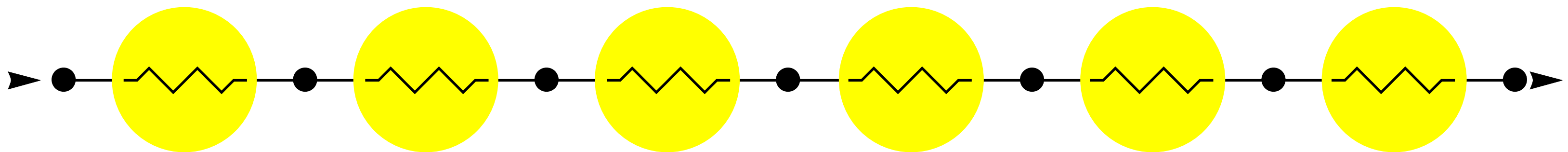
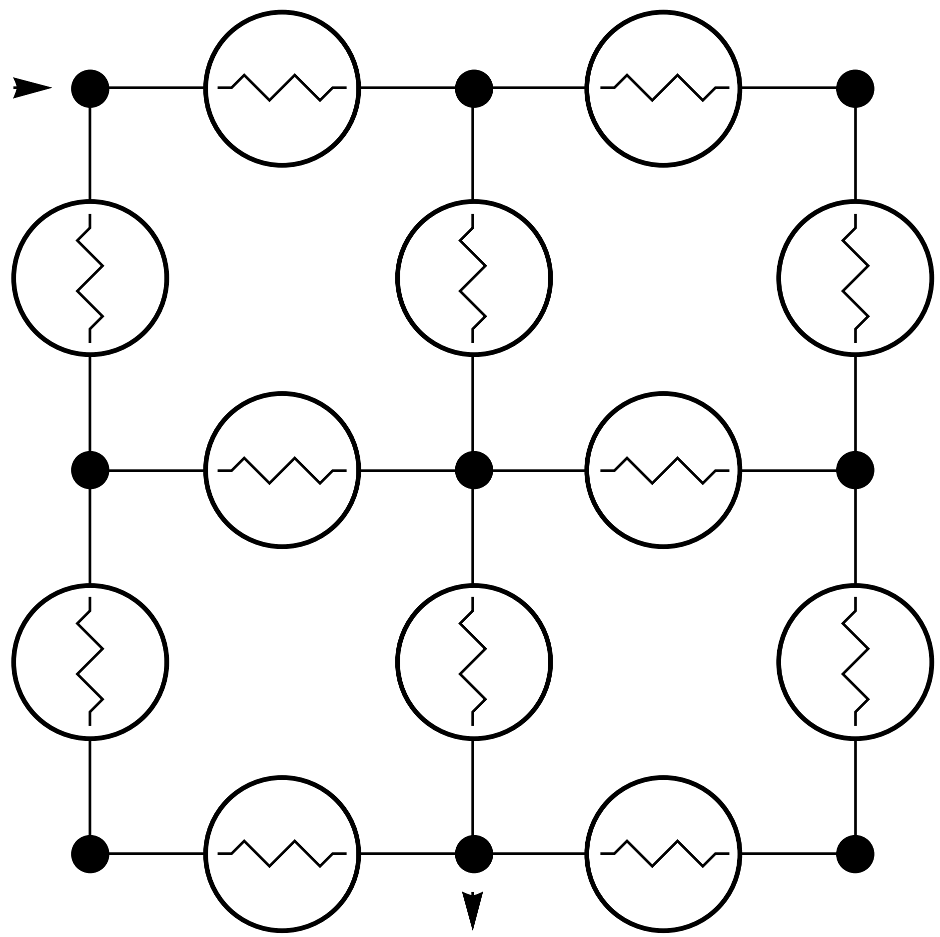
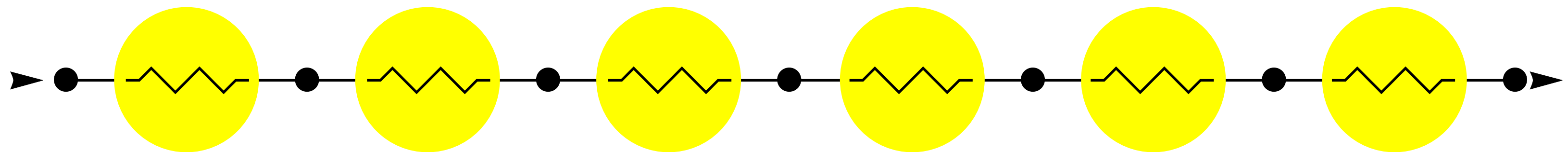
Fixed-energy harmonic functions

