The circle packing of random hyperbolic triangulations

Asaf Nachmias (TAU and UBC)

Joint work with subsets of:
{Angel, Barlow, Gurel-Gurevich, Hutchcroft and Ray}

20th Itzykson conference, June 12th, 2015

Some Classical Analysis

Consider Brownian motion (X_t) on the hyperbolic plane $\mathbb{D} = \{|z| < 1\}$.

- Almost surely $X_t \to X_\infty \in \partial\mathbb{D}$.
- If f is bounded harmonic on $\mathbb{D}$ then $f(x) = \mathbf{E}_x g(X_\infty)$ for some bounded g on $\partial\mathbb{D}$.

For an invariant event A, $\mathbf{P}_x(A)$ is bounded harmonic, so bounded harmonic functions encode invariant events.
In $\mathbb{D}$, all invariant events have the form $\{X_\infty \in A\}$ for some $A \subset \partial\mathbb{D}$.

More classical facts

If M is any Riemann surface homeomorphic to $\mathbb{D}$ then either

> *Brownian motion on M is recurrent, M is conformally equivalent to $\mathbb{C}$, and all bounded harmonic functions are constant,*

or

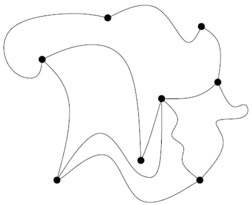
> *Brownian motion on M is transient, M is conformally equivalent to $\mathbb{D}$, and any bounded g on $\partial\mathbb{D}$ extends to M.*

Circle packing

Let G be a finite simple planar graph.

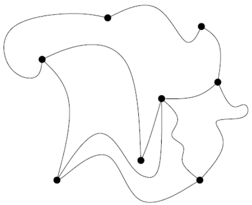
Circle packing

Let G be a finite simple planar graph.



Circle packing

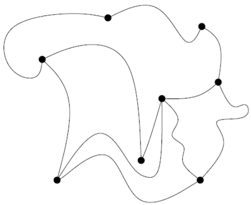
Let G be a finite simple planar graph.



The **Circle Packing Theorem** gives us a canonical way to draw G.

Circle packing

Let G be a finite simple planar graph.



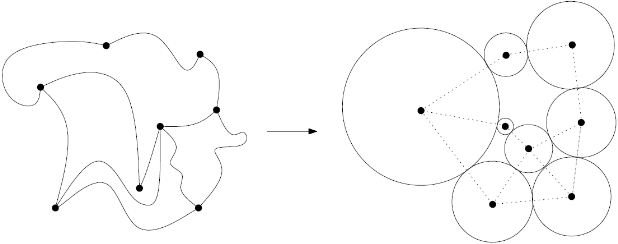
The **Circle Packing Theorem** gives us a canonical way to draw G.

Theorem (Koebe 1936, Andreev 1970, Thurston 1985)

Every finite simple planar graph is the tangency graph of a circle packing. If G is a triangulation, then the circle packing is unique up to Möbius transformations and reflections.

Circle packing

Let G be a finite simple planar graph.



The **Circle Packing Theorem** gives us a canonical way to draw G.

Theorem (Koebe 1936, Andreev 1970, Thurston 1985)

Every finite simple planar graph is the tangency graph of a circle packing. If G is a triangulation, then the circle packing is unique up to Möbius transformations and reflections.

Circle packing definitions

- A **circle packing** $P = \{C_v\}$ is a set of circles in the plane with disjoint interiors.

Circle packing definitions

- A **circle packing** $P = \{C_v\}$ is a set of circles in the plane with disjoint interiors.
- The **tangency graph** of P is a graph $G(P)$ in which the vertex set is the set of circles, and two circles are adjacent when they are tangent.

Circle packing definitions

- A **circle packing** $P = \{C_v\}$ is a set of circles in the plane with disjoint interiors.
- The **tangency graph** of P is a graph $G(P)$ in which the vertex set is the set of circles, and two circles are adjacent when they are tangent.
- The **carrier** of P is the union of all the circles of the packing, together with the curved triangular regions bounded between each triplet of mutually tangent circles corresponding to a face.

Circle packing definitions

- A **circle packing** $P = \{C_v\}$ is a set of circles in the plane with disjoint interiors.
- The **tangency graph** of P is a graph $G(P)$ in which the vertex set is the set of circles, and two circles are adjacent when they are tangent.
- The **carrier** of P is the union of all the circles of the packing, together with the curved triangular regions bounded between each triplet of mutually tangent circles corresponding to a face.
- We call a circle packing of an infinite triangulation a packing **in the disc** if its carrier is the unit disc $\mathbb{D}$, and **in the plane** if its carrier is $\mathbb{C}$.

Circle packing theorems

Theorem (Koebe-Andreev-Thurston)

Any finite planar graph G has a circle packing. If G is a sphere triangulation, the circle packing is unique (up to Möbius).

Circle packing theorems

Theorem (Koebe-Andreev-Thurston)

Any finite planar graph G has a circle packing. If G is a sphere triangulation, the circle packing is unique (up to Möbius).

Theorem (Rodin-Sullivan; Thurston's conjecture)

Certain circle packings converge to conformal maps.

Circle packing theorems

Theorem (Koebe-Andreev-Thurston)

Any finite planar graph G has a circle packing. If G is a sphere triangulation, the circle packing is unique (up to Möbius).

Theorem (Rodin-Sullivan; Thurston's conjecture)

Certain circle packings converge to conformal maps.

Theorem (He-Schramm '95)

Any plane triangulation can be circle packed in *(i.e., with carrier=) the plane $\mathbb{C}$ or the unit disc $\mathbb{D}$, but not both (CP parabolic vs. CP Hyperbolic).*

Circle packing theorems

Theorem (Koebe-Andreev-Thurston)

Any finite planar graph G has a circle packing. If G is a sphere triangulation, the circle packing is unique (up to Möbius).

Theorem (Rodin-Sullivan; Thurston's conjecture)

Certain circle packings converge to conformal maps.

