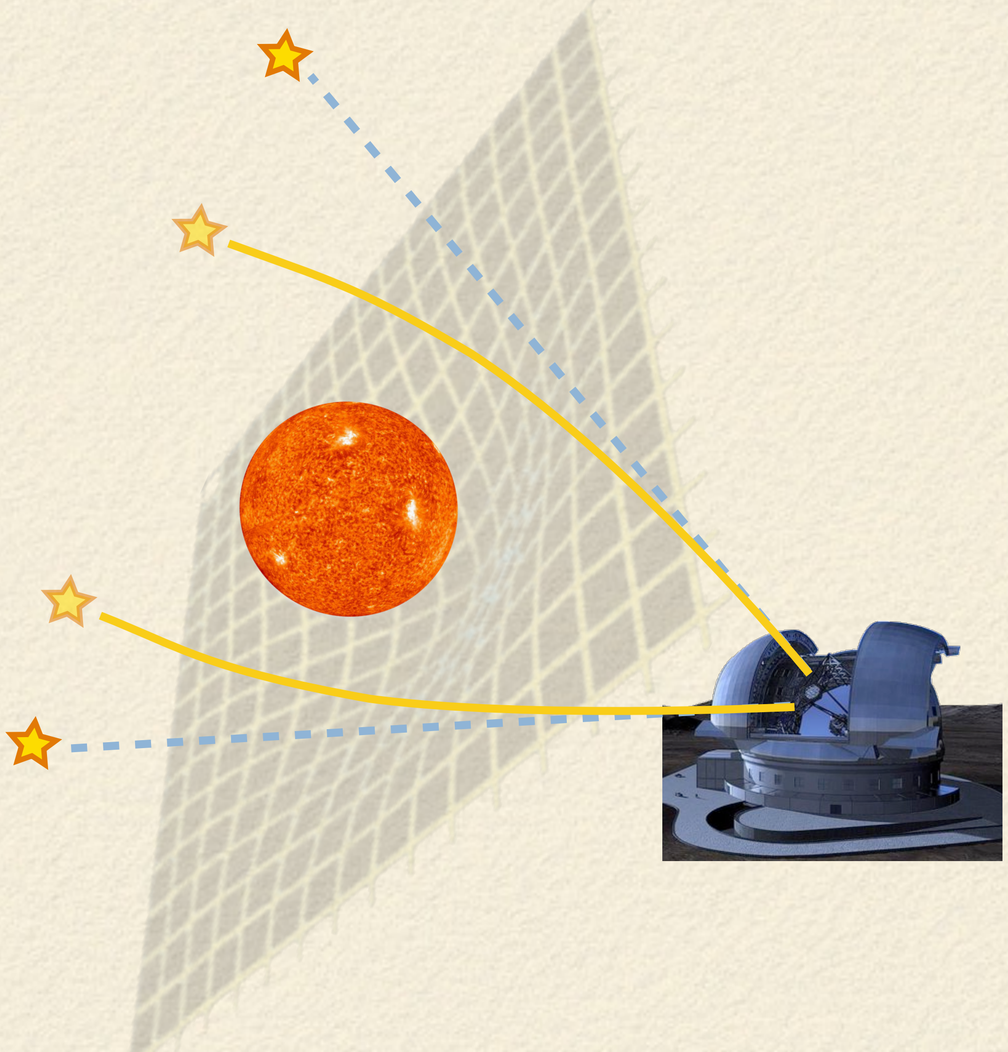
- Answer : the energy-momentum content of spacetime !
  - this includes mass
- Einstein Field Equations :

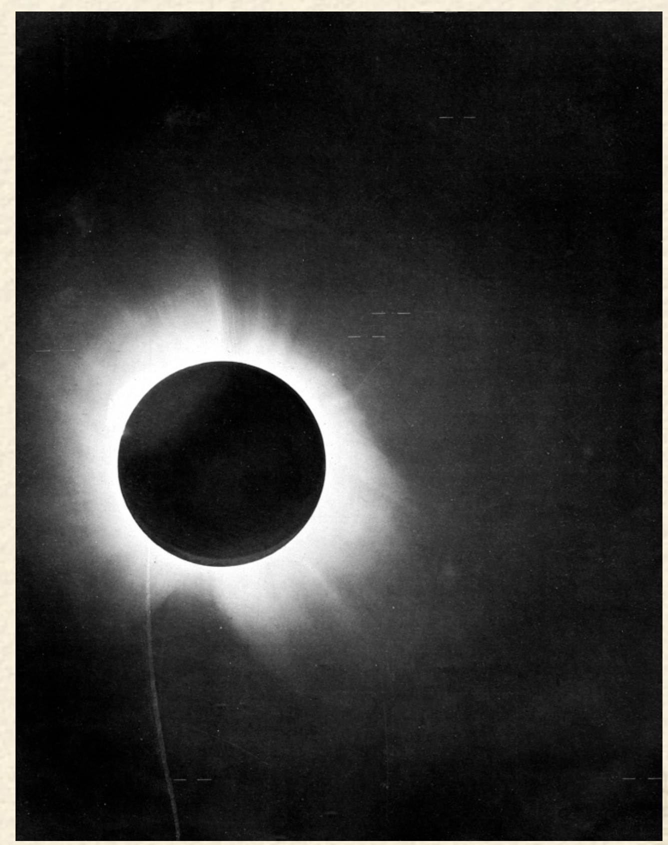
$$\underbrace{\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)}_{\text{curvature term}} = \frac{8\pi G}{c^4} \underbrace{\left(T_{\mu\nu}\right)}_{\text{energy-momentum term}}$$

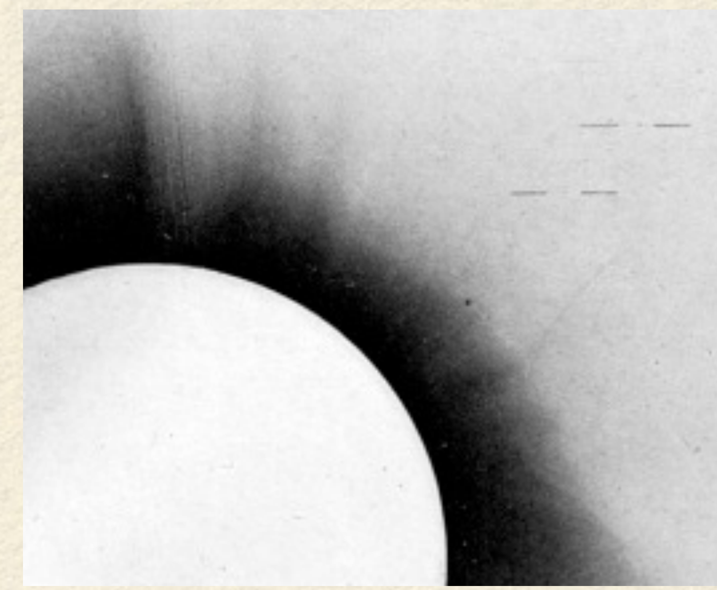
- Energy-momentum bends spacetime
  - being far from some energy density doesn't mean there is no bending !
- Spacetime tells mass (energy momentum) how to move
- These equations are non-linear

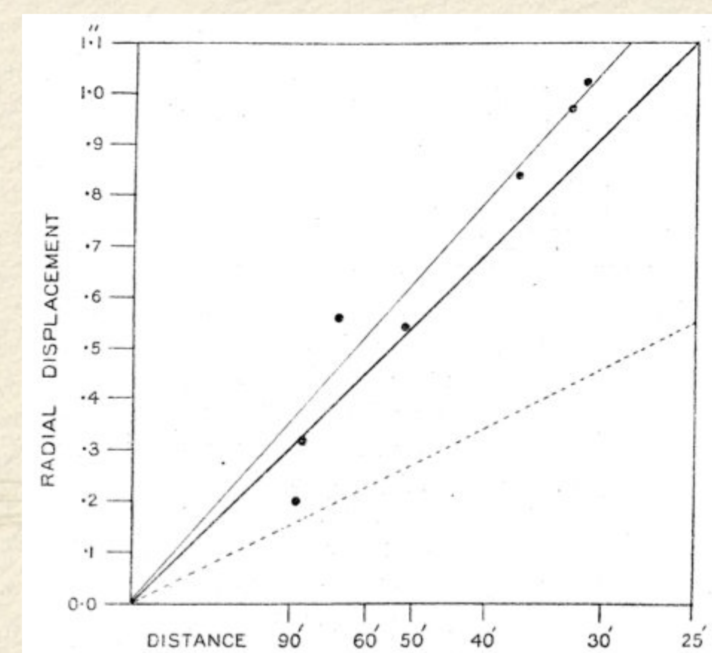
Novelties of the General Relativity

- New effects (w.r.t. Newtonian mechanics) but faint
  - the trajectory of some celestial bodies is modified (Mercury)
  - light follows the geodesics of space-time,
    its trajectory is curved nearby the sun
    (or any other body)









Eddington, 1919

Novelties of the General Relativity

Linearized gravity

- General Relativity (Einstein, 1916)
- Minkowski flat space-time with a small perturbation of the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

- where (Minkowski flat space-time metric) :

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- and $h_{\mu\nu} \ll 1$ is a perturbation of this metric
- then...

Linearized gravity

- start from the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- First step
  - linearization of all the constituents
    - replace $g_{\mu\nu}$ by $\eta_{\mu\nu} + h_{\mu\nu}$
    - remove the higher order terms in $h_{\mu\nu}$
- One obtains an equivalent equation which is still complicated
- Have to change variable

$$h_{\mu\nu} \ \rightarrow \ \bar{h}_{\mu\nu}$$

- where the trace reverse is defined as :

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h$$

Linearized gravity

- In General Relativity, the physics doesn't depend on the choice of coordinate system (gauge)
- Choose a particular set of coordinate systems where a certain condition is met, that simplifies the equations

$$\frac{\partial}{\partial x^\mu}\bar{h}^{\mu\nu} = \partial_\mu \bar{h}^{\mu\nu} = 0$$

Gauge condition

- The field equations may be written as :

$$\Box \bar{h}^{\mu\nu} = -2\frac{8\pi G}{c^4}T^{\mu\nu}$$

- Where $\Box$ is the d'Alembertian (or the wave operator) :

$$\Box \quad \Leftrightarrow \quad \{\nabla^2 - \frac{\partial^2}{\partial t^2}\} \quad \Leftrightarrow \quad \{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}\}$$

Gravitational waves

- In vacuum ($T_{\mu\nu} = 0$), the Einstein field equations are equivalent to a wave equation :

$$\Box\,\bar{h}_{\mu\nu} = 0 \quad \Leftrightarrow \quad \{\nabla^2 - \frac{\partial^2}{\partial\, t^2}\}\bar{h}_{\mu\nu} = 0$$

with a gauge condition : $\partial_\mu \bar{h}^{\mu\nu} = 0$

- where c = 1 and a harmonic gauge choice

- in the following, consider solutions

$$\bar{h}_{\mu\nu} = Re\,\{A_{\mu\nu}\exp\left(-ik_\rho x^\rho\right)\}$$

Gravitational waves

- Constraints
  - Satisfying the wave equation $\Box\,\bar{h}_{\mu\nu} = 0$:

$$k_\rho k^\rho = 0$$

$$\Rightarrow \quad \omega^2 = c^2|\vec{k}|^2 \quad \Longrightarrow \text{ the wave propagates at the speed of light c}$$

  - Use the gauge conditions $\partial_\mu \bar{h}^{\mu\nu} = 0$:

$$k_\rho A^{\rho\sigma} = 0$$

$\Longrightarrow$ 6 remaining independent elements in the amplitude tensor

Gravitational waves

- Can still simplify the expression of the amplitude
- Among the set of coordinate systems, choose a particular one such that

$$A_{0\sigma} = 0 \qquad \Rightarrow \text{number of independent elements further reduced to 2}$$

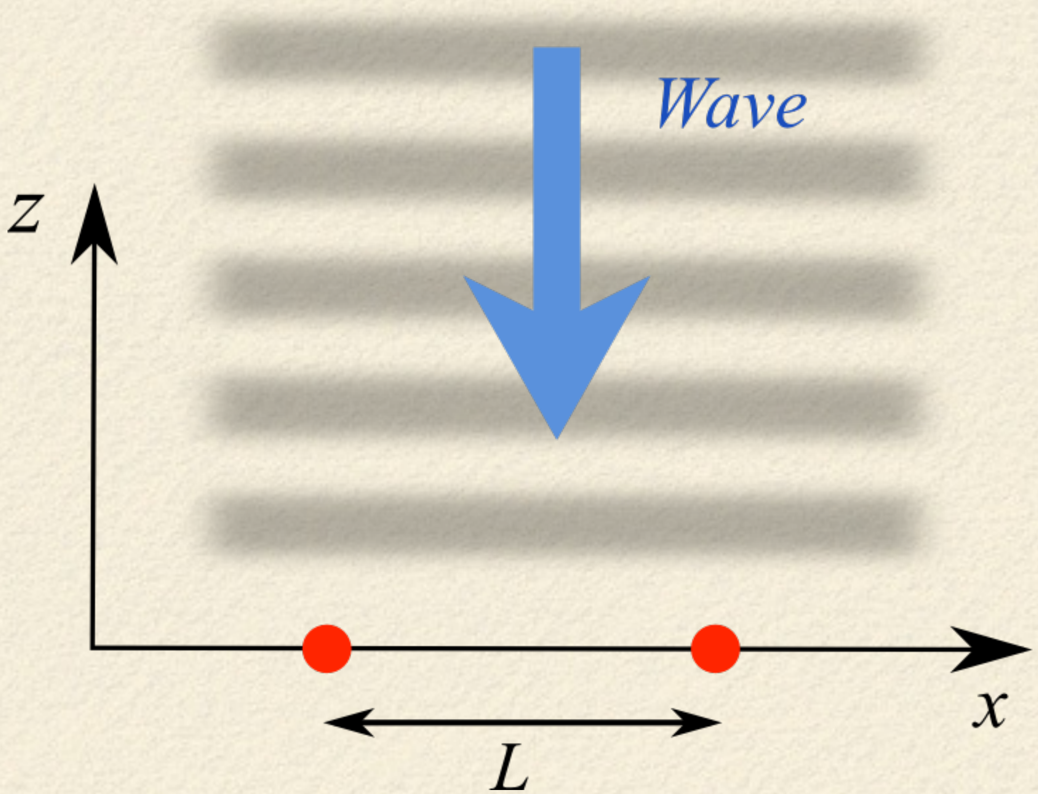
- for a wave traveling along the z axis, the amplitude is then :

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- choice , polarization called "+"
- choice $A_{11} = 1,\ A_{12} = 0$, polarization called "ξ"
- $A_{11} = 0,\ A_{12} = 1$
- This particular gauge (coordinate system) is called Transverse Traceless (TT)