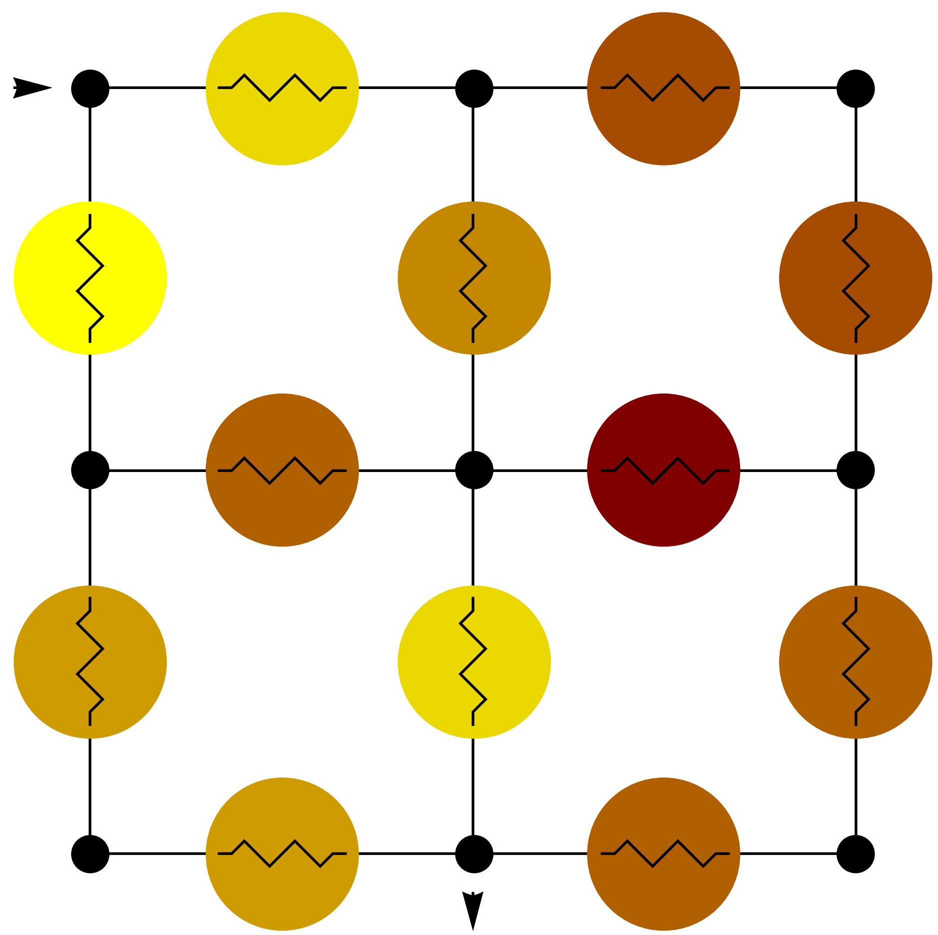
or, totally real dessins d'enfants

A. Abrams (Washington and Lee)
R. Kenyon (Brown)









Can we adjust *edge conductances* so that all bulbs burn with the same brightness?

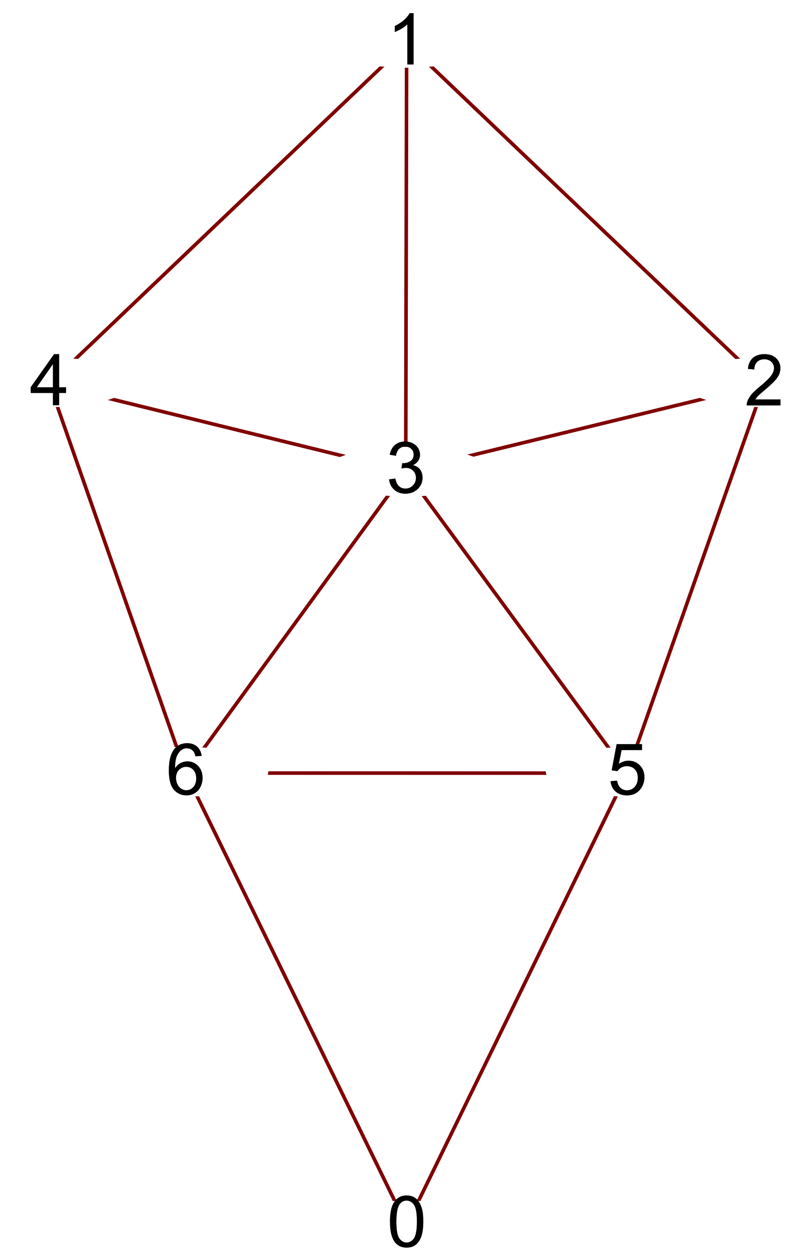
The Dirichlet problem

A graph $G = (V, E)$

$c : E \to \mathbb{R}_{>0}$ the edge conductances

$B \subset V$ boundary vertices

$u : B \to \mathbb{R}$ boundary values



Find $f : V \to \mathbb{R}$ harmonic on $V \setminus B$

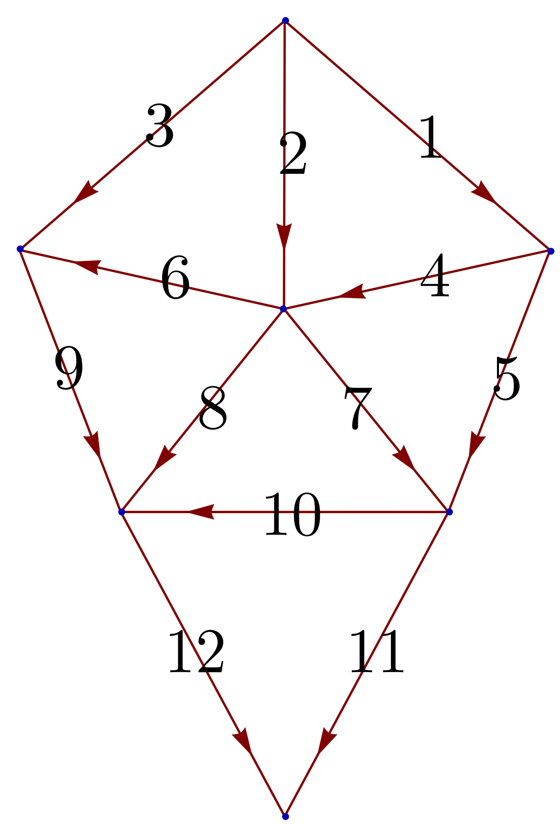
and $f|_B = u$.