Theorem (He-Schramm '95)

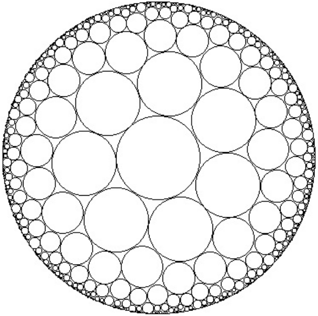
Any plane triangulation can be circle packed in *(i.e., with carrier=) the plane $\mathbb{C}$ or the unit disc $\mathbb{D}$, but not both (CP parabolic vs. CP Hyperbolic).*

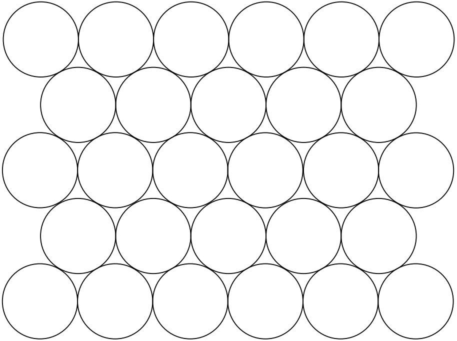
Theorem (Schramm's rigidity '91)

The above circle packing is unique up to Möbius.

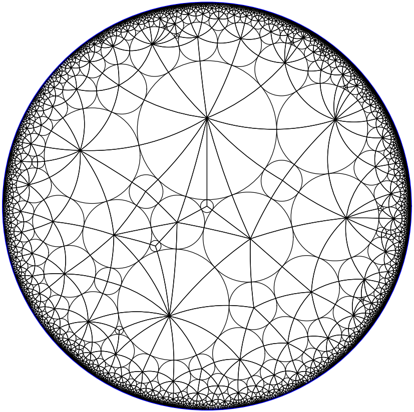
Examples

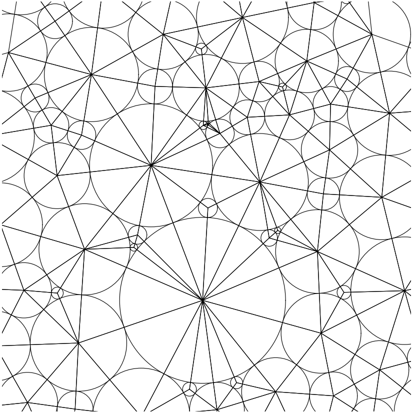
The 7-regular hyperbolic triangulation (CP hyperbolic) and the triangular lattice (CP parabolic).





Circle packing also gives us a drawing of the graph with either straight lines or hyperbolic geodesics depending on the type





In the bounded degree case, the type of the packing encapsulates a lot probabilistic information: recurrence/transience of the random walk, existence of non-trivial bounded harmonic functions, resistance estimates, etc.

In the bounded degree case, the type of the packing encapsulates a lot probabilistic information: recurrence/transience of the random walk, existence of non-trivial bounded harmonic functions, resistance estimates, etc.

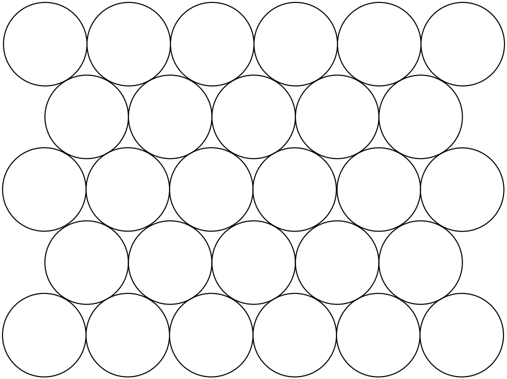
Theorem (He-Schramm '95)

If G is bounded degree, CP parabolicity is equivalent to recurrence of the simple random walk on G.

In the bounded degree case, the type of the packing encapsulates a lot probabilistic information: recurrence/transience of the random walk, existence of non-trivial bounded harmonic functions, resistance estimates, etc.

Theorem (He-Schramm '95)

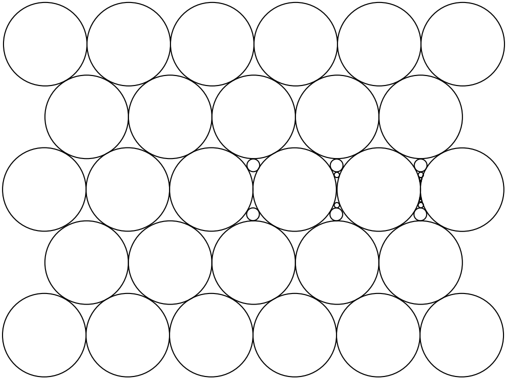
If G is bounded degree, CP parabolicity is equivalent to recurrence of the simple random walk on G.



In the bounded degree case, the type of the packing encapsulates a lot probabilistic information: recurrence/transience of the random walk, existence of non-trivial bounded harmonic functions, resistance estimates, etc.

Theorem (He-Schramm '95)

If G is bounded degree, CP parabolicity is equivalent to recurrence of the simple random walk on G.



A dichotomy for bounded degree plane triangulations

Theorem (Benjamini-Schramm '96)

Let G be CP Hyperbolic with bounded degrees. Then $X_n \to X_\infty \in \partial\mathbb{D}$.

A dichotomy for bounded degree plane triangulations

Theorem (Benjamini-Schramm '96)

Let G be CP Hyperbolic with bounded degrees. Then $X_n \to X_\infty \in \partial\mathbb{D}$.

If G has bounded degrees, CP Hyperbolic is equivalent to transience, so the dichotomy holds: Either

> *Random walk on G is recurrent, G is CP parabolic and all bounded harmonic functions are constant,*

or

> *Random walk on G is transient, G is CP hyperbolic and any bounded g on $\partial\mathbb{D}$ extends to G.*

Are there any other harmonic functions on G?

Characterization of harmonic functions

Theorem (Angel, Barlow, Gurel-Gurevich, N. 13)

No.

Characterization of harmonic functions

Theorem (Angel, Barlow, Gurel-Gurevich, N. 13)

No. For any bounded harmonic function $h : V \to \mathbb{R}$ there exists a measurable function $g : \partial\mathbb{D} \to \mathbb{R}$ such that

$$h(x) = \mathbf{E}_x g(\lim z(X_n)) .$$

In other words, $\partial\mathbb{D}$ is a realization of the Poisson-Furstenberg boundary.

- Intuition: $\lim z(X_n)$ contains all the invariant information of the random walk on G.

Characterization of harmonic functions

Theorem (Angel, Barlow, Gurel-Gurevich, N. 13)

No. For any bounded harmonic function $h : V \to \mathbb{R}$ there exists a measurable function $g : \partial\mathbb{D} \to \mathbb{R}$ such that

$$h(x) = \mathbf{E}_x g(\lim z(X_n)) .$$

In other words, $\partial\mathbb{D}$ is a realization of the Poisson-Furstenberg boundary.

- Intuition: $\lim z(X_n)$ contains all the invariant information of the random walk on G.
- This is not the case if we would pack in other domains, say, a slit domain.