Gravitational waves

- Proper length between two test masses in free fall



$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$$

if dt=0 (proper length) and
look only along the "x" direction

$$\Rightarrow \quad ds^2 = g_{xx} dx^2$$

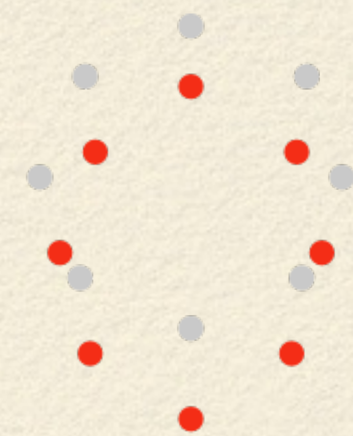
$$L = \int_0^{\Delta x} dx \sqrt{g_{xx}} = \int_0^{\Delta x} dx \sqrt{1 + h_{xx}^{TT}(t, z=0)}$$

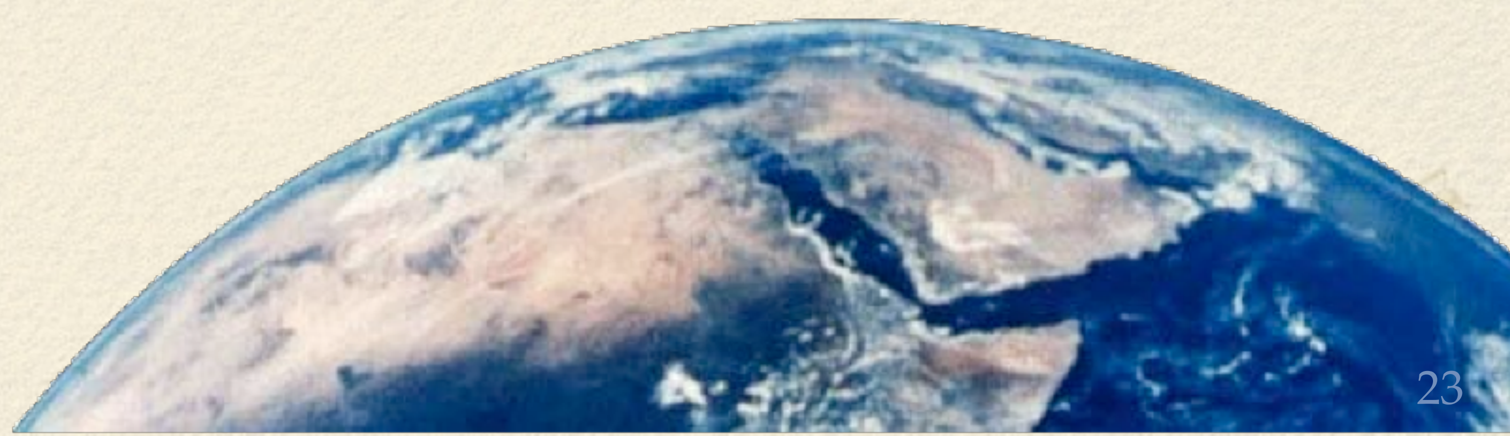
$$\approx \int_0^{\Delta x} dx \left[1 + \frac{1}{2} h_{xx}^{TT}(t, z=0)\right] = \Delta x \left[1 + \frac{1}{2} h_{xx}^{TT}(t, z=0)\right]$$

- h is the relative variation in proper length between the two test masses

Effect of spacetime curvature

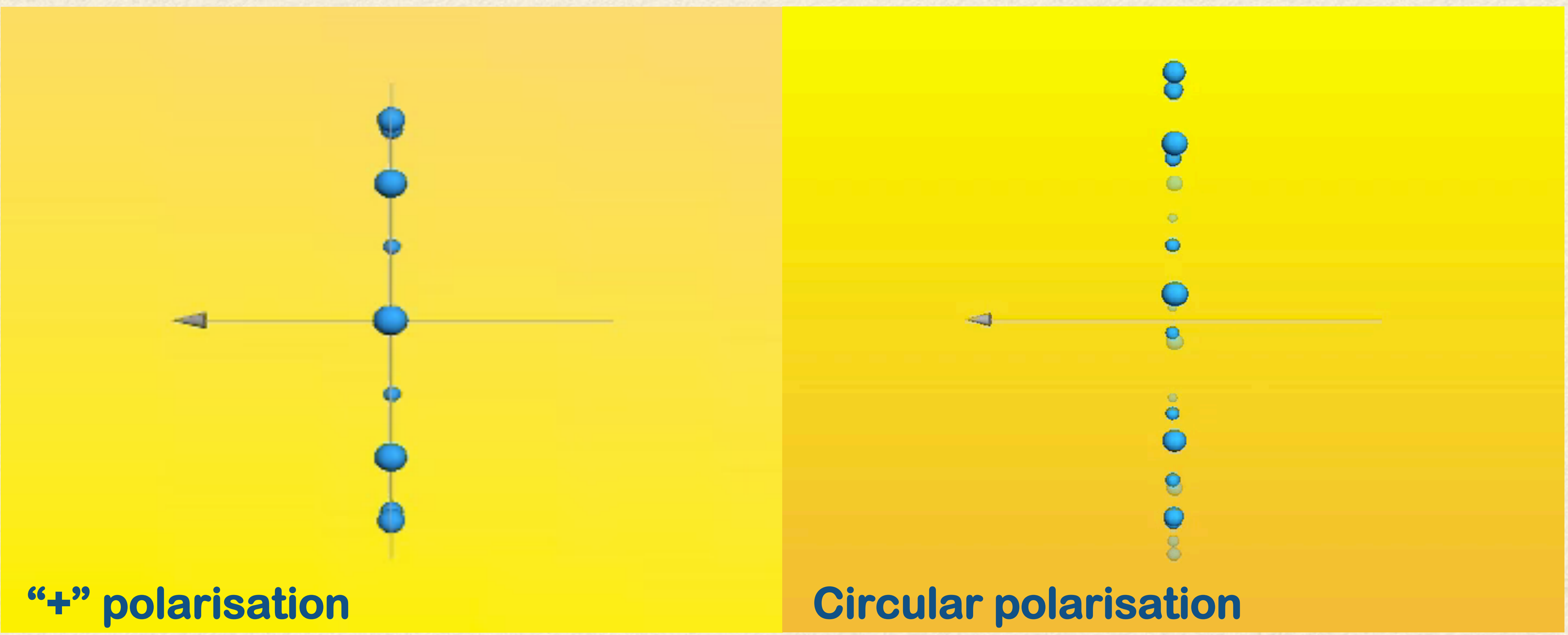
- Set of test masses
  - distributed on a circle
  - non interacting among themselves
  - freely floating above Earth's surface (static curvature)
- Effect of curvature on the set :
  - Lengthen in one dimension
  - Shrink in the perpendicular one





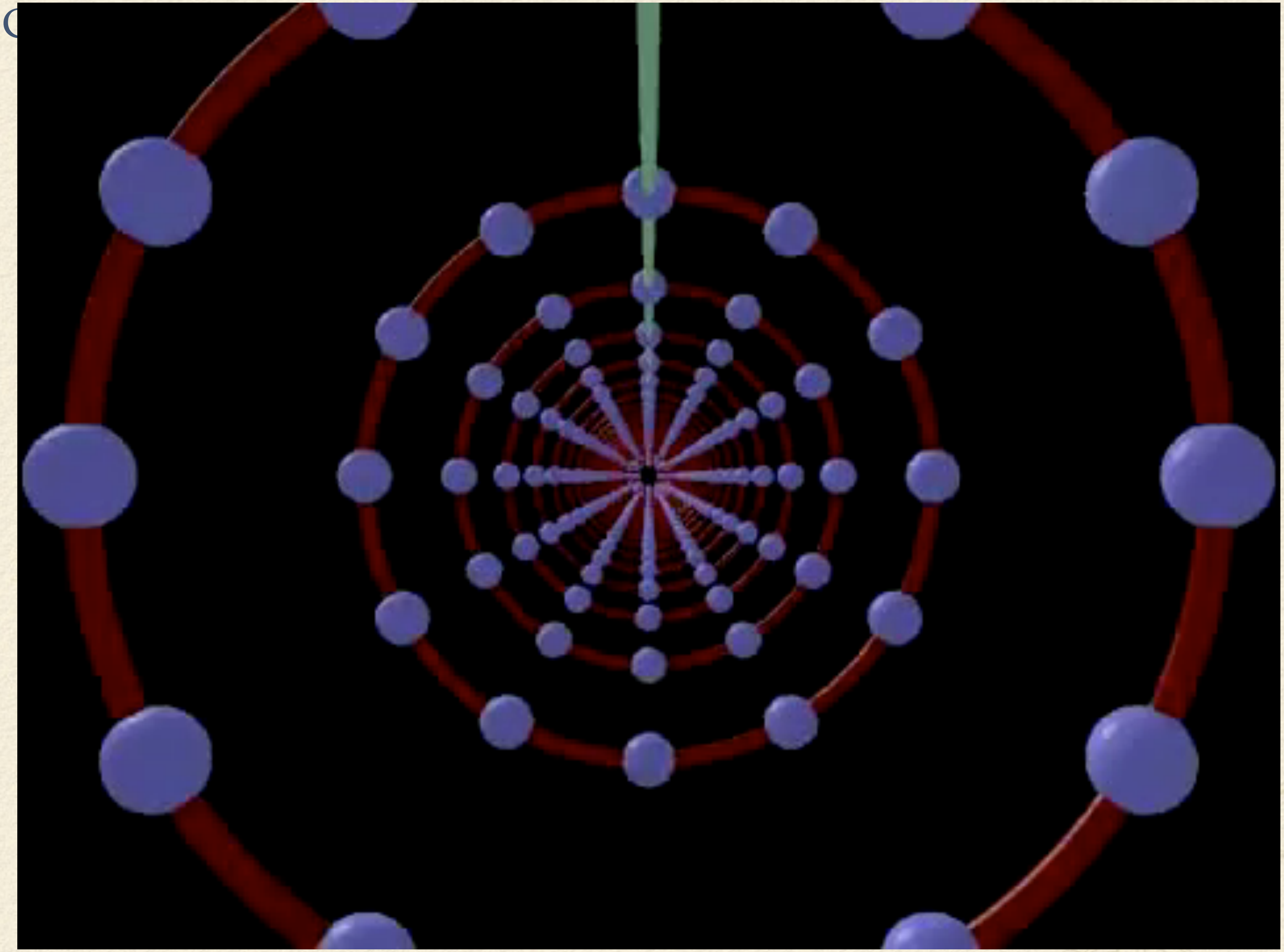
Effect of the gravitational waves

- Effect of GW on matter
- Same set of free falling test particles distributed on a circle
  illustration of the metric variation



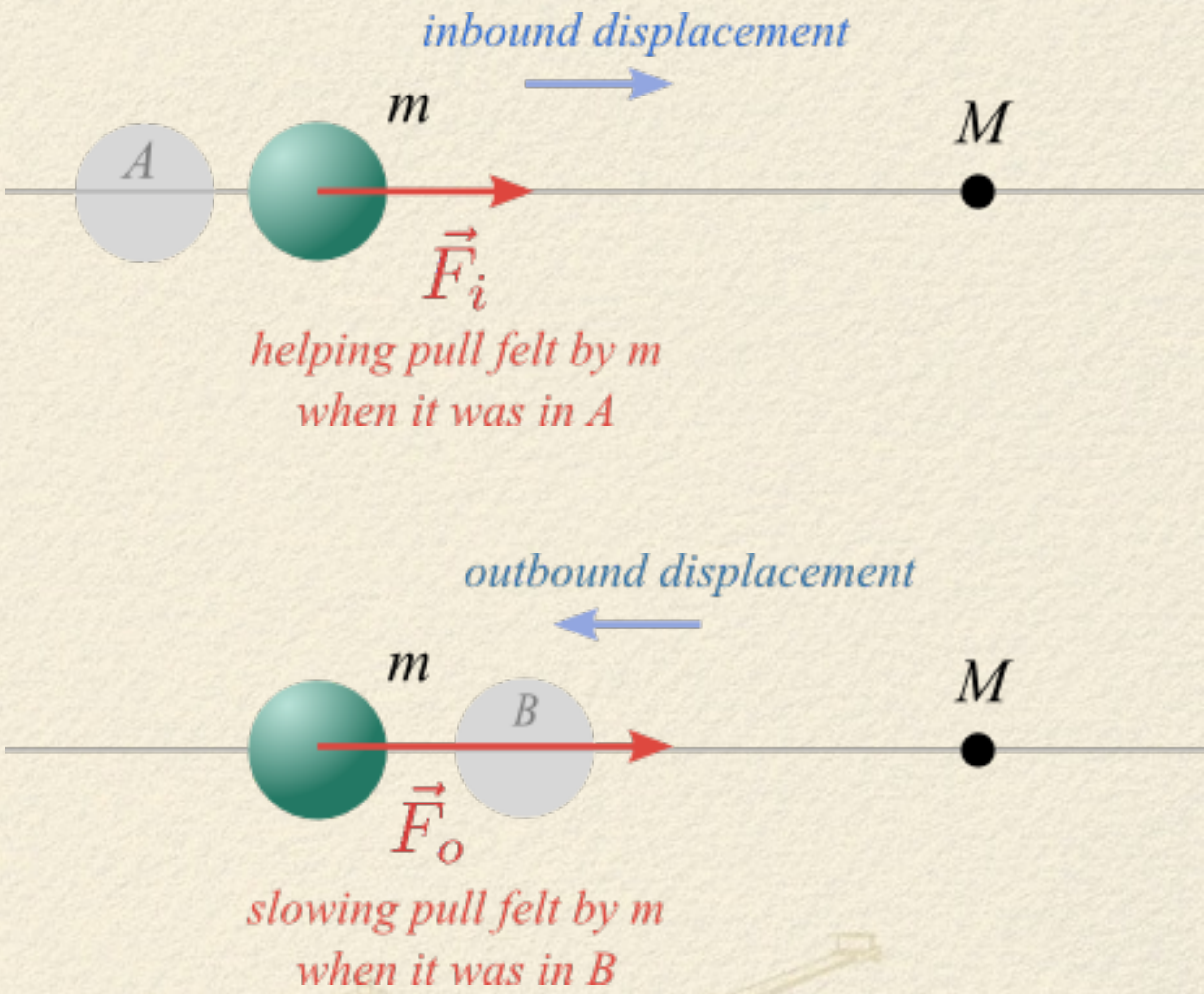
Effect of the gravitational waves

- C



Why does a massive system lose energy ?

- Argument by Kalckar and Ulfbeck, central point : time delay
- System : a mass m oscillating around a fixed center of attraction of mass M
  - in 1 dimension
- Time needed for gravity to propagate
  - when traveling inbound (towards M)
    force $\vec{F}_i$ felt by m when it was farther
  - when traveling outbound (away from M)
    force $\vec{F}_o$ felt by m when it was closer
- $\Longrightarrow \left|\vec{F}_o\right| > \left|\vec{F}_i\right|$
- the oscillating motion of m is damped
- energy of the system reduces
- energy was taken away by gravity !