$$0 = \Delta f(x) = \sum_{y \sim x} c_e (f(x) - f(y))$$

f is the unique function minimizing the Dirichlet energy

$$\mathcal{E}(f) = \sum_{e=xy} \underbrace{c_e(f(x) - f(y))^2}_{\text{edge energy}}$$

A harmonic function induces an *orientation* of the edges:



Let Σ be the set of acyclic orientations compatible with u:

$$\Sigma = \{\sigma \mid \exists f,\ f_B = u \text{ and no interior extrema, where } \mathrm{sgn}(df) = \sigma\}.$$

Let $\Psi : (0,\infty)^E \to [0,\infty)^E$ be the map from conductances to energies.

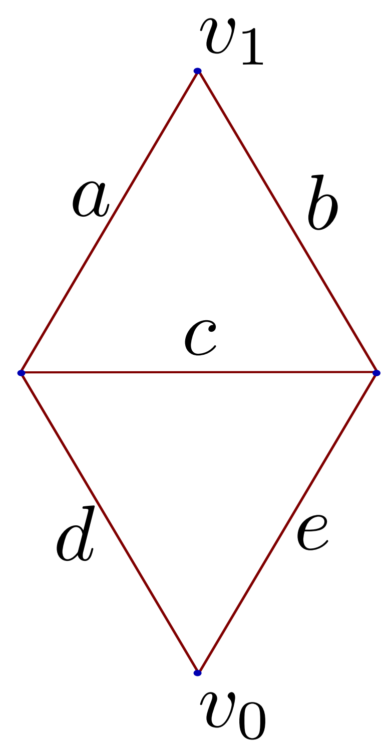
Theorem 1: For any $\sigma \in \Sigma$ and $\{\mathcal{E}_e\}$ there is a unique choice of conductances $\{c_e\}$ for which the associated harmonic function realizes this data.

Theorem 2: The rational map $\Psi : \mathbb{C}^n \to \mathbb{C}^n$ has degree $|\Sigma|$.

Cor. For rational energies and u, the Galois group of $\mathbb{Q}^{tr}$ over $\mathbb{Q}$ permutes the solutions.

(totally real alg. #s)

Example



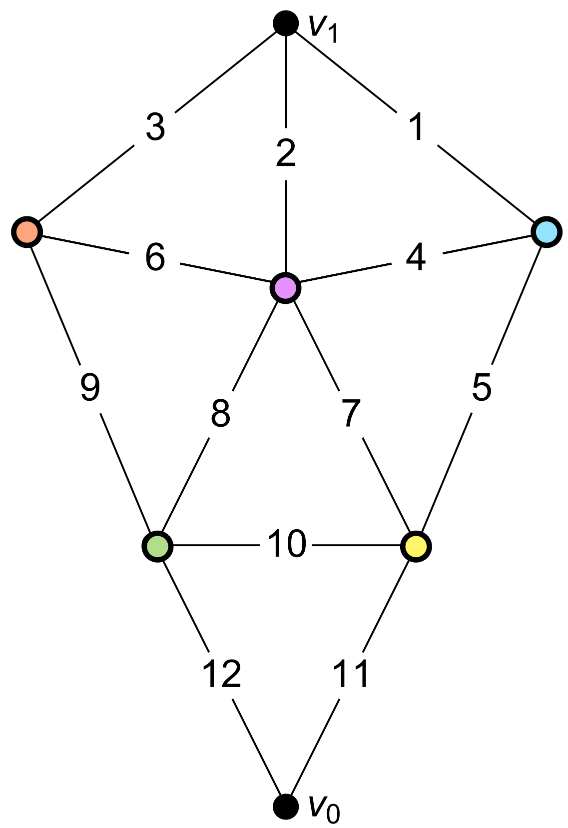
$$\Psi(a,b,c,d,e) = \Big(\frac{a(bd+cd+ce+de)^2}{(ab+ac+ae+bc+bd+cd+ce+de)^2}, \frac{b(ae+cd+ce+de)^2}{(ab+ac+ae+bc+bd+cd+ce+de)^2},$$