We wanted to rebuild the theory for *random* triangulations without a bounded degree assumption. This required a new approach.

We wanted to rebuild the theory for *random* triangulations without a bounded degree assumption. This required a new approach.

- Question 1: Is there an analogue of the He-Schramm Theorem to characterise the CP type of a random graph by probabilistic properties?
- Question 2: Can we easily determine the CP type of a given random triangulation?

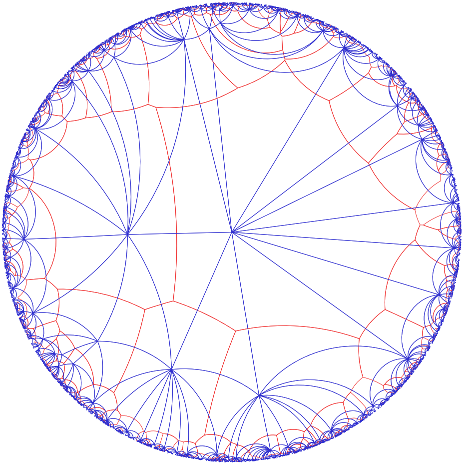
We wanted to rebuild the theory for *random* triangulations without a bounded degree assumption. This required a new approach.

- Question 1: Is there an analogue of the He-Schramm Theorem to characterise the CP type of a random graph by probabilistic properties?
- Question 2: Can we easily determine the CP type of a given random triangulation?

And, in the hyperbolic case,

- Question 3: Does the walker converge to a point in the boundary of the disc? Does the law of the limit have full support and no atoms almost surely?
- Question 4: Is the unit circle a realisation of the Poisson boundary?

Example 1: Hyperbolic Poisson-Voronoi triangulation



Benjamini-Schramm convergence of graphs was introduced to study questions of the following form

> *What does a typical triangulation of the sphere with a large number of vertices looks like microscopically near a typical point?*

Random Triangulations of the Sphere

Benjamini-Schramm convergence of graphs was introduced to study questions of the following form

> *What does a typical triangulation of the sphere with a large number of vertices looks like microscopically near a typical point?*

Take a sequence of finite graphs G_n, and for each n choose a root vertex ρ_n of G_n uniformly at random.

Random Triangulations of the Sphere

Benjamini-Schramm convergence of graphs was introduced to study questions of the following form

> *What does a typical triangulation of the sphere with a large number of vertices looks like microscopically near a typical point?*

Take a sequence of finite graphs G_n, and for each n choose a root vertex ρ_n of G_n uniformly at random.

The G_n Benjamini-Schramm converge to a random rooted graph (G, ρ) if for each fixed r, the balls of radius r converge in distribution:

$$B_r(G_n, \rho_n) \xrightarrow{d} B_r(G, \rho)$$

Examples of Benjamini-Schramm convergence

- Large tori $\mathbb{Z}^d/n\mathbb{Z}^d$ and large boxes $[-n,n]^d$ Benjamini-Schramm converge to the lattice $\mathbb{Z}^d$.

Examples of Benjamini-Schramm convergence

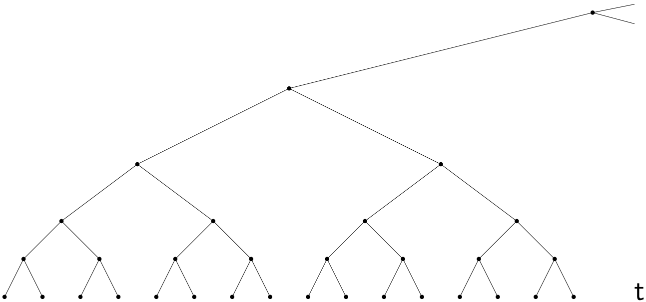
- Large tori $\mathbb{Z}^d/n\mathbb{Z}^d$ and large boxes $[-n, n]^d$ Benjamini-Schramm converge to the lattice $\mathbb{Z}^d$.
- Critical Erdös-Rényi random graphs $G(\frac{c}{n}, n)$ converge to Poisson(c) Galton-Watson trees.

Examples of Benjamini-Schramm convergence

- Large tori $\mathbb{Z}^d/n\mathbb{Z}^d$ and large boxes $[-n,n]^d$ Benjamini-Schramm converge to the lattice $\mathbb{Z}^d$.
- Critical Erdös-Rényi random graphs $G(\frac{c}{n},n)$ converge to Poisson(c) Galton-Watson trees.
- The height n binary tree converges to...

Examples of Benjamini-Schramm convergence

- Large tori $\mathbb{Z}^d/n\mathbb{Z}^d$ and large boxes $[-n,n]^d$ Benjamini-Schramm converge to the lattice $\mathbb{Z}^d$.
- Critical Erdös-Rényi random graphs $G(\frac{c}{n}, n)$ converge to Poisson(c) Galton-Watson trees.
- The height n binary tree converges to...



the **canopy tree**.

Theorem (Benjamini and Schramm '01)

Every Benjamini-Schramm limit of finite simple triangulations is CP parabolic.

Theorem (Angel and Schramm '03)

Let T_n be a uniformly random triangulation of the sphere. The Benjamini-Schramm limit of T_n as $n \to \infty$ exists. We call this limit the UIPT – Uniform Infinite Plane Triangulation.

Theorem (Benjamini and Schramm '01)

Every Benjamini-Schramm limit of finite simple triangulations is CP parabolic.

Theorem (Angel and Schramm '03)

Let T_n be a uniformly random triangulation of the sphere. The Benjamini-Schramm limit of T_n as $n \to \infty$ exists. We call this limit the UIPT – Uniform Infinite Plane Triangulation.

Theorem (Gurel-Gurevich, N. 2013)

The UIPT is almost surely recurrent.

Example 2: hyperbolic triangulations with Markov property

The UIPT has a natural Markov property – if we explore the UIPT, revealing more of it by peeling away at the boundary, the law of the part we haven't uncovered yet depends only on the length of the boundary of the piece we've already revealed.

Example 2: hyperbolic triangulations with Markov property

The UIPT has a natural Markov property – if we explore the UIPT, revealing more of it by peeling away at the boundary, the law of the part we haven't uncovered yet depends only on the length of the boundary of the piece we've already revealed.

Are there any other triangulations with this property?