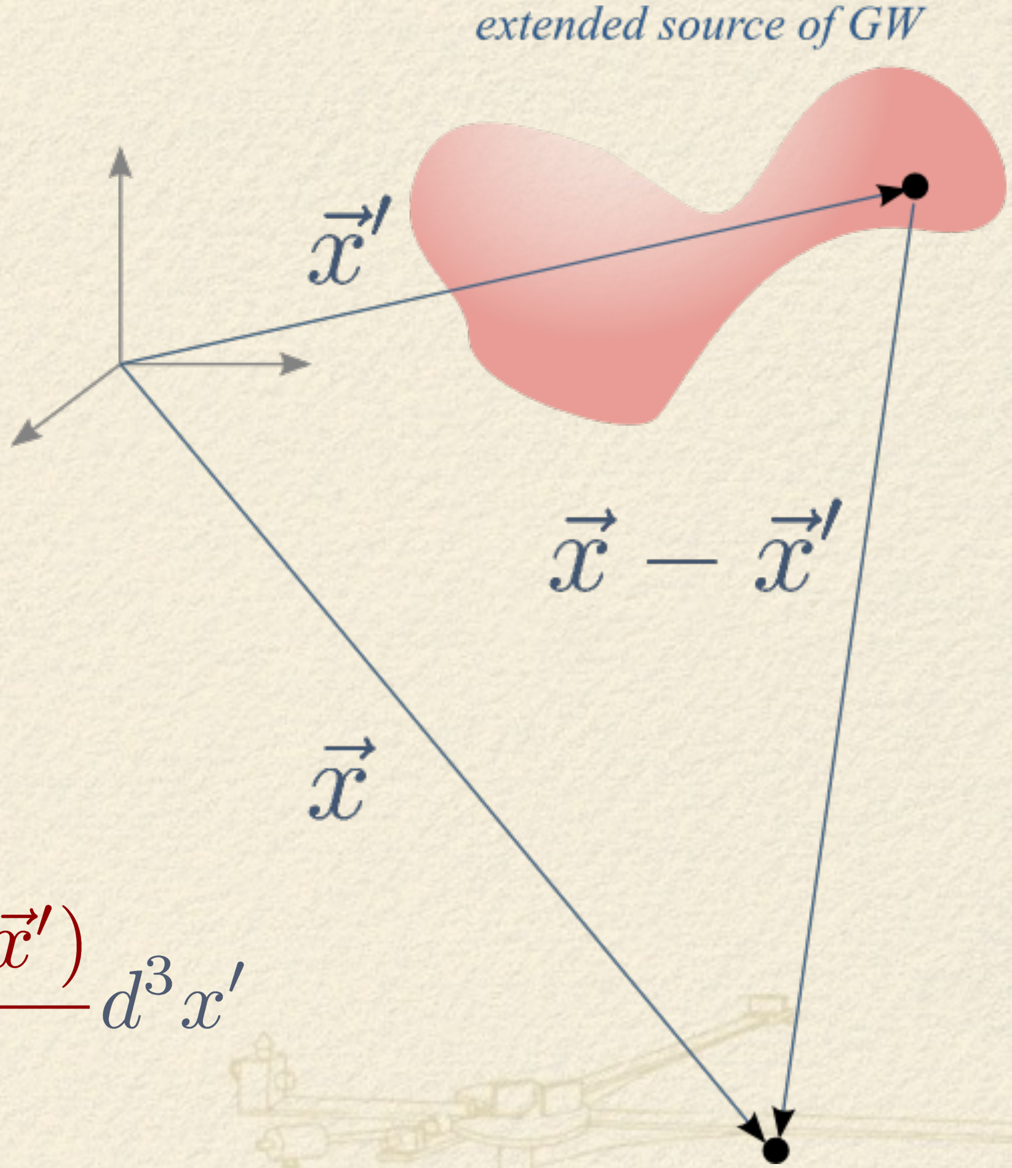


Generation of GW

- Emission of the gravitational waves
  - Linearized Einstein equations with a stress-energy tensor

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

  - Use Green functions
    - Solutions of the wave equation in the presence of a point source
  - Retarded potential



$$\bar{h}_{\mu\nu}(t,\vec{x}) = -\frac{4G}{c^4} \int_{source} \frac{T_{\mu\nu}(t - \frac{|\vec{x}-\vec{x}'|}{c}, \vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'$$

Generation of GW

- Approximations :
  - isolated source
  - compact source
  - observer far from the source ($R = |\vec{x} - \vec{x}'|$ >> typical size of the source)
- Amplitude of the wave written as a function of I_{ij}

$$\bar{h}_{ij}(t) = \frac{2G}{Rc^4} \frac{d^2 I_{ij}}{d\,t^2}\left(t - \frac{R}{c}\right)$$

I_{ij} = reduced quadrupolar moment of the source

$$= \int_{source} d\vec{x}\ x_i x_j\ T_{00}(t, \vec{x})$$

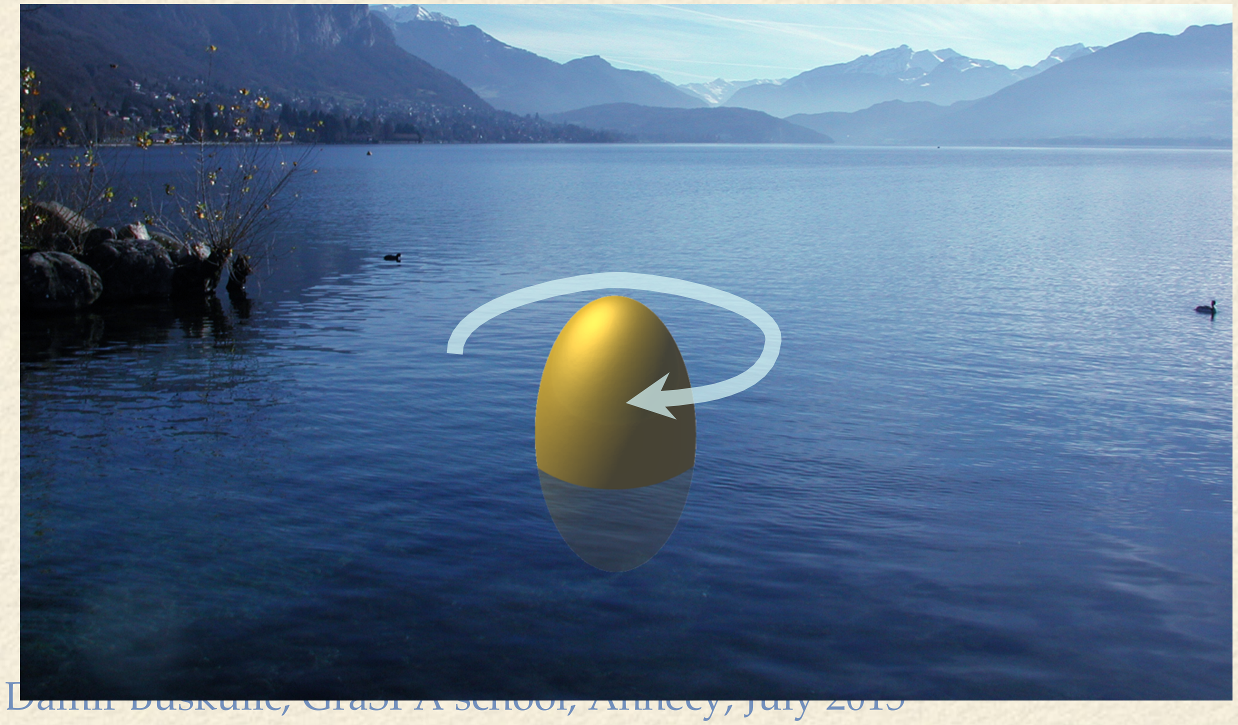
$$\frac{G}{c^4} \approx 8.24 \times 10^{-45}\ \mathrm{s}^2 \cdot \mathrm{m}^{-1} \cdot \mathrm{kg}^{-1}$$

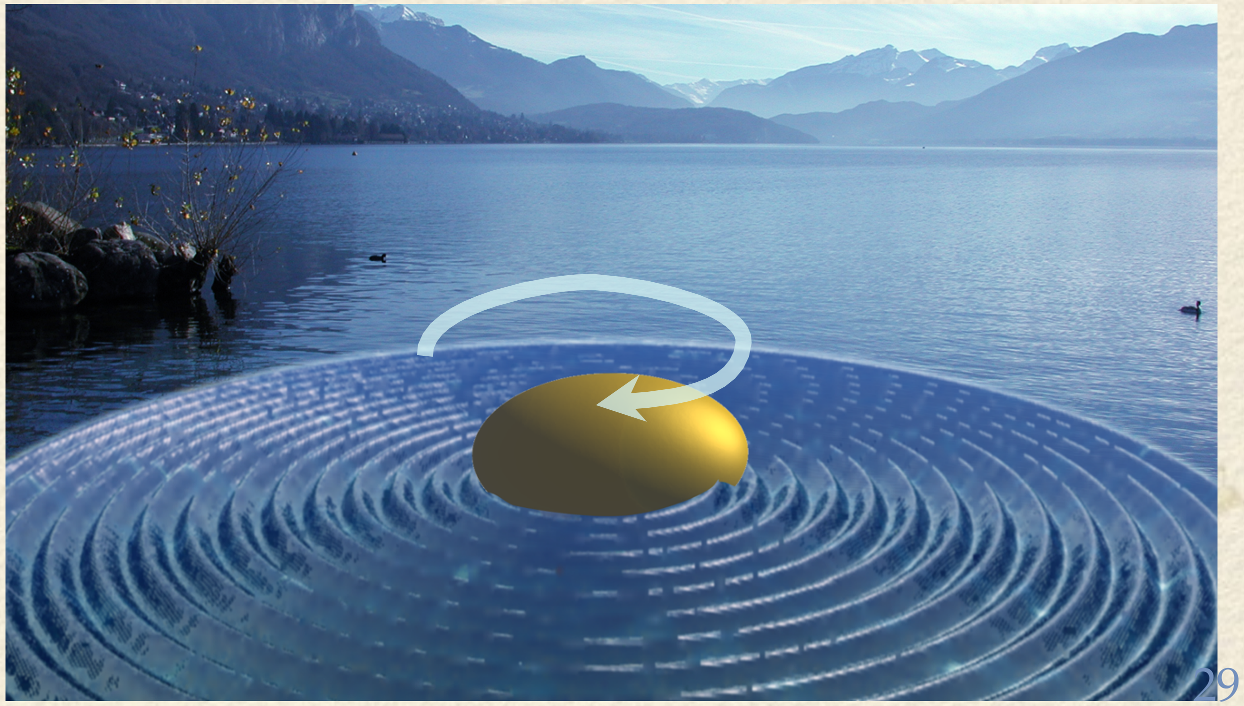
- Remark :
  Need a quadrupolar moment to generate a GW, the dipolar case is impossible (because of momentum conservation).

Generation of GW

- Example of a source : rotating neutron star
  - but not completely spherical (like a «rugby ball»)
- Or two neutron stars orbiting around each other
- Everything that rotates around an axis that is not a cylindrical symmetry axis

- ... raising your hand should generate gravitational waves !





Scientific goals

- Confirmation of GW
- Study properties, test GR
  - Speed = c ? Really quadrupolar ?
- Measure the Hubble constant
  - Coalescing binaries should be standard candles if the redshift and distance are known

Detectors on earth, in space

Detectors on earth, in space

Detectors on earth, in space

Impossible with only one detector for most of the sources

Build a worldwide observatory of gravitational waves

Scientific goals

- Study characteristics
  - of neutron stars
  - of solar mass black holes (BH)
    - Ellipticity, vibration modes, higher order moments
- Study supermassive black holes
  - cartography of space-time around a supermassive BH (Kerr).
  - study of their distribution, galactic evolution
- Stochastic background of GW :
  - first moments of the universe ?
- ...

Detectors on earth

Detectors in space

Detectors on earth ?

Detectors in space ?

Detection of gravitational waves by optical interferometry

- Historical introduction
- Principle of the interferometric detectors
- The noise makes the detector
- From Virgo to Advanced Virgo (AdV)
- A world wide detector network

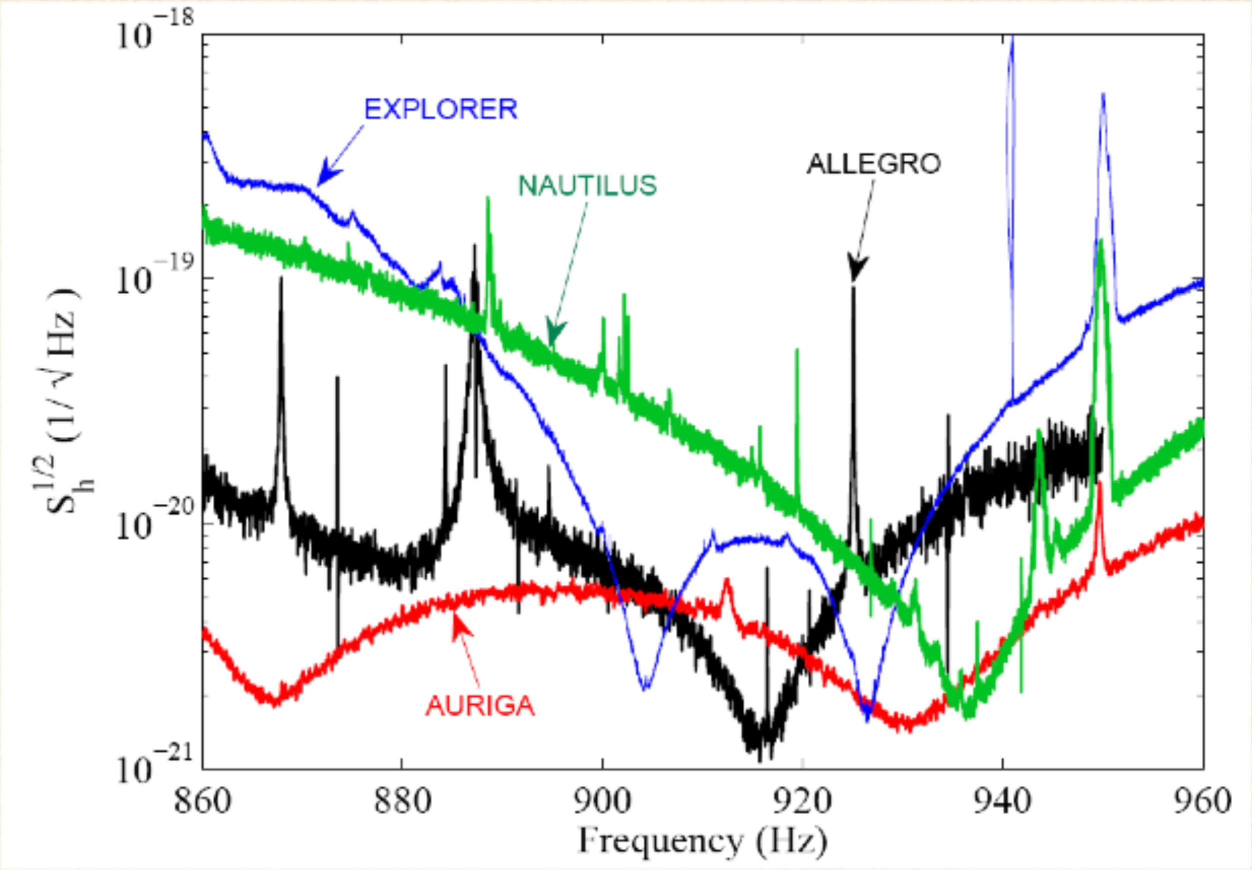
A rather complete and very pedagogical introduction :
"Fundamentals of Interferometric Gravitational Wave Detectors", P.R. Saulson, World Scientific, 1994