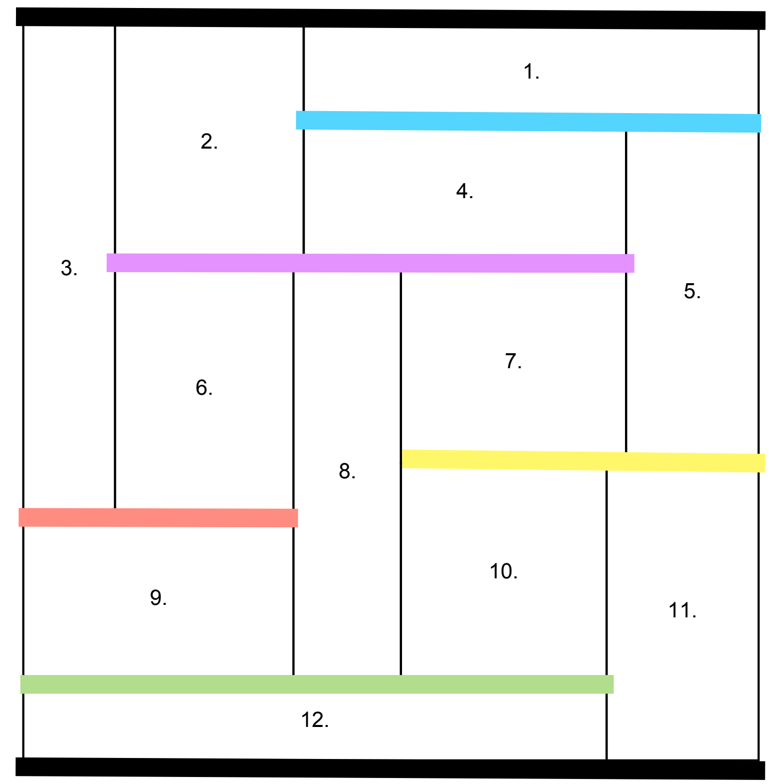
$$\frac{c(bd-ae)^2}{(ab+ac+ae+bc+bd+cd+ce+de)^2}, \frac{d(ab+ac+ae+bc)^2}{(ab+ac+ae+bc+bd+cd+ce+de)^2},$$

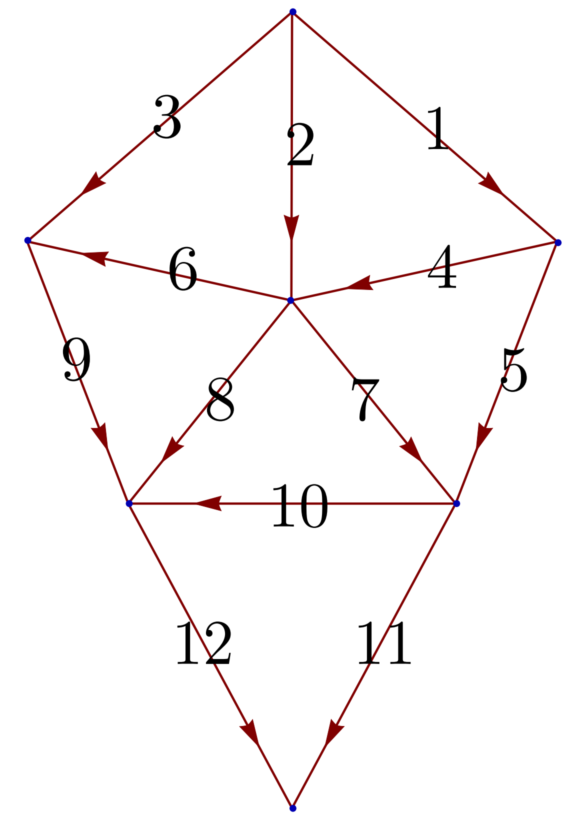
$$\frac{e(ab+ac+bc+bd)^2}{(ab+ac+ae+bc+bd+cd+ce+de)^2} \Big)$$

$$\begin{aligned} 0 = {} & e_1\left(e_1^2+e_3^2+e_4^2\right)\left(e_1^2+e_2^2+e_3^2+e_4^2+e_5^2\right) \\ & -(2e_1^4+2e_2^2e_1^2+3e_3^2e_1^2+2e_4^2e_1^2+e_5^2e_1^2+e_3^4+e_2^2e_3^2+e_2^2e_4^2+e_3^2e_4^2+e_3^2e_5^2)\sqrt{a} \\ & +e_1\left(e_1^2+e_2^2+e_3^2\right)a \end{aligned}$$

Smith diagram of a planar network
(with a harmonic function)