Theorem (Angel and Ray '13, Curien '13)

Yes. The laws Markovian plane triangulations form a one-parameter parameter family T_κ, $\kappa \in (0, 2/27]$. *The endpoint* $\kappa = 2/27$ *is the UIPT, all the others have 'hyperbolic flavour'.*

Example 2: hyperbolic triangulations with Markov property

The UIPT has a natural Markov property – if we explore the UIPT, revealing more of it by peeling away at the boundary, the law of the part we haven't uncovered yet depends only on the length of the boundary of the piece we've already revealed.

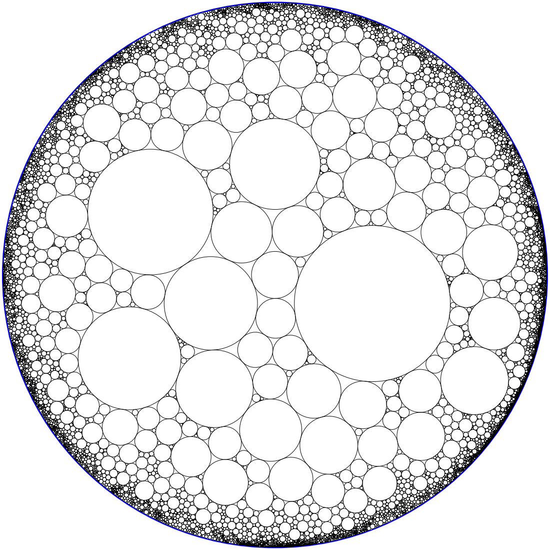
Are there any other triangulations with this property?

Theorem (Angel and Ray '13, Curien '13)

Yes. The laws Markovian plane triangulations form a one-parameter parameter family T_κ, $\kappa \in (0, 2/27]$. The endpoint $\kappa = 2/27$ is the UIPT, all the others have 'hyperbolic flavour'.

Conjecturally, the hyperbolic triangulations are Benjamini-Schramm limits of uniform triangulations with n vertices of surfaces of genus cn.

Random Hyperbolic Triangulations



Distributional limits of finite planar triangulations

A random rooted graph (G, ρ) is called unimodular if the *mass transport principle* holds: for any *automorphism invariant* $f : \mathcal{G}^{**} \to \mathbb{R}_+$,

$$\mathbf{E} \sum_v f(G, \rho, v) = \mathbf{E} \sum_v f(G, v, \rho).$$

That is, "expected mass out equals expected mass in."

A random rooted graph (G, ρ) is called unimodular if the *mass transport principle* holds: for any *automorphism invariant* $f : \mathcal{G}^{**} \to \mathbb{R}_+$,

$$\mathbf{E} \sum_v f(G, \rho, v) = \mathbf{E} \sum_v f(G, v, \rho).$$

That is, "expected mass out equals expected mass in."
This is a property of the law of (G, ρ) (not of a particular graph)

Distributional limits of finite planar triangulations

A random rooted graph (G,ρ) is called unimodular if the *mass transport principle* holds: for any *automorphism invariant* $f : \mathcal{G}^{**} \to \mathbb{R}_+$,

$$\mathbf{E}\sum_v f(G,\rho,v) = \mathbf{E}\sum_v f(G,v,\rho).$$

That is, "expected mass out equals expected mass in."
This is a property of the law of (G,ρ) (not of a particular graph)
Examples: Cayley graphs; Finite graphs with uniform root ρ; Distributional limits of finite graphs ("sofic" graphs; are these all?)

Distributional limits of finite planar triangulations

A random rooted graph (G, ρ) is called unimodular if the *mass transport principle* holds: for any *automorphism invariant* $f : \mathcal{G}^{**} \to \mathbb{R}_+$,

$$\mathbf{E} \sum_v f(G, \rho, v) = \mathbf{E} \sum_v f(G, v, \rho).$$

That is, "expected mass out equals expected mass in."
This is a property of the law of (G, ρ) (not of a particular graph)
Examples: Cayley graphs; Finite graphs with uniform root ρ; Distributional limits of finite graphs ("sofic" graphs; are these all?)

Theorem (Angel, Hutchcroft, N., Ray 2014)

Let G be a unimodular plane triangulation. Then either

- *G is CP parabolic and $\mathbf{E}deg(\rho) = 6$, or*
- *G is CP hyperbolic and $\mathbf{E}deg(\rho) > 6$.*

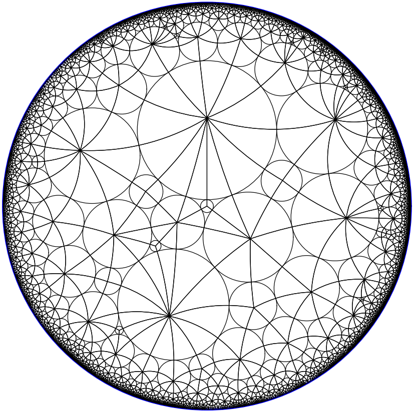
Corollary

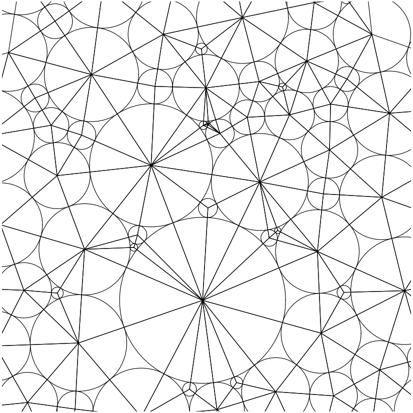
Theorem (Benjamini, Schramm 1996)

Any distributional limit of finite planar triangulations is CP-parabolic. Hence, if the degrees are bounded, the resulting graph is almost surely recurrent for the simple random walk.

Our proof avoids the use of a powerful, yet rather technical, lemma of Benjamini-Schramm known as the "magical lemma".

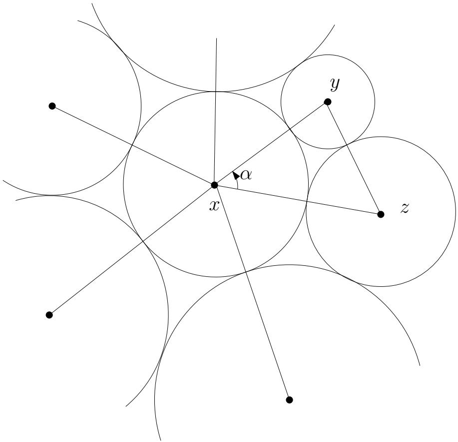
Circle packing also gives us a drawing of the graph with either straight lines or hyperbolic geodesics depending on the type





Proof: CP type and average degree (parabolic case)

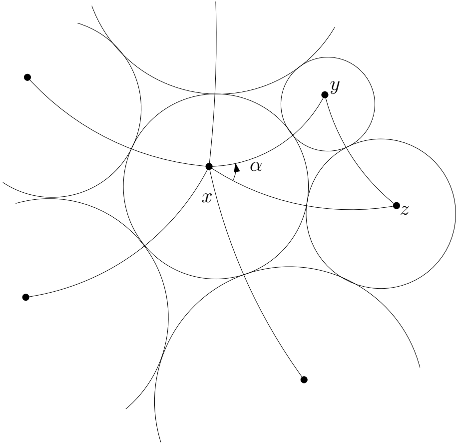
For each corner, send α from x to each of x, y, z.