History

- First idea : resonant bars or spheres
  - J. Weber 1966 : the GW changes the resonance condition of a resonant bar of a few tons
  - Claims the detection of GW (1968-1969)
    - Various problems, other experiments have not confirmed the discovery
- Other idea : measure the time of flight of photons between two test masses, Michelson interferometer
  - Gertsenshtein & Pustovoit (1962)
  - First interferometer for GW :
  - R. L. Forward & al (1971)
  - Foundations for the modern interferometers : Rainer Weiss (1972)

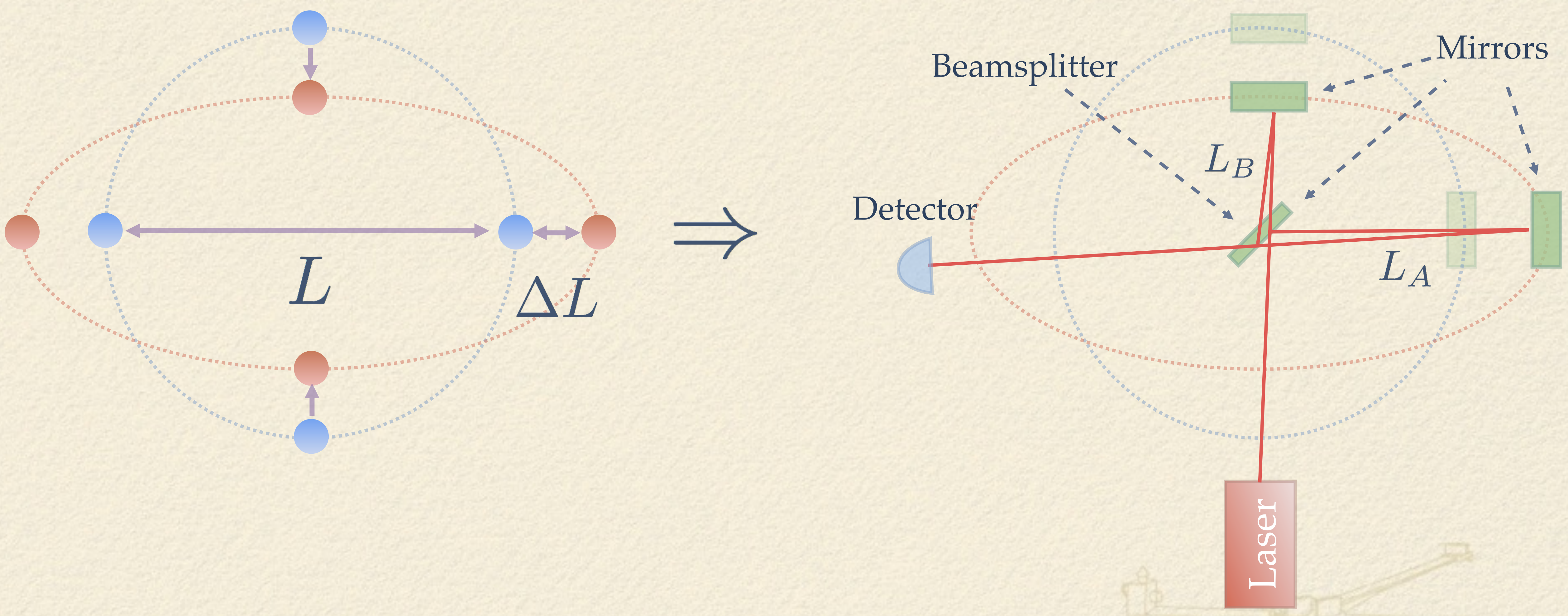
Resonant detectors

- J. Weber 1966 : the GW changes the resonance condition of a resonant bar of a few tons
- Many other experiments until the mid-2000's
- $f_s = 700 - 1000$ Hz, $\Delta f = 50 - 200$ Hz
- sensitivity : $h \approx 10^{-19}$ à 10^{-21}



Interferometric detectors : Principle of detection

Detection principle

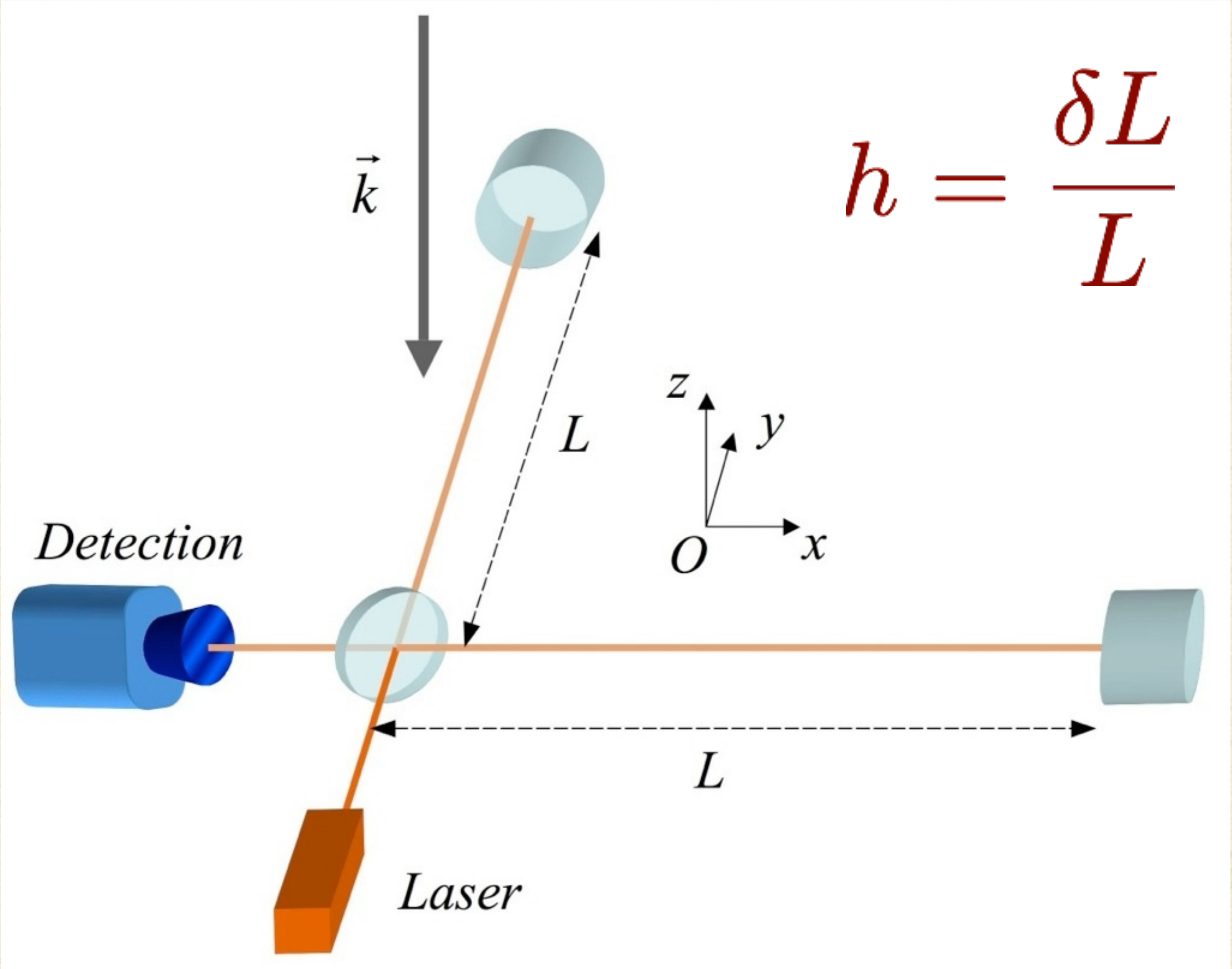


Detection principle

- Michelson interferometer

The detector measures the optical path length difference between the two arms

most of the elements (mirrors, injection and detection systems) are suspended and behave like free falling masses in the interferometer plane
(for $f \gg f_{pend}$)



Detection principle

- Photon in a field , general case

$$ds^2 = 0 = g_{\alpha\beta}dx^\alpha dx^\beta = \eta_{\alpha\beta}dx^\alpha dx^\beta + h_{\alpha\beta}dx^\alpha dx^\beta$$

- Particular case : wave along z, polarization "+" along one of the arms

$$ds^2 = 0 = -c^2dt^2 + (1 + h_+(t))dx^2 + (1 - h_+(t))dy^2 + dz^2$$

- Round trip time of the photons,
integration on the path, for example for the arm along x

$$\frac{1}{c}\int_0^L dx = \int_0^{\tau_{aller}} \frac{1}{\sqrt{1 + h_+(t)}}\,dt \approx \int_0^{\tau_{aller}} \left(1 - \frac{1}{2}h_+(t)\right) dt$$

- Consider
  - round trip in one arm
  - wavelength of the GW >> length of one arm
  - period of the GW << round trip time of the light in one arm

$$\lambda_{OG} \gg L \quad \Rightarrow \quad h_+(t) \text{ independent of the position along the arm}$$

$$\Rightarrow \quad h_+(t) = cte = h_+$$

Detection principle

- For the arm along the "x" direction

$$\int_0^{\tau_{arx}} \left(1 - \frac{1}{2}h_+(t)\right) dt \approx \frac{1}{c}\left(\int_0^L dx - \int_L^0 dx\right) = \frac{2L_x}{c}$$

$$= \tau_{arx} - \frac{1}{2}\int_0^{\tau_{arx}} h_+(t)dt = \tau_{arx} - \frac{1}{2}\int_0^{\frac{2L_x}{c}} h_+(t)dt$$

$$\Rightarrow \quad \tau_{arx} = \frac{2L_x}{c} + \frac{1}{2}\int_0^{\frac{2L_x}{c}} h_+(t)dt$$

- arm along "y" :

$$\Rightarrow \quad \tau_{ary} = \frac{2L_y}{c} - \frac{1}{2}\int_0^{\frac{2L_y}{c}} h_+(t)dt$$

- time difference (suppose h constant) if :

$$L_x = L_y = L$$

$$\delta\tau_{ar} = \frac{1}{2}h_+\left(\frac{2L_x}{c} + \frac{2L_y}{c}\right) = h_+\frac{2L}{c} \quad \Rightarrow \quad \frac{c\cdot\delta\tau_{ar}}{2} = \delta L = h_+\cdot L \quad \Rightarrow \quad h_+ = \frac{\delta L}{L}$$

- accumulated phase difference :

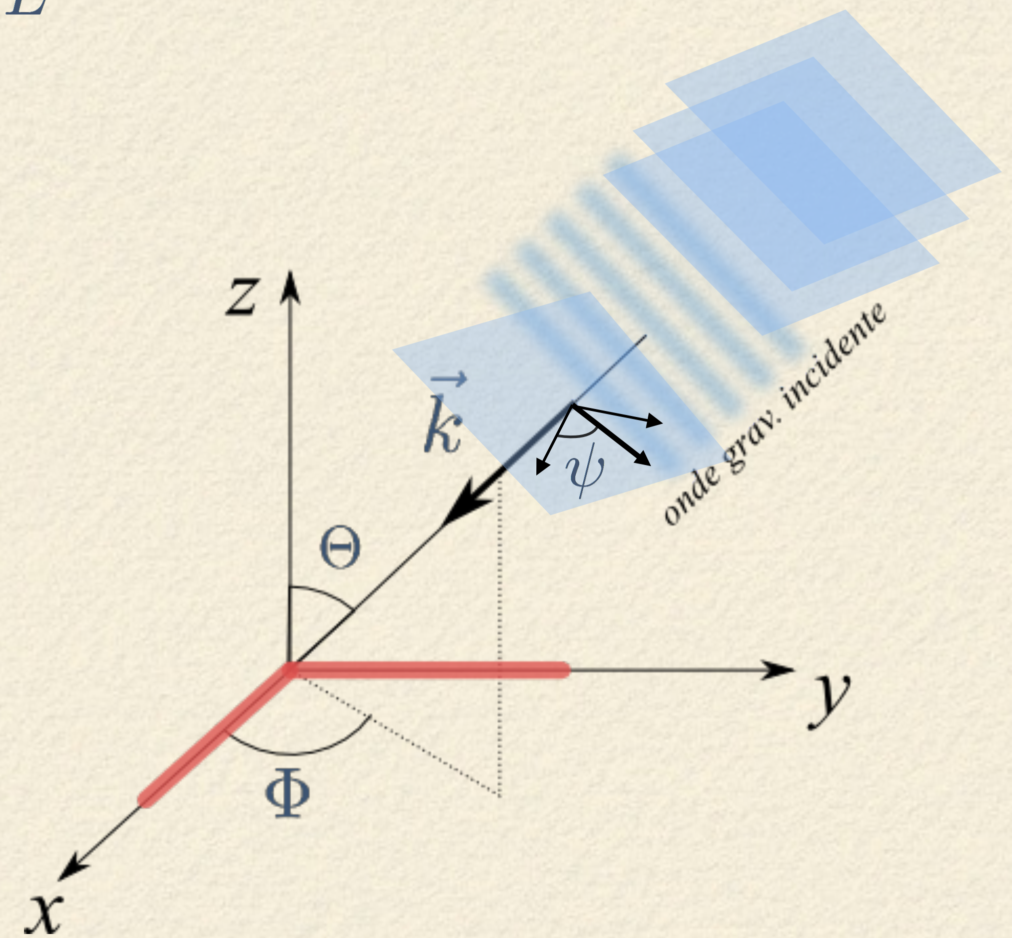
$$\delta\phi = \omega_{\text{laser}}\,\delta\tau_{ar} = \frac{4\pi}{\lambda_{\text{laser}}}Lh_+$$