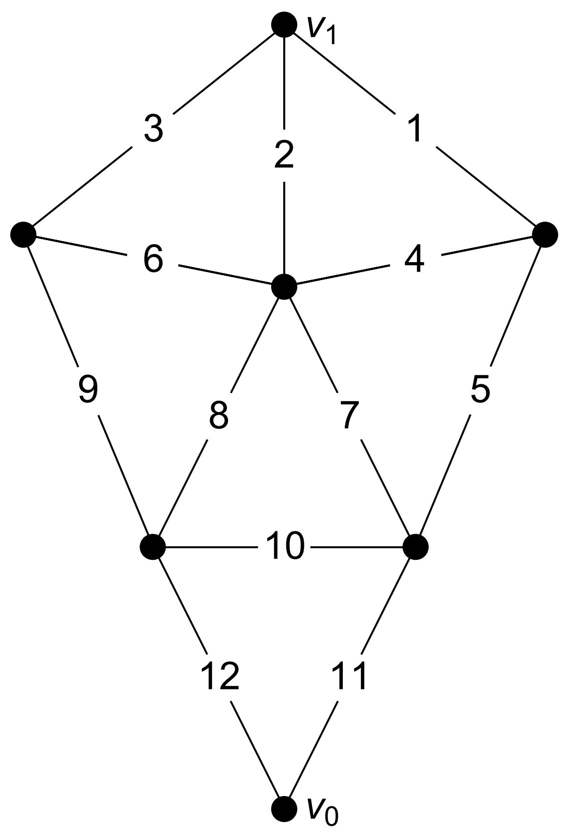


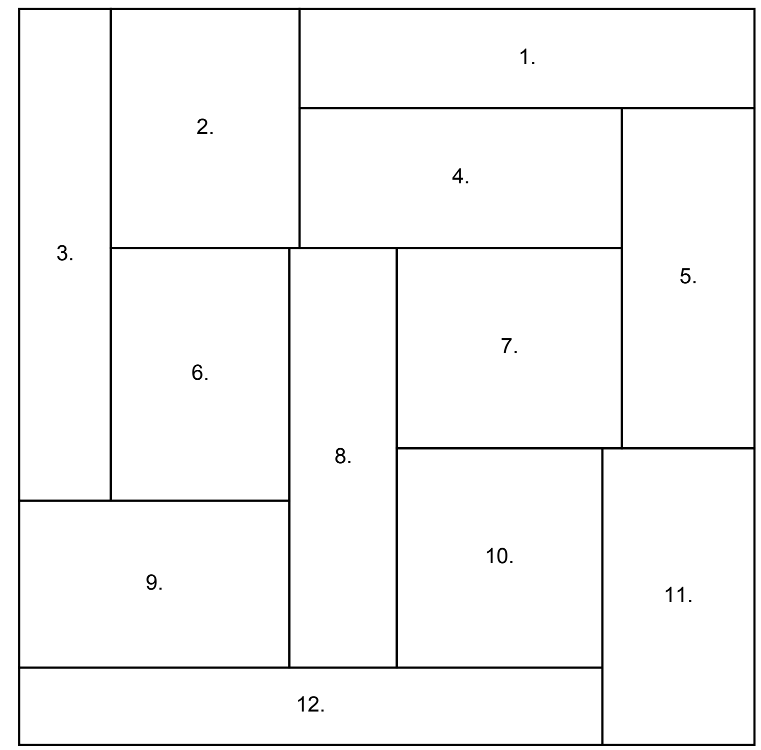


$$\text{voltage} = y\text{-coordinate}$$
$$\text{edge} = \text{rectangle}$$
$$\text{current} = \text{width}$$
$$\text{conductance} = \text{aspect ratio}$$
$$\text{energy} = \text{area}$$

This graph has 12 acyclic orientations with source at 1 and sink at 0.

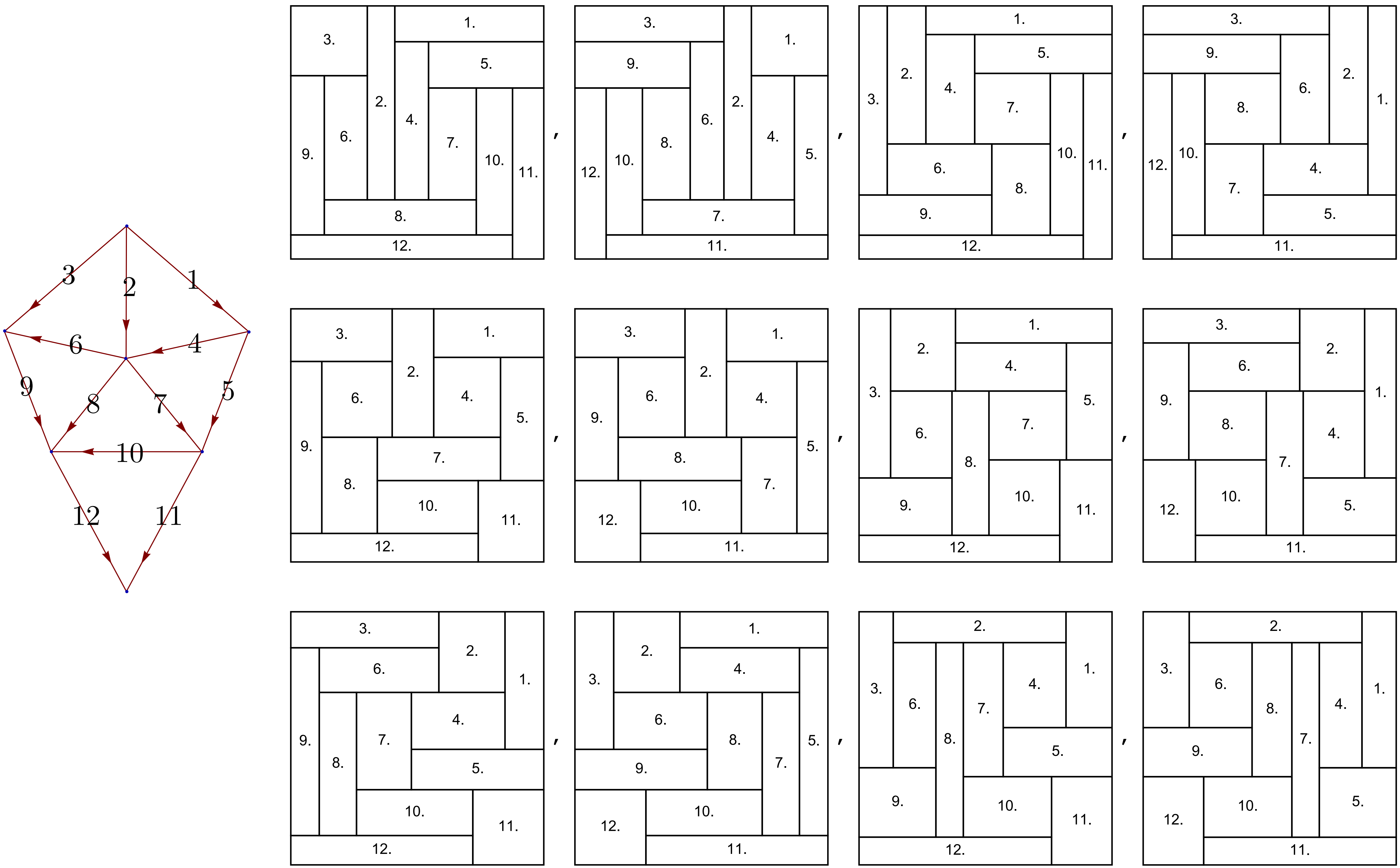
$(|\Sigma| = 12.)$





width(1) is the root of a polynomial:

$$2315250000z^{12} - 107438625000z^{11} + 2230924692500z^{10} - 27361273241750z^9 + 220350695004825z^8 - 1225394593409700z^7 + 4817113876088640z^6 - 13468300499707200z^5 + 26554002301384704z^4 - 35985219877131264z^3 + 31817913970765824z^2 - 16489700865736704z + 3791571715620864 = 0$$


3 2 1 6 4 9 8 7 5 10 12 11
3. 1. 5. 2. 4. 9. 6. 7. 10. 11. 8. 12.
3. 1. 9. 2. 6. 4. 12. 10. 8. 5. 7. 11.
1. 5. 2. 4. 3. 7. 10. 11. 6. 8. 9. 12.
3. 9. 2. 6. 1. 8. 12. 10. 4. 7. 5. 11.
3. 1. 2. 6. 4. 9. 5. 7. 8. 10. 11. 12.
3. 1. 2. 6. 4. 9. 8. 5. 7. 12. 10. 11.
1. 2. 4. 3. 5. 6. 7. 8. 9. 10. 11. 12.
3. 2. 6. 9. 1. 8. 4. 7. 12. 10. 5. 11.
3. 2. 6. 1. 4. 9. 7. 8. 5. 10. 11. 12.
1. 2. 4. 3. 6. 8. 5. 9. 7. 12. 10. 11.
2. 1. 3. 6. 4. 7. 8. 5. 9. 10. 11. 12.
2. 3. 6. 1. 8. 4. 7. 9. 5. 12. 10. 11.