Mass out is 6π. Mass in is $\pi \deg(x)$.

Proof: CP type and average degree (hyperbolic case)

For each corner, send α from x to each of x, y, z.



Mass out is 6π. Mass in is less than $\pi \deg(x)$.

Non-amenability

- Recall that the (edge) Cheeger constant of an infinite graph G is defined to be

$$\iota_E(G) = \inf\{\frac{|\partial_E W|}{|W|} : W \subset V(G) \text{ finite}\},$$

where $|W| = \sum_{v \in W} \deg(v)$.

Non-amenability

- Recall that the (edge) Cheeger constant of an infinite graph G is defined to be

$$\iota_E(G) = \inf\{\frac{|\partial_E W|}{|W|} : W \subset V(G) \text{ finite}\},$$

where $|W| = \sum_{v \in W} \deg(v)$.

- G is said to be amenable if $\iota_E(G) = 0$ and non-amenable otherwise.

Non-amenability

- Recall that the (edge) Cheeger constant of an infinite graph G is defined to be

$$\iota_E(G) = \inf\{\frac{|\partial_E W|}{|W|} : W \subset V(G) \text{ finite}\},$$

  where $|W| = \sum_{v \in W} \deg(v)$.
- G is said to be amenable if $\iota_E(G) = 0$ and non-amenable otherwise.
- $\mathbb{Z}^d$ is amenable, but the 3-regular tree is non-amenable.

Non-amenability

- Recall that the (edge) Cheeger constant of an infinite graph G is defined to be
$$\iota_E(G) = \inf\{\frac{|\partial_E W|}{|W|} : W \subset V(G) \text{ finite}\},$$
where $|W| = \sum_{v\in W} \deg(v)$.
- G is said to be amenable if $\iota_E(G) = 0$ and non-amenable otherwise.
- $\mathbb{Z}^d$ is amenable, but the 3-regular tree is non-amenable.
- Theorem (Kesten '59, Cheeger '70, Dodziuk '84): non-amenability is equivalent to the exponential decay of the heat-kernel, i.e.,
$$P_x(X_n = x) \le a^n \text{ for some } a < 1 \text{ and all } x \in G.$$

Non-amenability

- Recall that the (edge) Cheeger constant of an infinite graph G is defined to be
$$\iota_E(G) = \inf\{\frac{|\partial_E W|}{|W|} : W \subset V(G) \text{ finite}\},$$
where $|W| = \sum_{v \in W} \deg(v)$.
- G is said to be amenable if $\iota_E(G) = 0$ and non-amenable otherwise.
- $\mathbb{Z}^d$ is amenable, but the 3-regular tree is non-amenable.
- Theorem (Kesten '59, Cheeger '70, Dodziuk '84): non-amenability is equivalent to the exponential decay of the heat-kernel, i.e.,
$$P_x(X_n = x) \le a^n \text{ for some } a < 1 \text{ and all } x \in G.$$
- Are unimodular CP hyperbolic triangulations non-amenable?

Non-amenability

- Recall that the (edge) Cheeger constant of an infinite graph G is defined to be

$$\iota_E(G) = \inf\{\frac{|\partial_E W|}{|W|} : W \subset V(G) \text{ finite}\},$$

  where $|W| = \sum_{v \in W} \deg(v)$.
- G is said to be amenable if $\iota_E(G) = 0$ and non-amenable otherwise.
- $\mathbb{Z}^d$ is amenable, but the 3-regular tree is non-amenable.
- Theorem (Kesten '59, Cheeger '70, Dodziuk '84): non-amenability is equivalent to the exponential decay of the heat-kernel, i.e.,

$$P_x(X_n = x) \leq a^n \text{ for some } a < 1 \text{ and all } x \in G.$$

- Are unimodular CP hyperbolic triangulations non-amenable?
  No, the condition is too strong.

Invariant non-amenability (Aldous-Lyons '07)

- A percolation on a unimodular random graph (G, ρ) is a random subgraph ω of G such that (G, ρ, ω) is unimodular.

Invariant non-amenability (Aldous-Lyons '07)

- A percolation on a unimodular random graph (G, ρ) is a random subgraph ω of G such that (G, ρ, ω) is unimodular. The component $K_\omega(\rho)$ of ω at ρ is the connected component of ω containing ρ.

Invariant non-amenability (Aldous-Lyons '07)

- A percolation on a unimodular random graph (G, ρ) is a random subgraph ω of G such that (G, ρ, ω) is unimodular. The component $K_\omega(\rho)$ of ω at ρ is the connected component of ω containing ρ. We say ω is finite if all of its connected components are finite almost surely.

Invariant non-amenability (Aldous-Lyons '07)

- A percolation on a unimodular random graph (G, ρ) is a random subgraph ω of G such that (G, ρ, ω) is unimodular. The component $K_\omega(\rho)$ of ω at ρ is the connected component of ω containing ρ. We say ω is finite if all of its connected components are finite almost surely.
- The invariant Cheeger constant of (G, ρ) is defined to be

$$\iota^{\text{inv}}(G) = \inf\left\{\mathbf{E}\left[\frac{|\partial_E K_\omega(\rho)|}{|K_\omega(\rho)|}\right] : \omega \text{ a finite percolation}\right\}.$$

Invariant non-amenability (Aldous-Lyons '07)

- A percolation on a unimodular random graph (G, ρ) is a random subgraph ω of G such that (G, ρ, ω) is unimodular. The component $K_\omega(\rho)$ of ω at ρ is the connected component of ω containing ρ. We say ω is finite if all of its connected components are finite almost surely.
- The invariant Cheeger constant of (G, ρ) is defined to be

$$\iota^{\mathrm{inv}}(G) = \inf \left\{ \mathbf{E}\left[\frac{|\partial_E K_\omega(\rho)|}{|K_\omega(\rho)|} \right] : \omega \text{ a finite percolation} \right\}.$$

- A unimodular graph (G, ρ) is said to be invariantly non-amenable iff $\iota^{\mathrm{inv}}(G) > 0$.