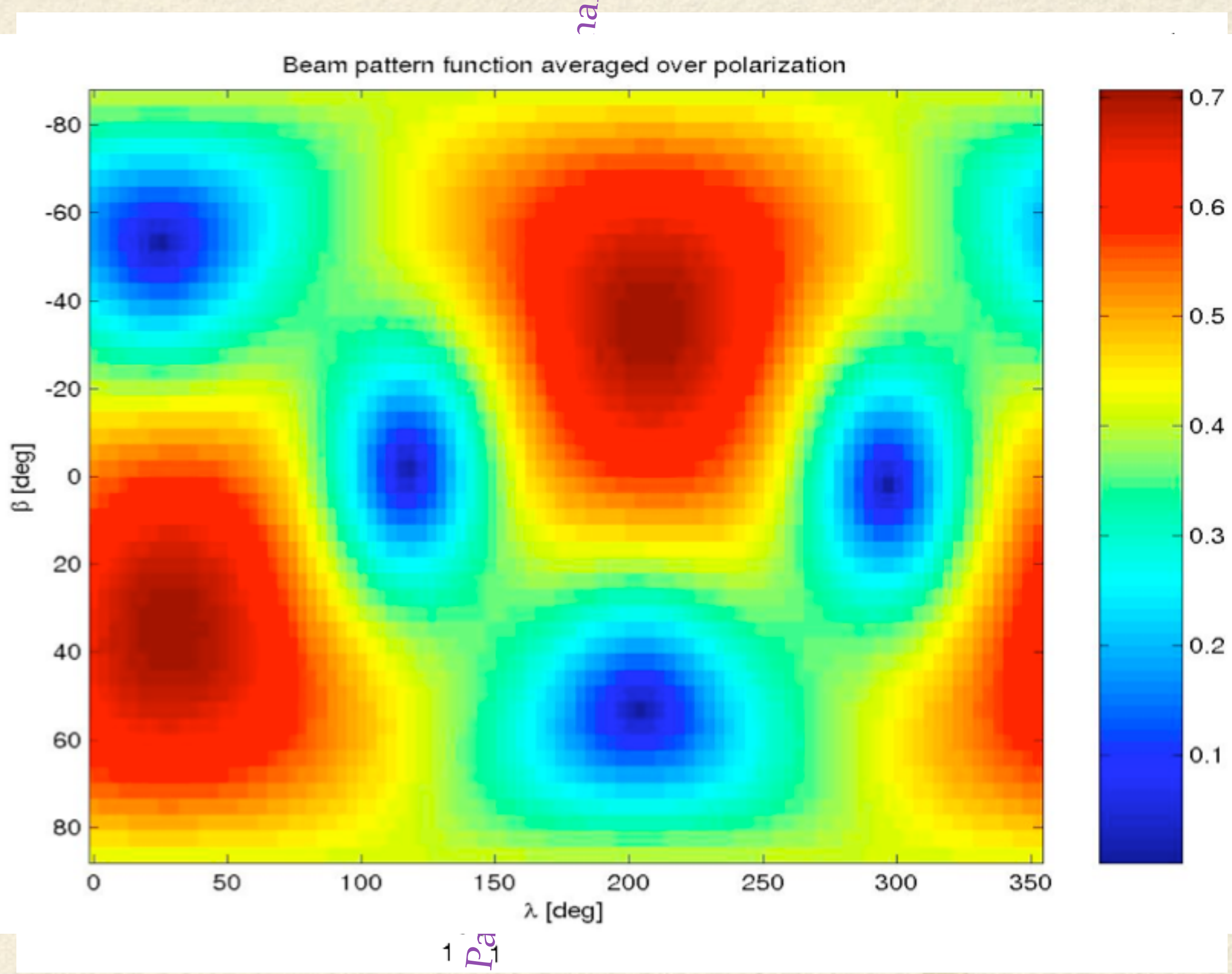
- proportional to h and L

Angular response

- Interferometer angular response
  - Average on the polarization of the incident wave

$$\frac{\Delta L}{L} = F_+(\Theta, \Phi, \psi)\, h_+ + F_\times(\Theta, \Phi, \psi)\, h_\times$$





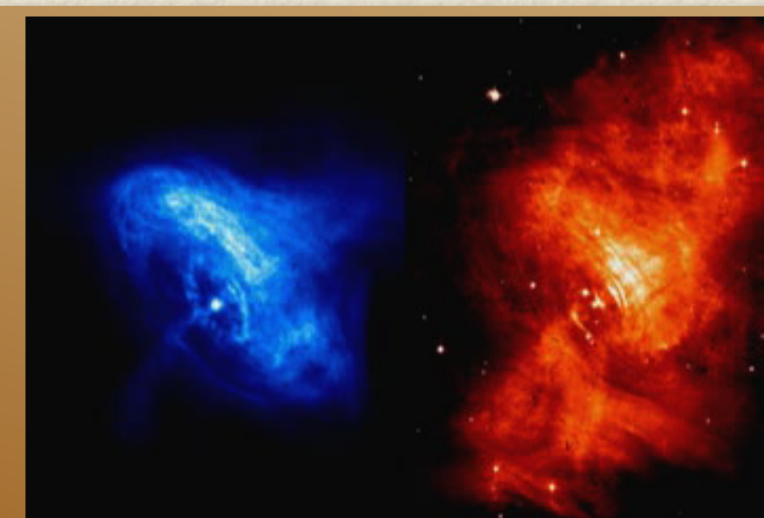
$$F_+ = -\frac{1}{2}(1+\cos^2\Theta)\cos 2\Phi \cos 2\psi - \cos\Theta \sin 2\Phi \sin 2\psi$$

$$F_\times = \frac{1}{2}(1+\cos^2\Theta)\cos 2\Phi \sin 2\psi - \cos\Theta \sin 2\Phi \cos 2\psi$$

"quasi" omni-directional detector

Sources of gravitational waves

Main sources



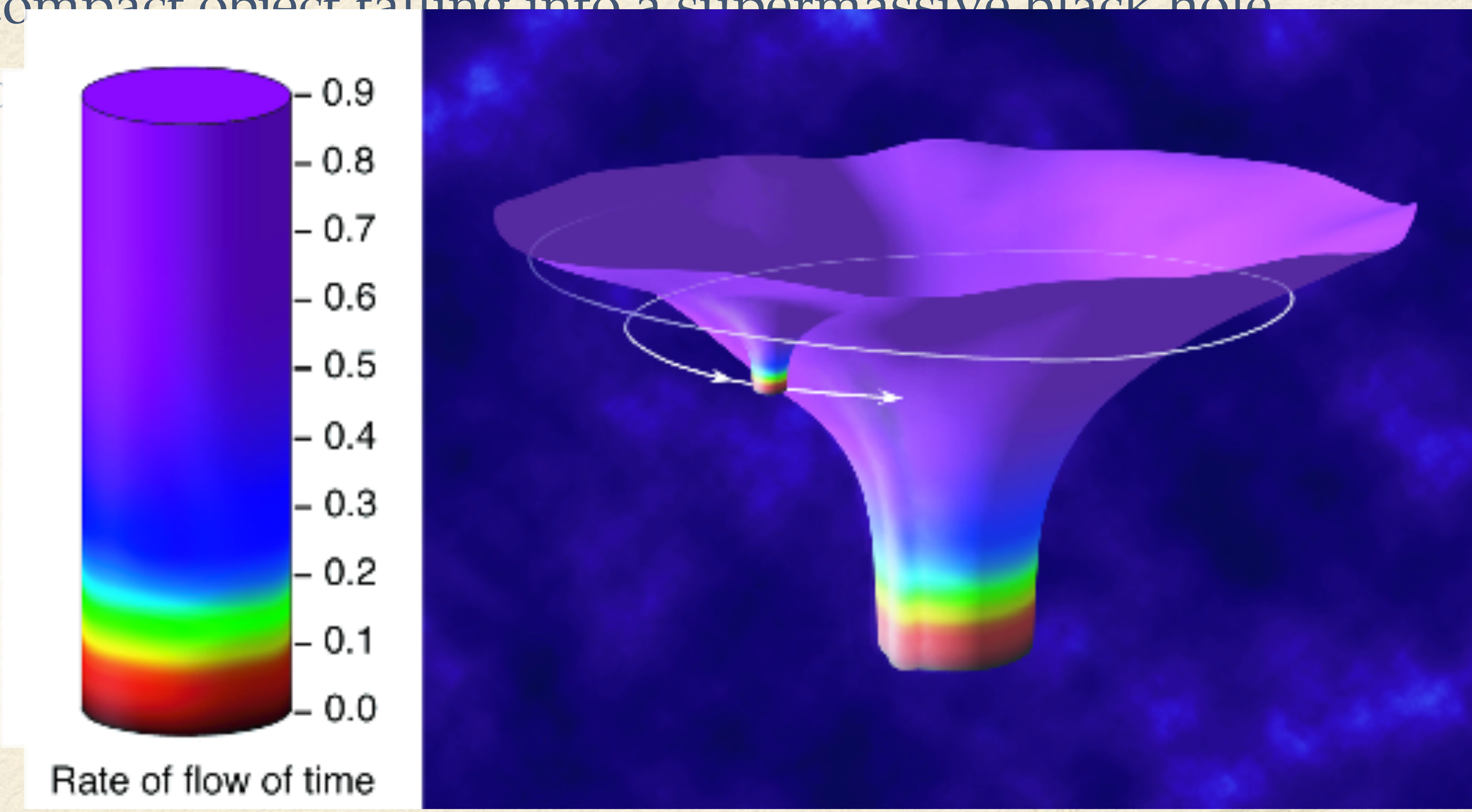
- Impulsive (or burst) sources
  - Supernovae $T \sim \text{ms},\ \nu \sim \text{ kHz},\ h \sim 10^{-21} - 10^{-24}$ à 15 Mpc
  - Compact binary system coalescences (neutron stars or black holes) $T \sim \text{mn},\ \nu \sim 10\text{ Hz - }1\text{ kHz},\ h \sim 10^{-23}$ à 10 Mpc
- Periodic sources
  - rotating neutrons stars $\nu \sim\ 1\text{ Hz - }1\text{ kHz},\ h \sim 10^{-25}$ à 3 kpc
- What about the Gamma Ray Bursts ?
  - short GRBs may be coalescences

D Eichler, M Livio, T Piran, and D Schramm. Nature, 340:126, 1989.

R Narayan, Paczynski, and T Piran. Astroph. J., 395:L83, 1992

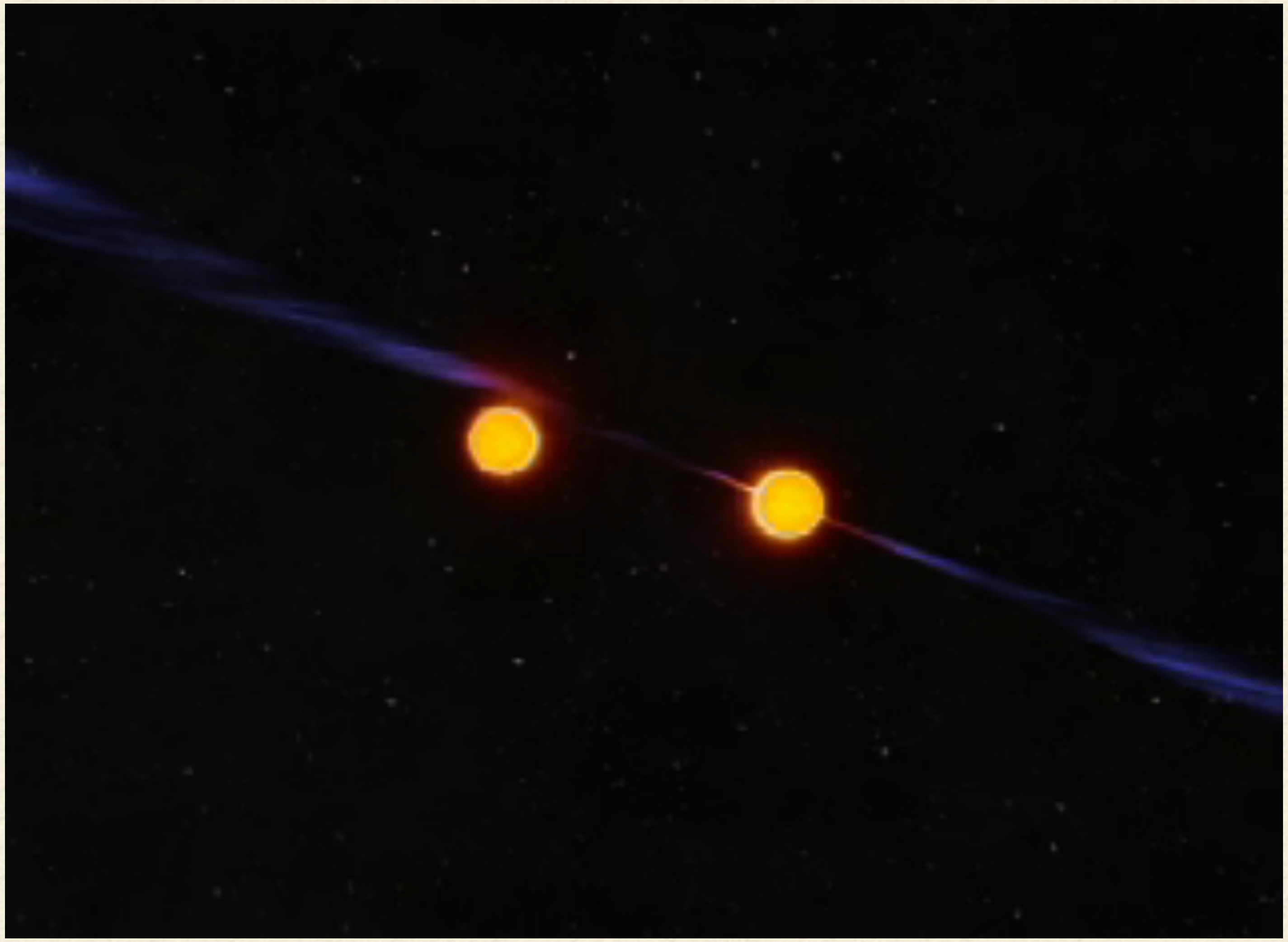
More "exotic" sources

- Stochastic (random) noise of gravitational waves
- Compact object falling into a supermassive black hole
- E



Sources : Coalescences

- Binary system of compact objects
- One of the most promising sources for detection :

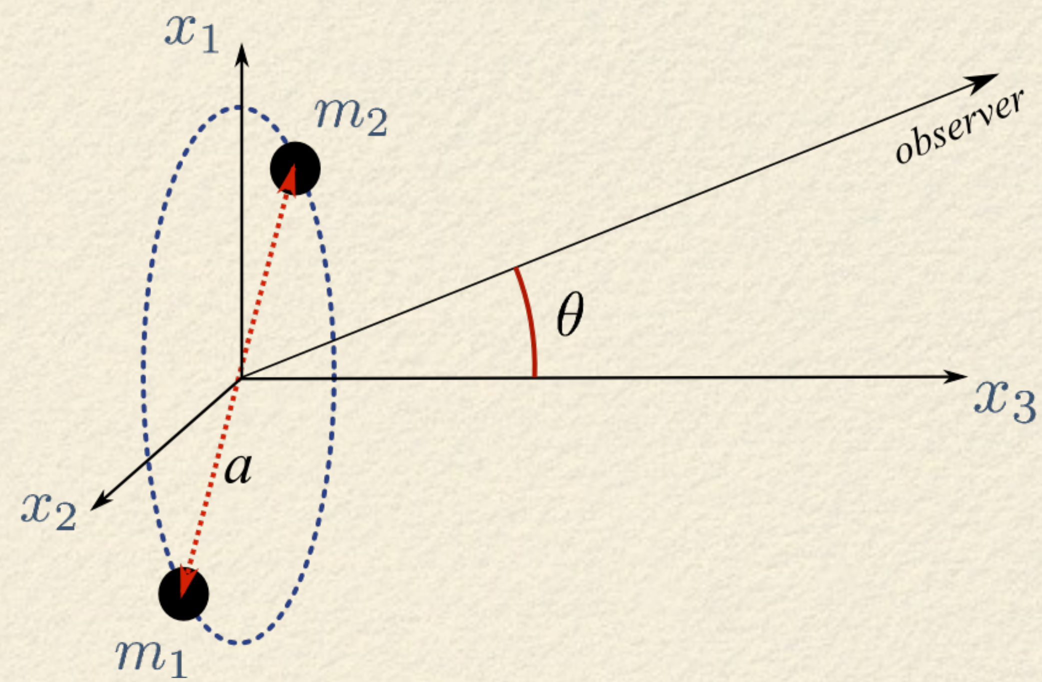


- Black hole - black hole (BHBH)
- BH - Neutron star (NS)
- NS - NS

$T \sim$ mn, $\nu \sim$ 10 Hz - 1 kHz,
$h \sim 10^{-23}$ à 10 Mpc

Generation : binary system

- Example : binary system of two compact objects
  - Masses m_1 and m_2
  - Distance between the objects : a
  - Total mass : $M = m_1 + m_2$
  - Reduced mass : $\mu = \frac{m_1 m_2}{M}$
- Newtonian approximation
  - 3[d] Kepler law : $\omega = \sqrt{\frac{GM}{a^3}}$
- Take circular orbits
- Compute h_+ and $h_\times$, the amplitude of the two modes of the emitted wave seen by an observer situated at a distance $R \gg a$
- Relative coordinates : $x_1(t) = a\cos\omega t$, $x_2(t) = a\sin\omega t$, $x_3(t) = 0$