What is $|\Sigma|$?

[Bernardi:] If $B = \{v_0, v_1\}$ connected by an edge,

then $|\Sigma|$ is the **chromatic invariant**.

$|\Sigma| = |\chi'_G(1)|$, where χ is the chromatic polynomial.

Equivalently, $|\Sigma| = T_x(0,0)$.

This is NP-hard to compute

Proof of Theorem 1:

$$0 = \Delta h(x) = \sum_{y \sim x} c_e(h(x) - h(y))$$

$$= \sum_{y \sim x} \frac{\mathcal{E}_e}{h(x) - h(y)}$$

the **enharmonic equation**

"$energy - harmonic$"

solutions of the enharmonic equation are critical points of the functional

$$M(h) = \prod_e |h(x) - h(y)|^{\mathcal{E}_e}.$$

Note $\log M(h)$ is strictly concave on each polytope $P_\sigma = \{h : \mathrm{sign}(dh) = \sigma\}$

$\square$

Proof of Theorem 2:

We just need to show that all solutions to enharmonic equation are real.

Gauss-Lucas Theorem:

Roots of $p'(z)$ are contained in the convex hull of roots of $p(z)$.

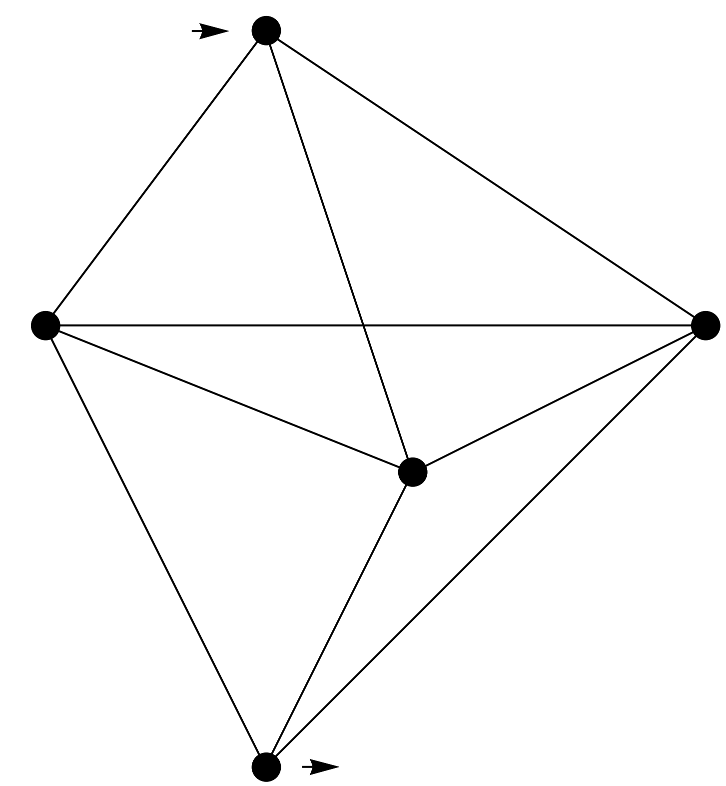
$$0 = \sum_{y \sim x} \frac{\mathcal{E}_e}{h(x) - h(y)}$$

$h(x)$ is a root of $p'(z)$, where $p(z) = \prod (z - h(y))^{\mathcal{E}_e}$

This implies $h(x)$ is in the convex hull of the neighboring values.

Since boundary values are real, all values are real. $\square$

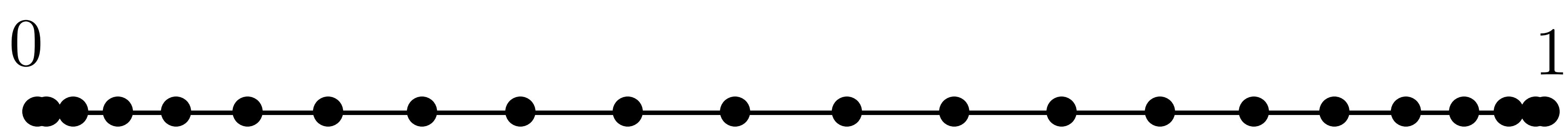
Example. Complete graph K_{n+2}
with energy 2 on each edge.



Maximize $\displaystyle\prod_{i<j}(x_i - x_j)^2$

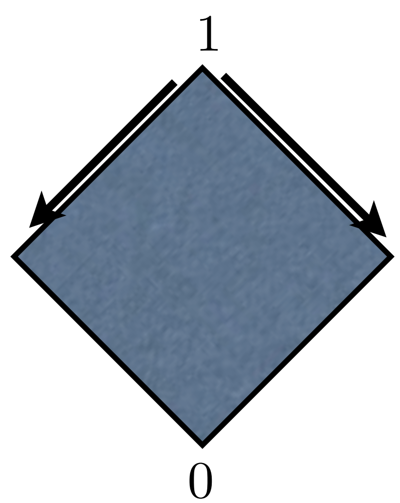
with $x_i \in [0,1]$, and $x_1 = 0, x_n = 1$.

$\Longrightarrow$ roots of Jacobi polynomial $P_n(x)$.