Invariant non-amenability (Aldous-Lyons '07)

- A percolation on a unimodular random graph (G, ρ) is a random subgraph ω of G such that (G, ρ, ω) is unimodular. The component $K_\omega(\rho)$ of ω at ρ is the connected component of ω containing ρ. We say ω is finite if all of its connected components are finite almost surely.
- The invariant Cheeger constant of (G, ρ) is defined to be

$$\iota^{\mathrm{inv}}(G) = \inf\left\{\mathbf{E}\left[\frac{|\partial_E K_\omega(\rho)|}{|K_\omega(\rho)|}\right] : \omega \text{ a finite percolation}\right\}.$$

- A unimodular graph (G, ρ) is said to be invariantly non-amenable iff $\iota^{\mathrm{inv}}(G) > 0$.
- Easy fact: $\iota^{\mathrm{inv}}(G) = \mathbf{E}\deg(\rho) - \alpha(G)$ where

$$\alpha(G) = \sup\left\{\mathbf{E}[\deg_\omega(\rho)] : \omega \text{ a finite percolation}\right\}.$$

Invariant amenability: examples

- $\mathbb{Z}^d$ is invariantly amenable.

Invariant amenability: examples

- $\mathbb{Z}^d$ is invariantly amenable.
- Take 3-regular tree and replace each edge e with a path of length L_e where $\{L_e\}_e$ are i.i.d. unbounded random variables with $\mathbf{E}L_e < \infty$. This is invariantly non-amenable, but is a.s. amenable.

Invariant amenability: examples

- $\mathbb{Z}^d$ is invariantly amenable.
- Take 3-regular tree and replace each edge e with a path of length L_e where $\{L_e\}_e$ are i.i.d. unbounded random variables with $\mathbf{E}L_e < \infty$. This is invariantly non-amenable, but is a.s. amenable.
- Critical Galton-Watson tree conditioned to survive is invariantly amenable.

Invariant amenability: examples

- $\mathbb{Z}^d$ is invariantly amenable.
- Take 3-regular tree and replace each edge e with a path of length L_e where $\{L_e\}_e$ are i.i.d. unbounded random variables with $\mathbf{E}L_e < \infty$. This is invariantly non-amenable, but is a.s. amenable.
- Critical Galton-Watson tree conditioned to survive is invariantly amenable.
- Fact (Aldous-Lyons '07): If (G, ρ) is unimodular and is a.s. recurrent, then it is invariantly amenable.

CP Hyperbolic triangulations

By Euler's formula the average degree of any *finite* planar triangulation is at most 6. Hence,

Theorem (Angel, Hutchcroft, N., Ray 2014)

Let G be a unimodular plane triangulation. Then either

- *G is CP parabolic and $\mathbf{E}deg(\rho) = 6$ and is invariantly amenable, or*
- *G is CP hyperbolic and $\mathbf{E}deg(\rho) > 6$ and is invariantly non-amenable.*

CP Hyperbolic triangulations

By Euler's formula the average degree of any *finite* planar triangulation is at most 6. Hence,

Theorem (Angel, Hutchcroft, N., Ray 2014)

Let G be a unimodular plane triangulation. Then either

- *G is CP parabolic and $\mathbf{E}\deg(\rho) = 6$ and is invariantly amenable, or*
- *G is CP hyperbolic and $\mathbf{E}\deg(\rho) > 6$ and is invariantly non-amenable.*

This is good news because:

Theorem (Benjamini-Lyons-Schramm '99)

Let (G, ρ) be an invariantly non-amenable ergodic unimodular random rooted graph with $E[\deg(\rho)] < \infty$. Then G admits an ergodic percolation ω so that $\iota_E(\omega) > 0$ and all vertices in ω have uniformly bounded degrees in G.

Boundary theory

Theorem (Angel, Hutchcroft, N., Ray '14)

Let (G,ρ) be a CP hyperbolic unimodular random planar triangulation with $\mathbb{E}[\deg^2(\rho)] < \infty$ and let $\mathcal{C}$ be a circle packing of G in the unit disc. The following hold conditional on (G,ρ) almost surely:

1. *The random walk almost surely has $X_n \to X_\infty \in \partial\mathbb{D}$*
2. *The law of X_∞ has full support and no atoms.*
3. *$\partial\mathbb{D}$ is a realisation of the Poisson-Furstenberg boundary of G.*

Proof of convergence: $X_n \to X_\infty \in \partial\mathbb{D}$

- Assume G is **really** non-amenable and has degrees bounded by M. Then for some $a < 1$ and any v

$$\mathbf{P}_\rho(X_n = v) \le M^{1/2} a^n \,.$$

Proof of convergence: $X_n \to X_\infty \in \partial\mathbb{D}$

- Assume G is **really** non-amenable and has degrees bounded by M. Then for some $a < 1$ and any v

$$\mathbf{P}_\rho(X_n = v) \leq M^{1/2} a^n\,.$$

- Since the total area of circles is at most π, there are at most $(1/a)^{n/2}$ circles of radius $\geq a^{n/4}$.

Proof of convergence: $X_n \to X_\infty \in \partial\mathbb{D}$

- Assume G is **really** non-amenable and has degrees bounded by M. Then for some $a < 1$ and any v
$$\mathbf{P}_\rho(X_n = v) \le M^{1/2} a^n \, .$$
- Since the total area of circles is at most π, there are at most $(1/a)^{n/2}$ circles of radius $\ge a^{n/4}$. So $\mathbf{P}_\rho(\text{radius}(X_n) \ge a^{n/4}) \le M^{1/2} a^{n/2}$.