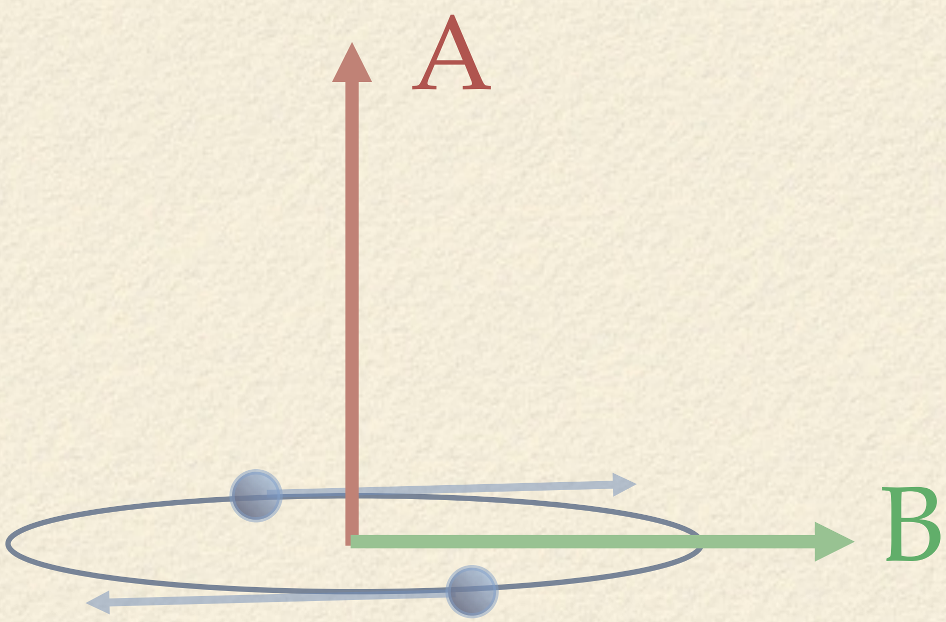


Generation : binary system

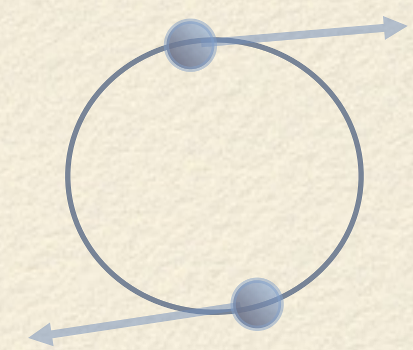
- One obtains

$$h_+(t) = \frac{4G\mu a^2\omega^2}{Rc^4}\frac{1+\cos^2\theta}{2}\cos 2\omega t$$

$$h_\times(t) = \frac{4G\mu a^2\omega^2}{Rc^4}\cos\theta\sin 2\omega t$$



Observer A : $\cos\theta = 1$
sees the two polarizations



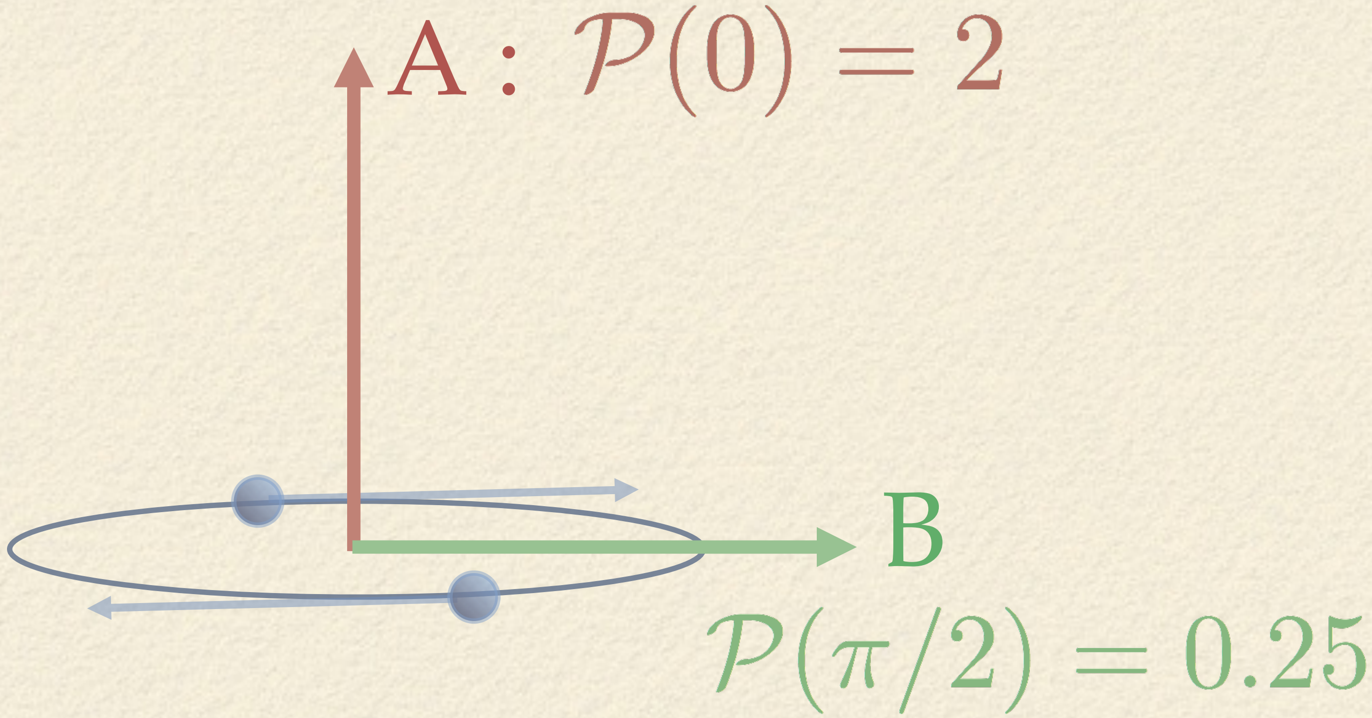
Observer B : $\cos\theta = 0$
sees a linear polarization

Generation : binary system

- Radiated power per unit solid angle $\frac{dP}{d\Omega} = \frac{2G\mu^2 a^4 \omega^6}{\pi c^5}\mathcal{P}(\theta)$

$$\mathcal{P}(\theta) = \frac{1}{4}(1 + 6\cos^2\theta + \cos^4\theta)$$

- Radiated power non zero whatever the direction of emission



- Total radiated power

$$\boxed{P = \frac{32G\mu^2 a^4 \omega^6}{5c^5}}$$

Generation : binary system

- Some examples
- Sun-Jupiter system

$$m_J = 1.9 \times 10^{27}\mathrm{kg}, \quad a = 7.8 \times 10^{11}\mathrm{m}, \quad \omega = 1.68 \times 10^{-7}\mathrm{s}^{-1}$$

$$\Rightarrow \quad P = 5 \times 10^3 \quad \mathrm{J/s}$$

  - Very small, compared to the light power emitted by the sun :

$$L_\odot \approx 3.8 \times 10^{26}\ \mathrm{J/s}$$

- Binary pulsar PSR1913+16 (Hulse and Taylor) :

$$P = 7.35 \times 10^{24}\ \mathrm{J/s}$$

Generation : binary system

- Consider simplified Newtonian case (so called "order 0").
- Radiated energy taken to the gravitational energy of the system
  - Grav. energy of the system decreases, radius of the orbits decreases
  - Frequency of the GW increases
  - Conservation of energy : $\frac{dE}{dt} = -P$ (E total energy of the system)
- Newtonian

$$E = -G\frac{m_1 m_2}{2a}, \quad \omega^2 = \frac{GM}{a^3}$$

- Hence

$$\dot{a} = -\frac{2}{3}(a\omega)\left(\frac{\dot{\omega}}{\omega^2}\right)$$

radial speed

tangential speed

adiabatic factor

Generation : binary system

- Goal : calculate the evolution of the frequency of the wave
- Calculation of the adiabatic factor

$$E = -G\frac{m_1 m_2}{2a}, \quad \omega^2 = \frac{GM}{a^3} \quad \Rightarrow \quad \dot{E} = -G^{2/3}\frac{m_1 m_2}{2M^{1/3}}\frac{2}{3}\,\dot{\omega}\,\omega^{-1/3}$$

$$\dot{E} = -P \quad \Rightarrow \quad G^{2/3}\frac{m_1 m_2}{2M^{1/3}}\frac{2}{3}\,\dot{\omega}\,\omega^{-1/3} = \frac{32G\mu^2 a^4 \omega^6}{5c^5}$$

- Replace a by its expression as a function of ω :

$$\frac{\dot{\omega}}{\omega^2} = \frac{96}{5}\frac{G^{5/3}}{c^5}\frac{\mu}{M}(M\omega)^{5/3}$$

- Frequency :

$$\dot{f}_{OG} = \frac{96}{5}\frac{G^{5/3}}{c^5}\pi^{8/3}\mathcal{M}^{5/3} f_{OG}^{11/3}$$

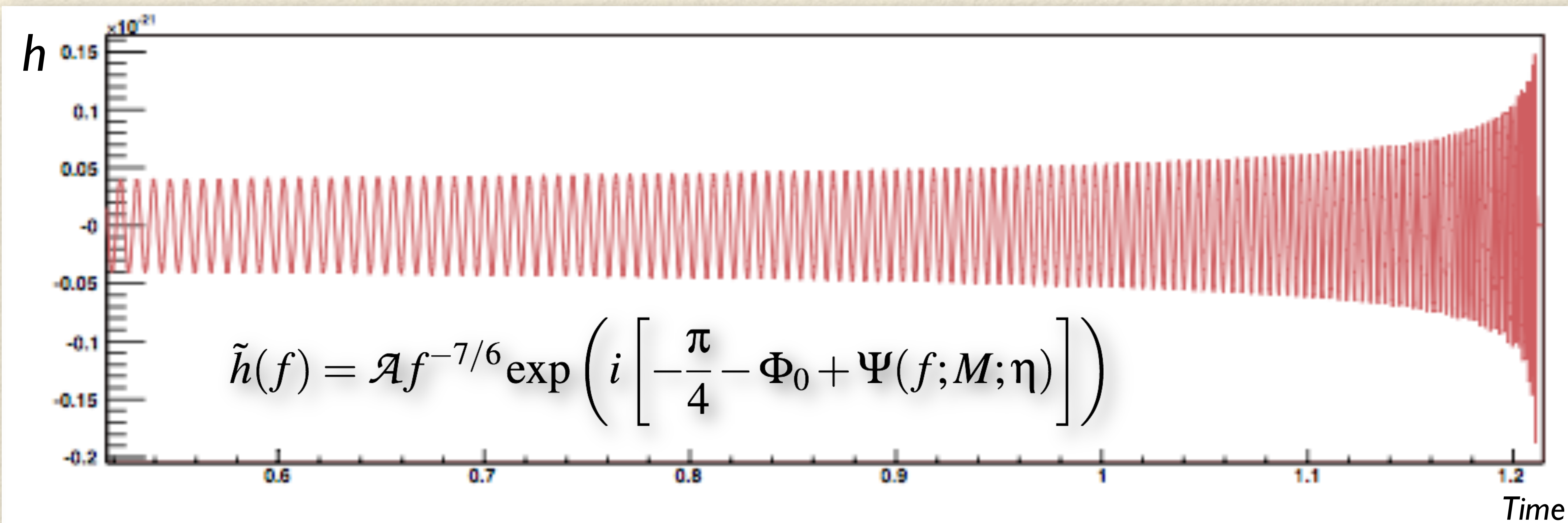
(since $2\pi f_{OG} = 2\omega$)

  - where we define the “chirp mass” :

$$\mathcal{M} = \mu^{3/5} M^{2/5}$$

Generation : binary system

- Calculated waveform before the “plunge” :



$$\tilde{h}(f) = \mathcal{A} f^{-7/6} \exp\left(i\left[-\frac{\pi}{4} - \Phi_0 + \Psi(f;M;\eta)\right]\right)$$

Generation : binary system

- GR $\longrightarrow$ post-newtonian corrections $\longrightarrow$ less simple !
- Development around the newtonian limit in $\epsilon = \left(\frac{v}{c}\right)^2$
  - v = relative speed of the two stars (dimensionless) $v = (M\omega)^{1/3}$
- For example, development of the orbital phase

$$\phi(t) = \phi_{ref} + \phi_N \sum_{k=0}^{n} \phi_{\frac{k}{2}PN} v^k$$

- The successive terms become more and more complex
  - higher order effect, spin-spin interaction, spin-orbit, radiation.

k	N	2	3	4	5
$\mathcal{F}_k$	$\frac{32\eta^2 v^{10}}{5}$	$-\frac{1247}{336} - \frac{35\eta}{12}$	4π	$-\frac{44711}{9072} + \frac{9271\eta}{504} + \frac{65\eta^2}{18}$	$-\left(\frac{8191}{672} + \frac{535\eta}{24}\right)\pi$
t_k^v	$-\frac{5m}{256\eta v^8}$	$\frac{743}{252} + \frac{11\eta}{3}$	$-\frac{32\pi}{5}$	$\frac{3058673}{508032} + \frac{5429\eta}{504} + \frac{617\eta^2}{72}$	$-\left(\frac{7729}{252} + \eta\right)\pi$
ϕ_k^v	$-\frac{1}{16\eta v^5}$	$\frac{3715}{1008} + \frac{55\eta}{12}$	-10π	$\frac{15293365}{1016064} + \frac{27145\eta}{1008} + \frac{3085\eta^2}{144}$	$\left(\frac{38645}{672} + \frac{15\eta}{8}\right)\pi \ln\left(\frac{v}{v_{\rm lso}}\right)$
ϕ_k^t	$-\frac{2}{\eta\theta^5}$	$\frac{3715}{8064} + \frac{55\eta}{96}$	$-\frac{3\pi}{4}$	$\frac{9275495}{14450688} + \frac{284875\eta}{258048} + \frac{1855\eta^2}{2048}$	$\left(\frac{38645}{21504} + \frac{15\eta}{256}\right)\pi \ln\left(\frac{\theta}{\theta_{\rm lso}}\right)$
F_k^t	$\frac{\theta^3}{8\pi m}$	$\frac{743}{2688} + \frac{11\eta}{32}$	$-\frac{3\pi}{10}$	$\frac{1855099}{14450688} + \frac{56975\eta}{258048} + \frac{371\eta^2}{2048}$	$-\left(\frac{7729}{21504} + \frac{3}{256}\eta\right)\pi$
τ_k	$\frac{3}{128\eta}$	$\frac{5}{9}\left(\frac{743}{84} + 11\eta\right)$	-16π	$2\phi_4^v$	$\frac{1}{3}\left(8\phi_5^v - 5t_5^v\right)$

Generation : binary system

- PSR 1913+16
  (Hulse et Taylor)
- points = observations,
  line = GR prediction

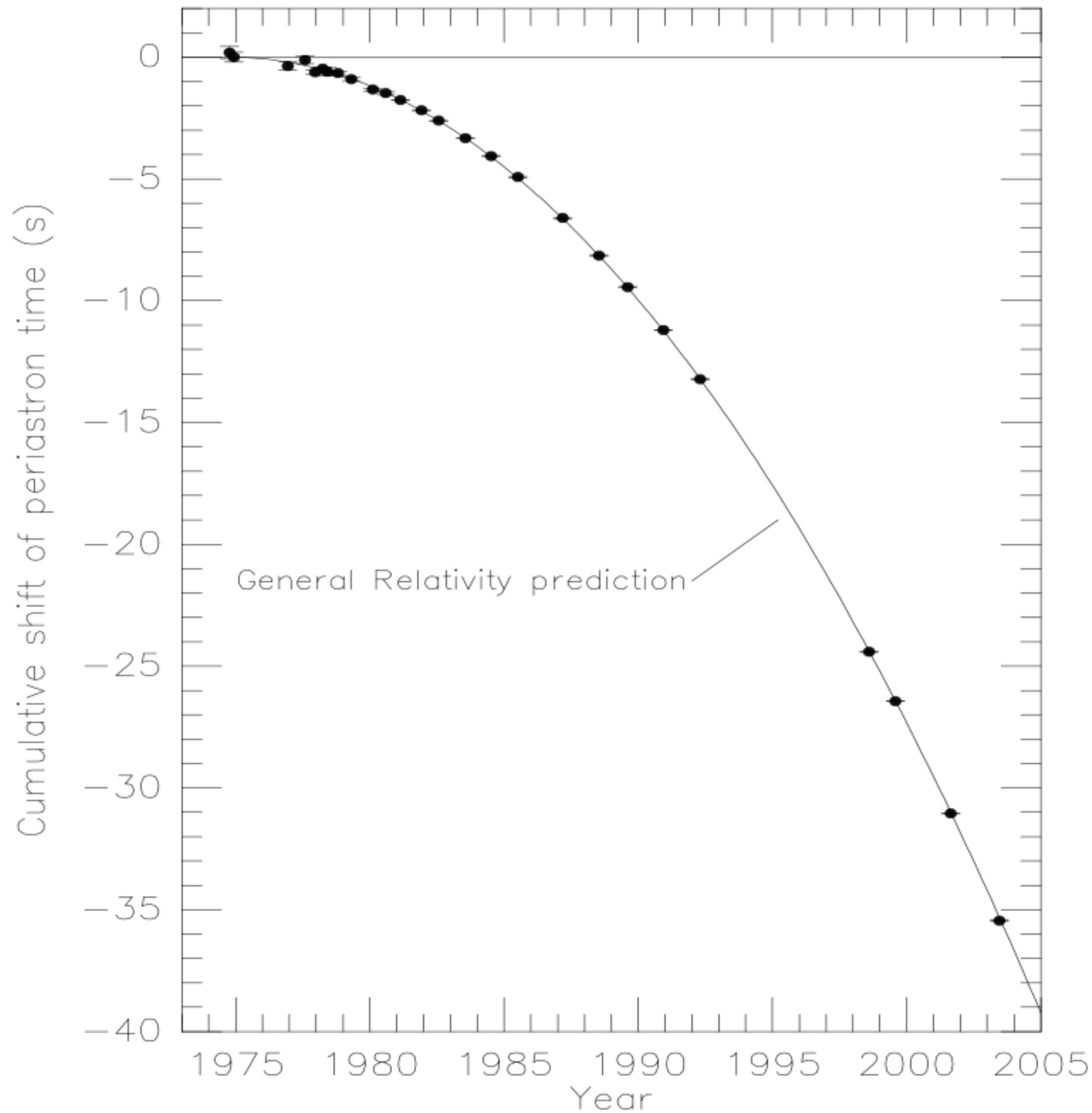
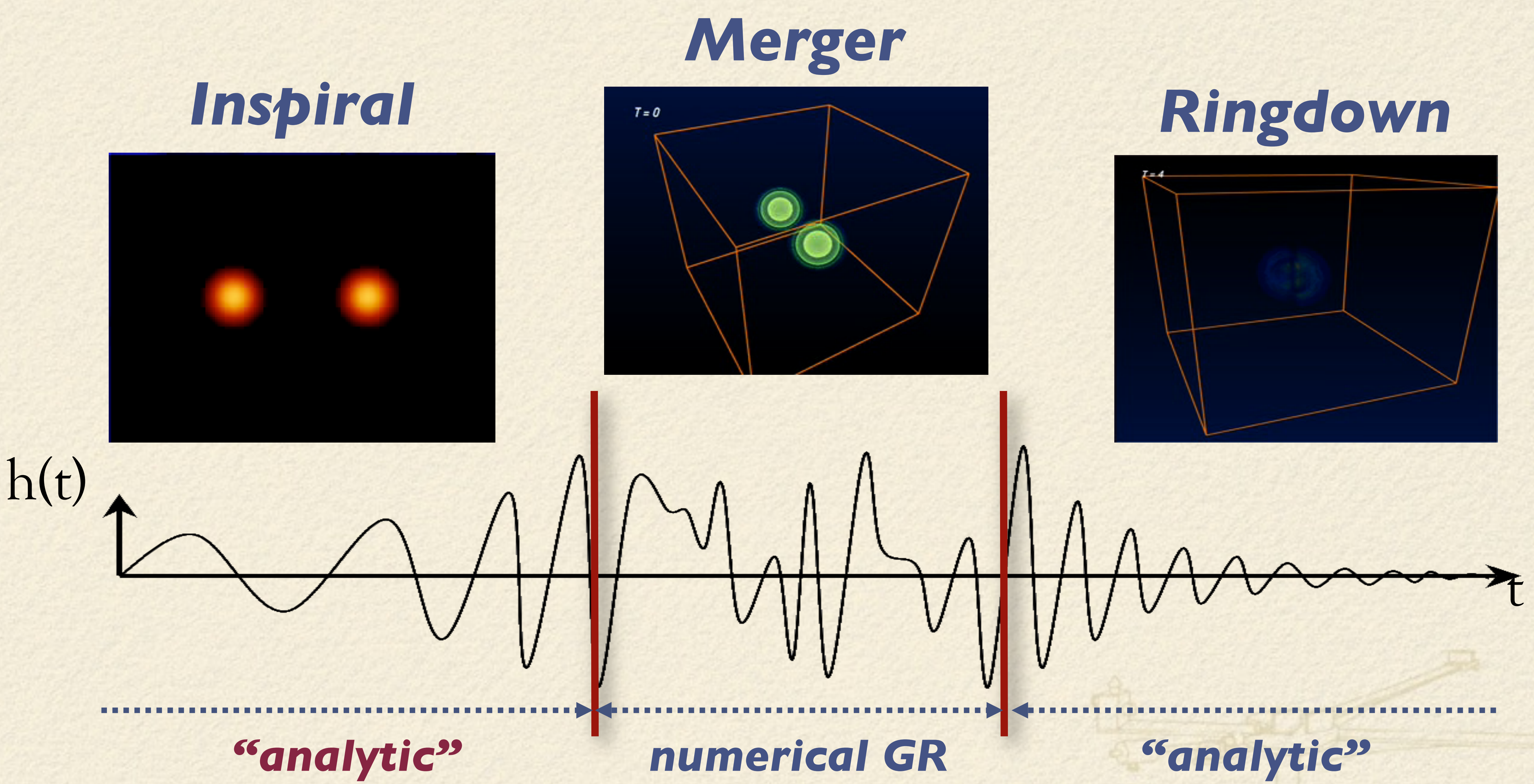


Figure 1. Orbital decay of PSR B1913+16. The data points indicate the observed change in the epoch of periastron with date while the parabola illustrates the theoretically expected change in epoch for a system emitting gravitational radiation, according to general relativity.

Sources : Coalescences

Phases of the coalescence of a binary system of compact objects
(neutron stars or black holes)



Sources : “Pulsars”

- rotating neutron stars

$\nu \sim$ 1 Hz - 1 kHz, $h \sim 10^{-25}$ à 3 kpc

Amplitude of the wave :

$$h_0 = \frac{4\pi^2 G}{c^4}\frac{I_{zz}\,\varepsilon\, f_{gw}^2}{d}$$

very poorly known or estimated

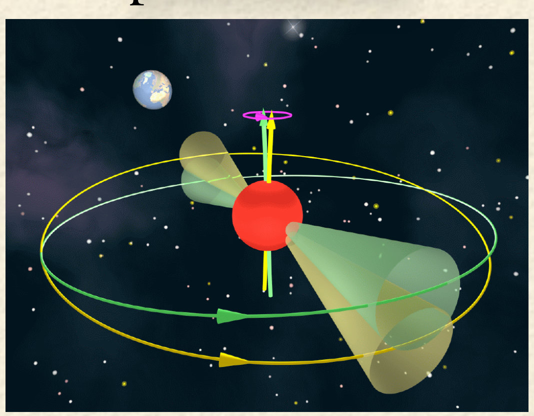
I_{zz} : moment of inertia along the axis of rotation

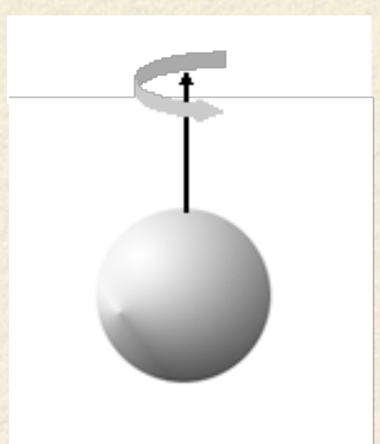
$$\varepsilon = \frac{I_{xx} - I_{yy}}{I_{zz}}$$

: ellipticity in the equatorial plane

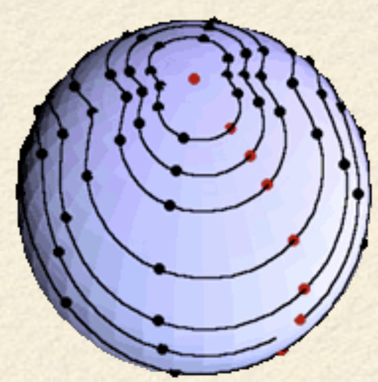
modulated (Doppler effect) by the motion and orientation of the detector around the sun

precession





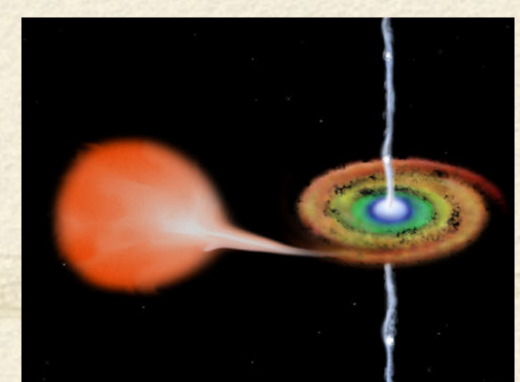
“mountains”



Oscillating modes

LMXB

(Low Mass X-ray Binaries)



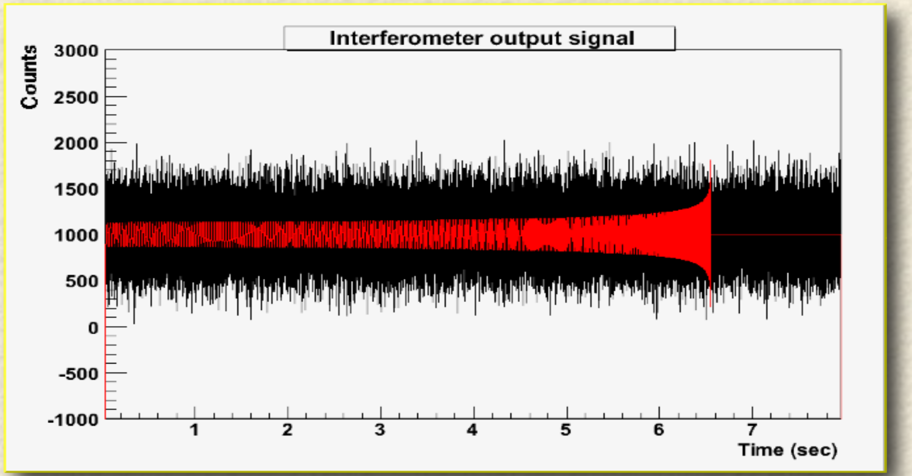
A glimpse of data analysis

The problem

- Signal buried into the noise

$$a(t) = n(t) + s(t)$$

Detector output = Noise + Signal



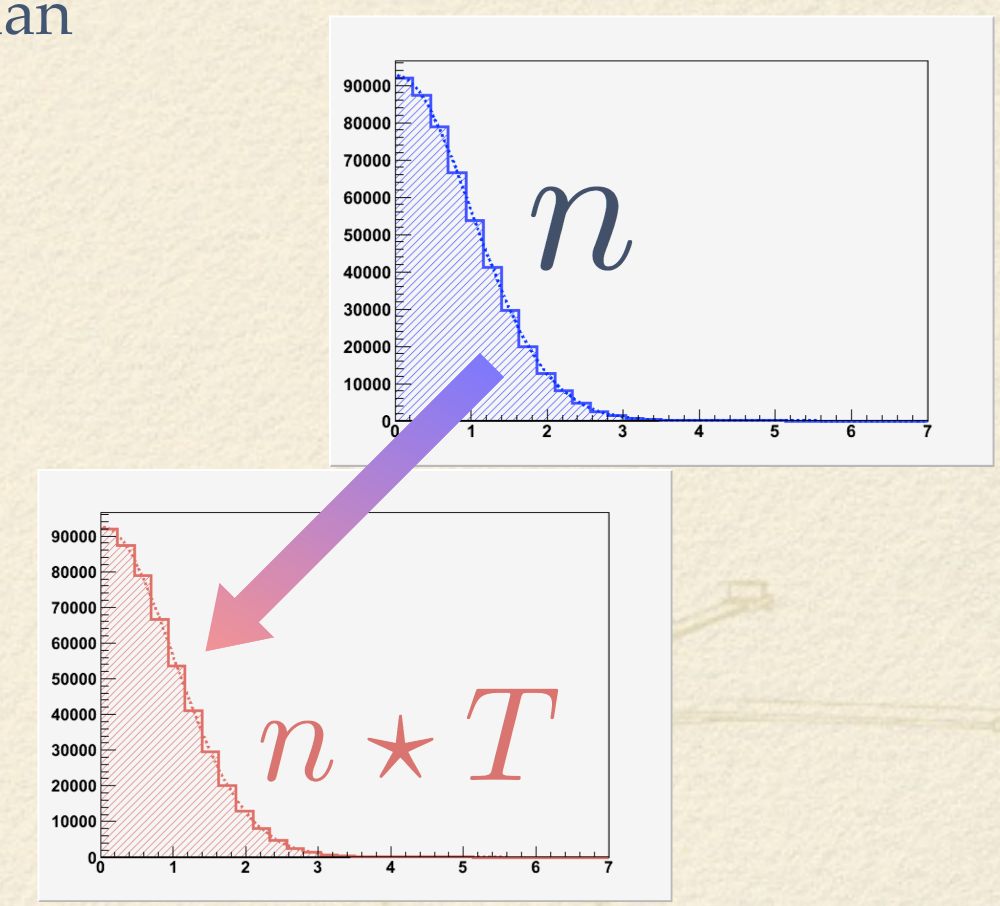
- Signal is deterministic $\Rightarrow$ expressed (in frequency domain) in $1/\mathrm{Hz}$
- Noise is stochastic $\Rightarrow$ expressed (in frequency domain) in $1/\sqrt{\mathrm{Hz}}$
- ***Not of the same nature !***
- How to recognize that a waveform is hidden in the noise ?