Proof of convergence: $X_n \to X_\infty \in \partial\mathbb{D}$

- Assume G is **really** non-amenable and has degrees bounded by M. Then for some $a < 1$ and any v

$$\mathbf{P}_\rho(X_n = v) \leq M^{1/2} a^n \,.$$

- Since the total area of circles is at most π, there are at most $(1/a)^{n/2}$ circles of radius $\geq a^{n/4}$. So $\mathbf{P}_\rho(\text{radius}(X_n) \geq a^{n/4}) \leq M^{1/2} a^{n/2}$.
- So $\mathbf{E}_\rho[\text{radius}(X_n)] \leq M^{1/2} a^{n/2} + a^{n/4}$,

Proof of convergence: $X_n \to X_\infty \in \partial\mathbb{D}$

- Assume G is **really** non-amenable and has degrees bounded by M. Then for some $a < 1$ and any v

$$\mathbf{P}_\rho(X_n = v) \le M^{1/2} a^n \,.$$

- Since the total area of circles is at most π, there are at most $(1/a)^{n/2}$ circles of radius $\ge a^{n/4}$. So $\mathbf{P}_\rho(\text{radius}(X_n) \ge a^{n/4}) \le M^{1/2} a^{n/2}$.
- So $\mathbf{E}_\rho[\text{radius}(X_n)] \le M^{1/2} a^{n/2} + a^{n/4}$, and in particular X_n converges since

$$\mathbf{E}_\rho\Big[\sum_n \text{radius}(X_n)\Big] < \infty \,.$$

Proof of convergence: $X_n \to X_\infty \in \partial\mathbb{D}$

- Assume G is **really** non-amenable and has degrees bounded by M. Then for some $a < 1$ and any v

$$\mathbf{P}_\rho(X_n = v) \leq M^{1/2} a^n \,.$$

- Since the total area of circles is at most π, there are at most $(1/a)^{n/2}$ circles of radius $\geq a^{n/4}$. So $\mathbf{P}_\rho(\text{radius}(X_n) \geq a^{n/4}) \leq M^{1/2} a^{n/2}$.
- So $\mathbf{E}_\rho[\text{radius}(X_n)] \leq M^{1/2} a^{n/2} + a^{n/4}$, and in particular X_n converges since

$$\mathbf{E}_\rho\Big[\sum_n \text{radius}(X_n)\Big] < \infty \,.$$

- When G is only invariantly non-amenable, perform the same argument on the "dense" non-amenable subgraph and argue that in the times the random walker is not in this subgraph things cannot go very badly.

Exponential decay of radii

This argument carries through to our setting with a little work, and in fact more is true:

Theorem (Angel, Hutchcroft, N., Ray '14)

Under the same setup as before, the Euclidean radii of the circles decay exponentially, the walk has positive speed in the hyperbolic metric, and the two rates agree:

$$\lim_{n\to\infty} \frac{d_{hyp}(z_h(\rho), z_h(X_n))}{n} = \lim_{n\to\infty} \frac{-\log r(X_n)}{n} > 0.$$

Exponential decay of radii

This argument carries through to our setting with a little work, and in fact more is true:

Theorem (Angel, Hutchcroft, N., Ray '14)

Under the same setup as before, the Euclidean radii of the circles decay exponentially, the walk has positive speed in the hyperbolic metric, and the two rates agree:

$$\lim_{n\to\infty} \frac{d_{hyp}(z_h(\rho), z_h(X_n))}{n} = \lim_{n\to\infty} \frac{-\log r(X_n)}{n} > 0.$$

This is not something that is necessarily true in the deterministic bounded degree case!

Proof of non-atomic exit measure

- Assume (G, ρ) is stationary (i.e., $(G, \rho) \overset{d}{=} (G, X_1)$) (easy to obtain from a unimodularity via degree biasing).
- Let us first show that a.s. there are either no atoms or one atom with mass 1.

Proof of non-atomic exit measure

- Assume (G, ρ) is stationary (i.e., $(G, \rho) \stackrel{d}{=} (G, X_1)$) (easy to obtain from a unimodularity via degree biasing).
- Let us first show that a.s. there are either no atoms or one atom with mass 1.
- For each atom ξ, let $h_\xi(v) = \mathbf{P}_x(X_\infty = \xi)$.

Proof of non-atomic exit measure

- Assume (G, ρ) is stationary (i.e., $(G, \rho) \stackrel{d}{=} (G, X_1)$) (easy to obtain from a unimodularity via degree biasing).
- Let us first show that a.s. there are either no atoms or one atom with mass 1.
- For each atom ξ, let $h_\xi(v) = \mathbf{P}_x(X_\infty = \xi)$. Then $h_\xi : G \to [0, 1]$ is harmonic, and by Levy's 0-1 law

$$h_\xi(X_n) \xrightarrow[n\to\infty]{\text{a.s.}} \mathbb{1}(X_\infty = \xi)$$

Proof of non-atomic exit measure

- Assume (G, ρ) is stationary (i.e., $(G, \rho) \stackrel{d}{=} (G, X_1)$) (easy to obtain from a unimodularity via degree biasing).
- Let us first show that a.s. there are either no atoms or one atom with mass 1.
- For each atom ξ, let $h_\xi(v) = \mathbf{P}_x(X_\infty = \xi)$. Then $h_\xi : G \to [0, 1]$ is harmonic, and by Levy's 0-1 law

$$h_\xi(X_n) \xrightarrow[n\to\infty]{\text{a.s.}} \mathbb{1}(X_\infty = \xi)$$

- Hence, define $M(v) := \max_\xi h_\xi(v)$, so that a.s. $\lim_n M(X_n) \in \{0, 1\}$.

Proof of non-atomic exit measure

- Assume (G, ρ) is stationary (i.e., $(G, \rho) \stackrel{d}{=} (G, X_1)$) (easy to obtain from a unimodularity via degree biasing).
- Let us first show that a.s. there are either no atoms or one atom with mass 1.
- For each atom ξ, let $h_\xi(v) = \mathbf{P}_x(X_\infty = \xi)$. Then $h_\xi : G \to [0, 1]$ is harmonic, and by Levy's 0-1 law

$$h_\xi(X_n) \xrightarrow[n\to\infty]{\text{a.s.}} \mathbb{1}(X_\infty = \xi)$$

- Hence, define $M(v) := \max_\xi h_\xi(v)$, so that a.s. $\lim_n M(X_n) \in \{0, 1\}$.
- But since (G, ρ) is stationary and by the CP rigidity, $M(X_n)$ is also stationary, hence $M(\rho) \in \{0, 1\}$ a.s.

Proof of non-atomic exit measure

- Assume (G, ρ) is stationary (i.e., $(G, \rho) \overset{d}{=} (G, X_1)$) (easy to obtain from a unimodularity via degree biasing).
- Let us first show that a.s. there are either no atoms or one atom with mass 1.
- For each atom ξ, let $h_\xi(v) = \mathbf{P}_x(X_\infty = \xi)$. Then $h_\xi : G \to [0, 1]$ is harmonic, and by Levy's 0-1 law

$$h_\xi(X_n) \xrightarrow[n\to\infty]{\text{a.s.}} \mathbb{1}(X_\infty = \xi)$$

- Hence, define $M(v) := \max_\xi h_\xi(v)$, so that a.s. $\lim_n M(X_n) \in \{0, 1\}$.
- But since (G, ρ) is stationary and by the CP rigidity, $M(X_n)$ is also stationary, hence $M(\rho) \in \{0, 1\}$ a.s.