Intercorrelation

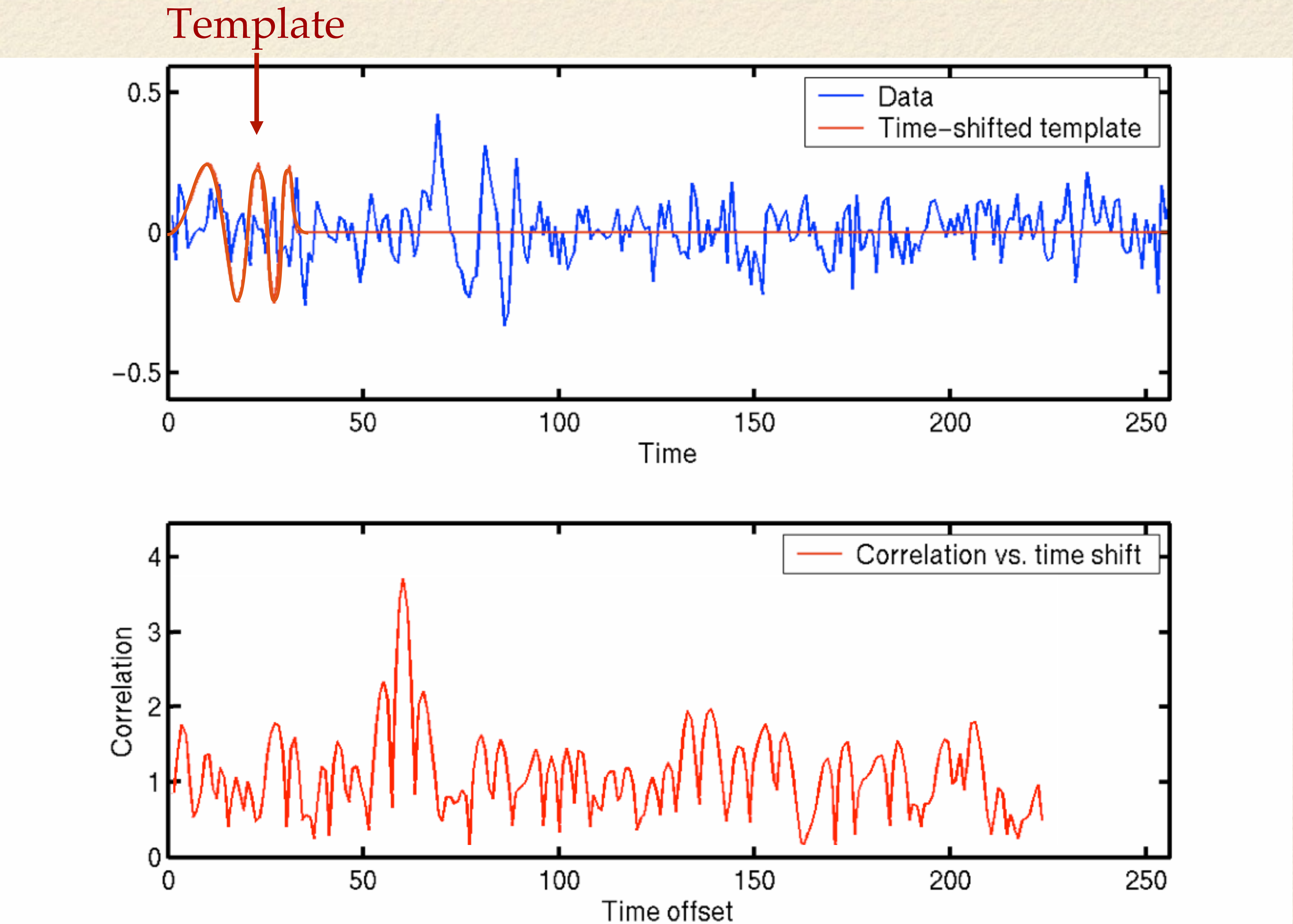
- Resemblance of two waveforms : intercorrelation

$$a_1 \star a_2(\tau) = \int_{-\infty}^{+\infty} a_1(t)\, a_2(t+\tau)\, dt$$

- If
  - The distribution of the noise values is gaussian
  - There is no signal present in the data
- Then
  - The distribution of the values of intercorrelation between the data and and a test signal (template) T is also gaussian

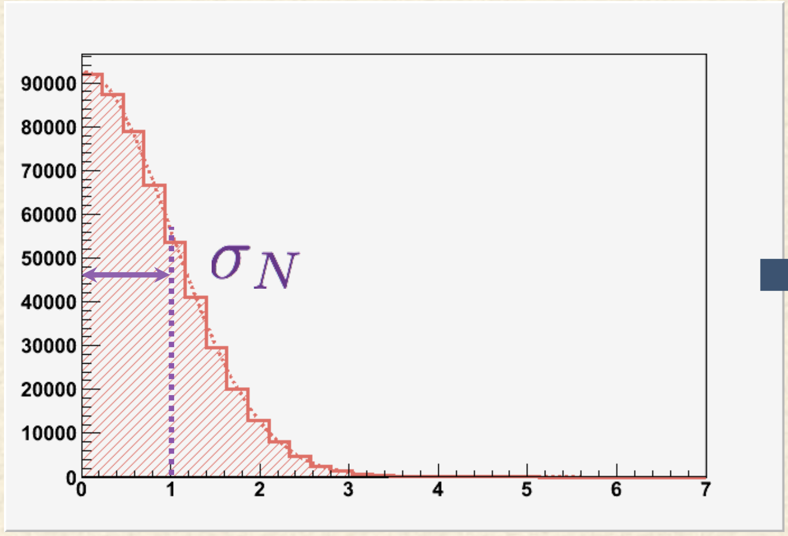


Intercorrelation

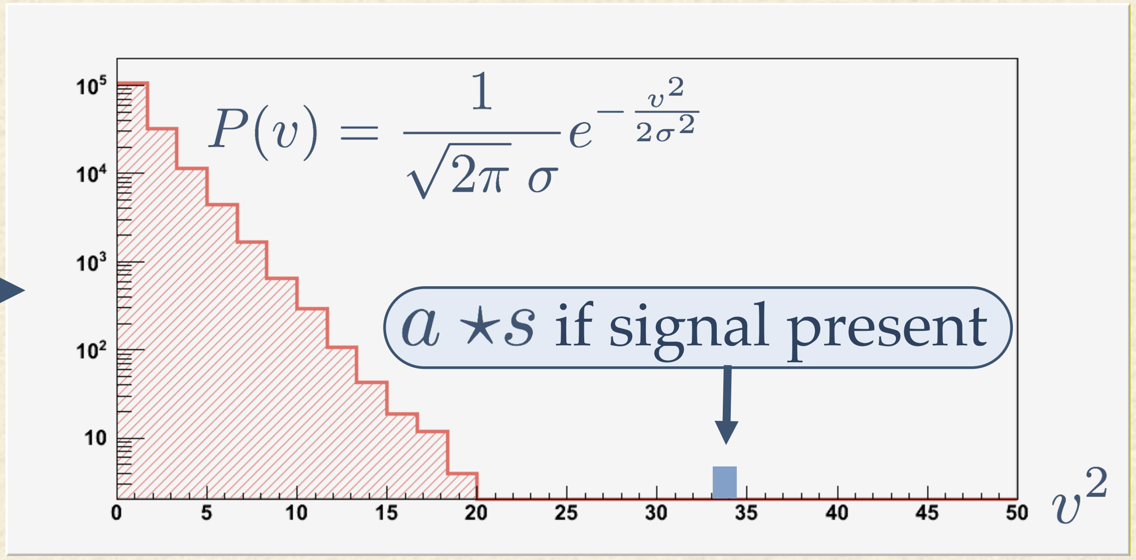


Signal over noise ratio

- If a signal s_0 is present in the noise
  - value of $a \star s$ high for s of the same shape as the signal s_0



$\ln(P)$
as a fonction of v^2



- width of the noise distribution :
- signal :
- Signal over Noise Ratio (SNR)

$$\sigma_N = \sqrt{\langle n \star s\ (\tau)^2 \rangle - \underbrace{\cancel{\langle n \star s(\tau) \rangle}}_{=0}{}^2}$$

$$S \equiv |a \star s\ (\tau)|$$

$$SNR = \frac{S}{\sigma_N}$$

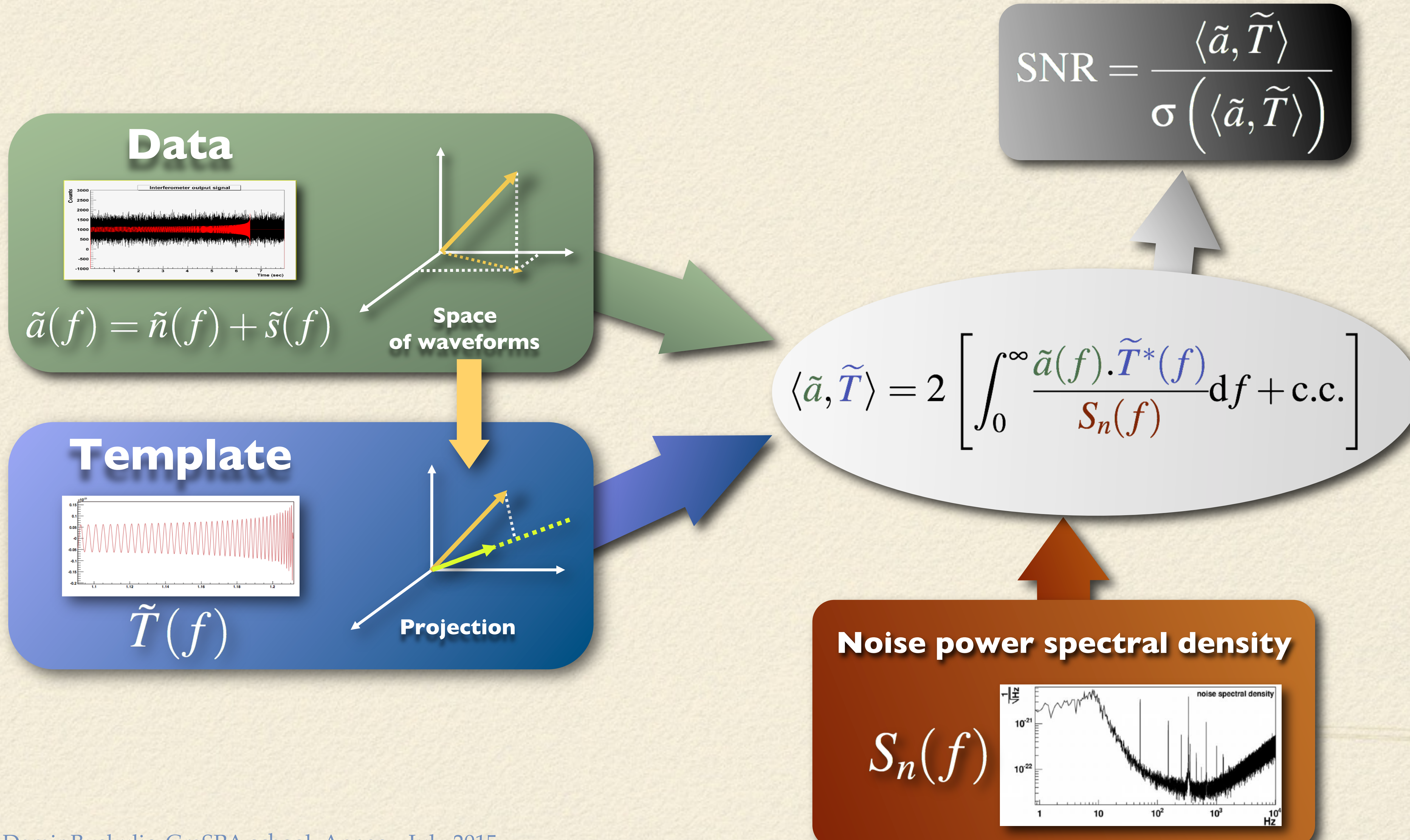
Optimal filtering

- In our case...
- Search for a waveform s buried in a noise n (radars in the 40')
- Intercorrelation gives an optimal SNR if n is a white noise (power spectral density $P_a(f) = cte$)
  - But in practice the noise is not white !
- One can show that an optimal filter can be found in the frequency domain

$$\langle \tilde{a}, \widetilde{T} \rangle = 2\left[\int_0^{\infty} \frac{\tilde{a}(f).\widetilde{T}^*(f)}{S_n(f)} \mathrm{d}f + \text{c.c.}\right]$$

  - c.c. = complex conjugate
  - can be interpreted
    - as a "weighted intercorrelation", where the weight is the noise power spectral density or
    - as a scalar product in the space of signals

Optimal filtering



Detection rate

$$\mathcal{R}_{det} \propto R.\Gamma.T$$

Observation time

Astrophysical rate

Number of reachable galaxies

- Estimated rates for current and future detectors
  (takes into account the sensitivity and galaxy distribution)

Realistic rates

TABLE V: Detection rates for compact binary coalescence sources.

IFO	Source[a]	$\dot{N}_{low}$ yr^{-1}	$\dot{N}_{re}$ yr^{-1}	$\dot{N}_{high}$ yr^{-1}	$\dot{N}_{max}$ yr^{-1}
Initial	NS-NS	2×10^{-4}	0.02	0.2	0.6
	NS-BH	7×10^{-5}	0.004	0.1	
	BH-BH	2×10^{-4}	0.007	0.5	
	IMRI into IMBH			$< 0.001^{b}$	0.01^{c}
	IMBH-IMBH			10^{-4d}	10^{-3e}
Advanced	NS-NS	0.4	40	400	1000
	NS-BH	0.2	10	300	
	BH-BH	0.4	20	1000	
	IMRI into IMBH			10^{b}	300^{c}
	IMBH-IMBH			0.1^{d}	1^{e}

Ze End

(as we say in french)