Proof of non-atomic exit measure (continued)

- Assume now that there is a single atom ξ in the exist measure with mass 1.

Proof of non-atomic exit measure (continued)

- Assume now that there is a single atom ξ in the exist measure with mass 1. This means that in some sense our graph is not really hyperbolic. Formally:
- Apply a Möbius transformation sending $\xi \to \infty$.

Proof of non-atomic exit measure (continued)

- Assume now that there is a single atom ξ in the exist measure with mass 1. This means that in some sense our graph is not really hyperbolic. Formally:
- Apply a Möbius transformation sending $\xi \to \infty$. This gives a circle packing of G in the upper half plane that is unique up to translations and scaling.

Proof of non-atomic exit measure (continued)

- Assume now that there is a single atom ξ in the exist measure with mass 1. This means that in some sense our graph is not really hyperbolic. Formally:
- Apply a Möbius transformation sending $\xi \to \infty$. This gives a circle packing of G in the upper half plane that is unique up to translations and scaling.
- In particular, this drawing is determined by the graph and hence the angles between straight line Euclidean geodesics are determined by the graph.

Proof of non-atomic exit measure (continued)

- Assume now that there is a single atom ξ in the exist measure with mass 1. This means that in some sense our graph is not really hyperbolic. Formally:
- Apply a Möbius transformation sending $\xi \to \infty$. This gives a circle packing of G in the upper half plane that is unique up to translations and scaling.
- In particular, this drawing is determined by the graph and hence the angles between straight line Euclidean geodesics are determined by the graph. We deduce that $\mathbf{E}\deg(\rho) = 6$.

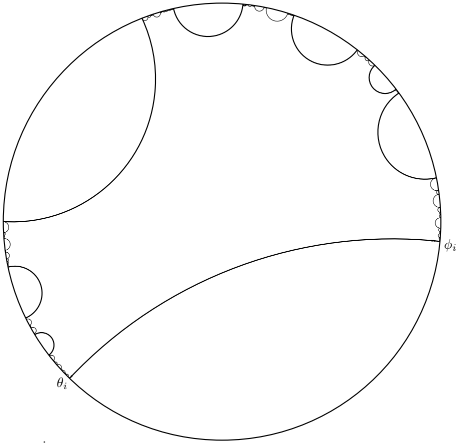
Full support

Suppose the exit measure does not have full support.

We will define a mass transport on G in which each vertex sends a mass of at most one, but some vertices receive infinite mass, contradicting the mass transport principle.

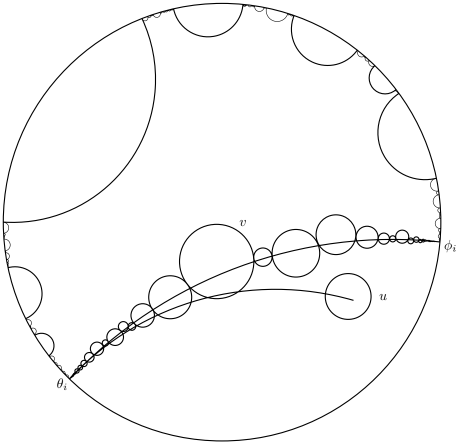
The transport will be defined in terms of the hyperbolic geometry and the support of the exit measure, so by Schramm's rigidity, it will not depend on the choice of the packing (and so it will be a legitimate mass transport).

The complement of the support of the exit measure may be written as a union of disjoint open intervals (θ_i, ϕ_i) in the circle. Let's draw the hyperbolic geodesic γ_i between the endpoints of each such interval.



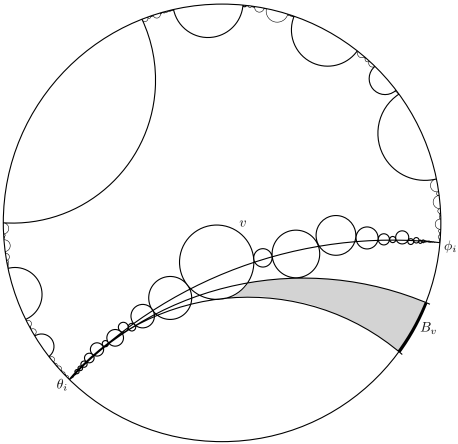
Write A_i for the set of circles enclosed by the geodesic between θ_i and ϕ_i.

Transport mass one from each u in A_i to the first circle intersected by the geodesic from the hyperbolic centre of u to θ_i that also intersects γ_i.

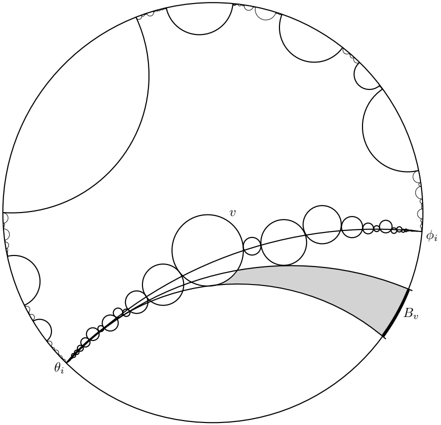


It might be that no such circle exists, in which case u sends no mass.

Consider the set of angles $B_v \subset (\theta_i, \phi_i)$ such that v is the first circle intersected by the geodesic from θ to θ_i that also intersects γ_i. For each v, this set is an interval.

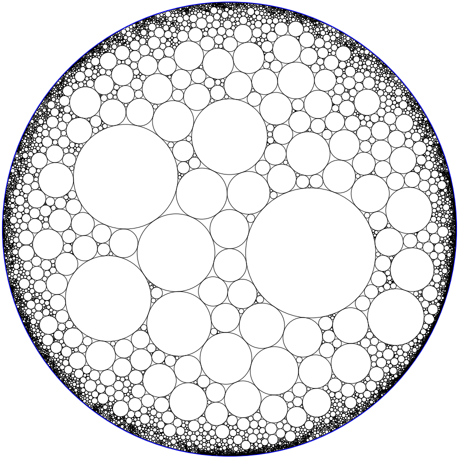


The union of the B_v's over all v intersecting γ_i is an interval of positive length, and hence, since there are only countably many circles, one of the intervals B_v has positive length.



Such a vertex receives infinite mass, since it is sent mass by every vertex with centres in some open neighbourhood of the boundary interval.

Thank you!



תודה