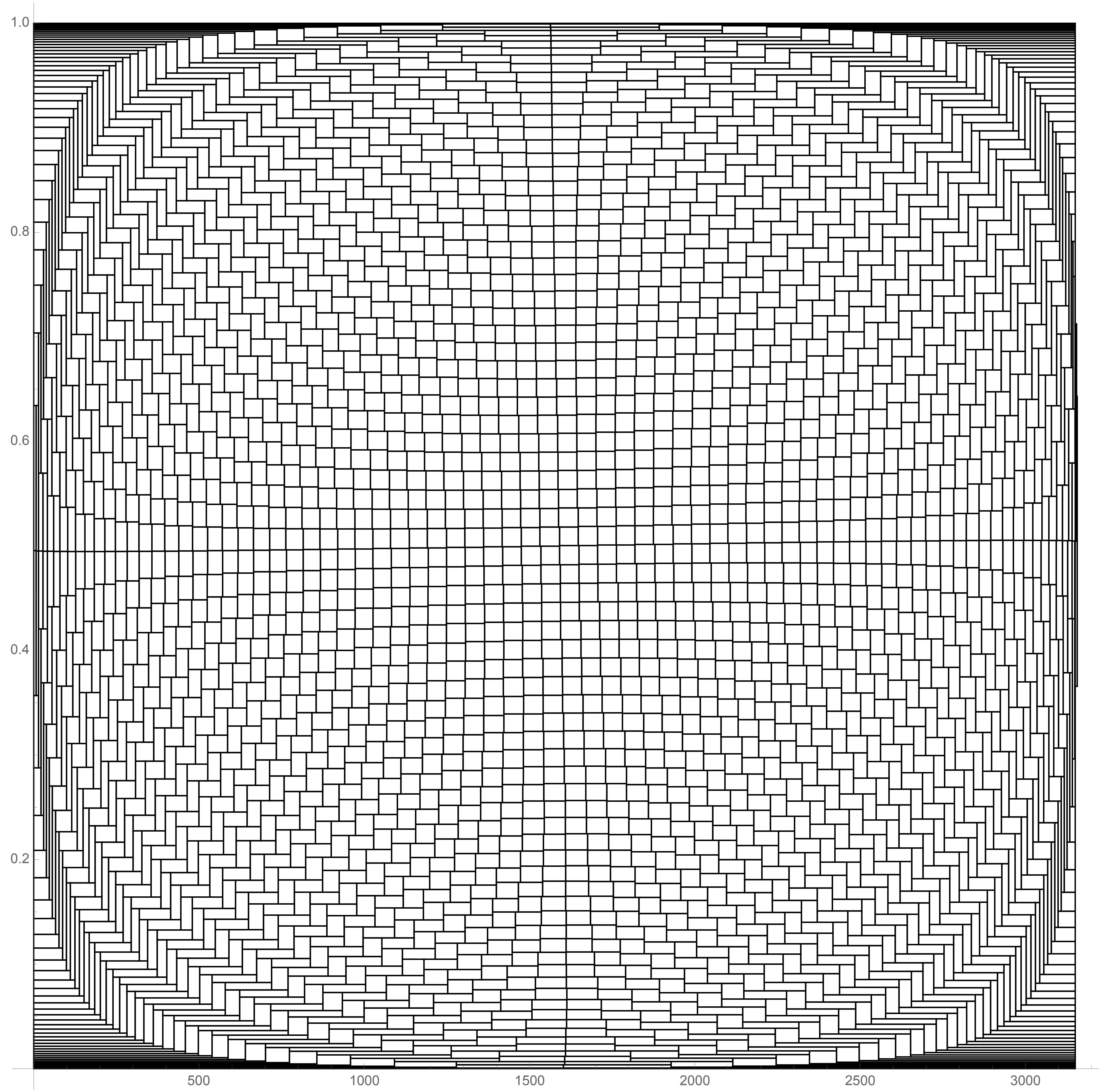


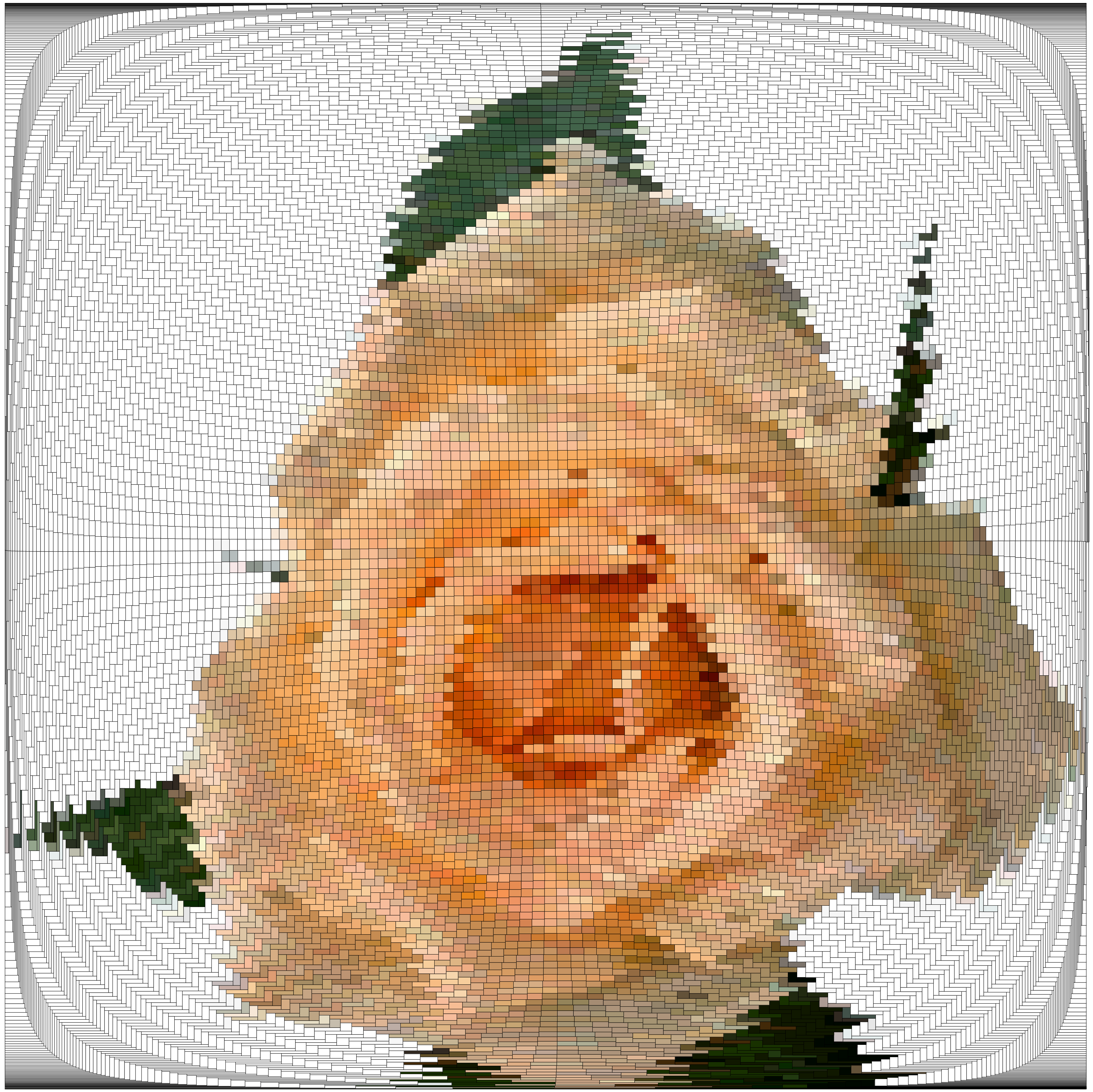
$n = 20$

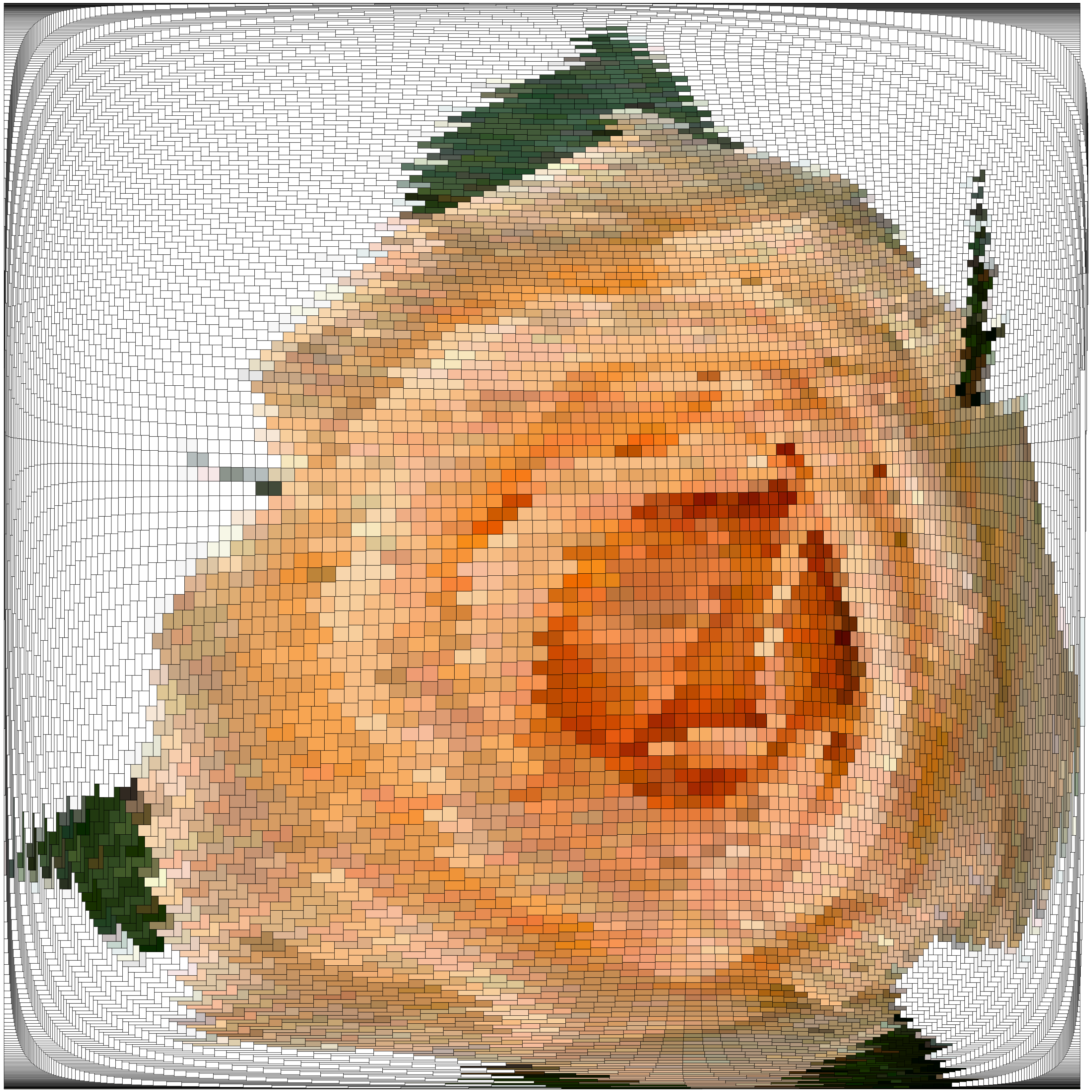
one of the gazillion area-1 rectangulations based on the 40X40 grid

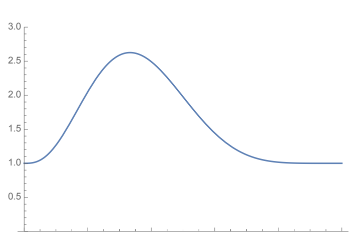


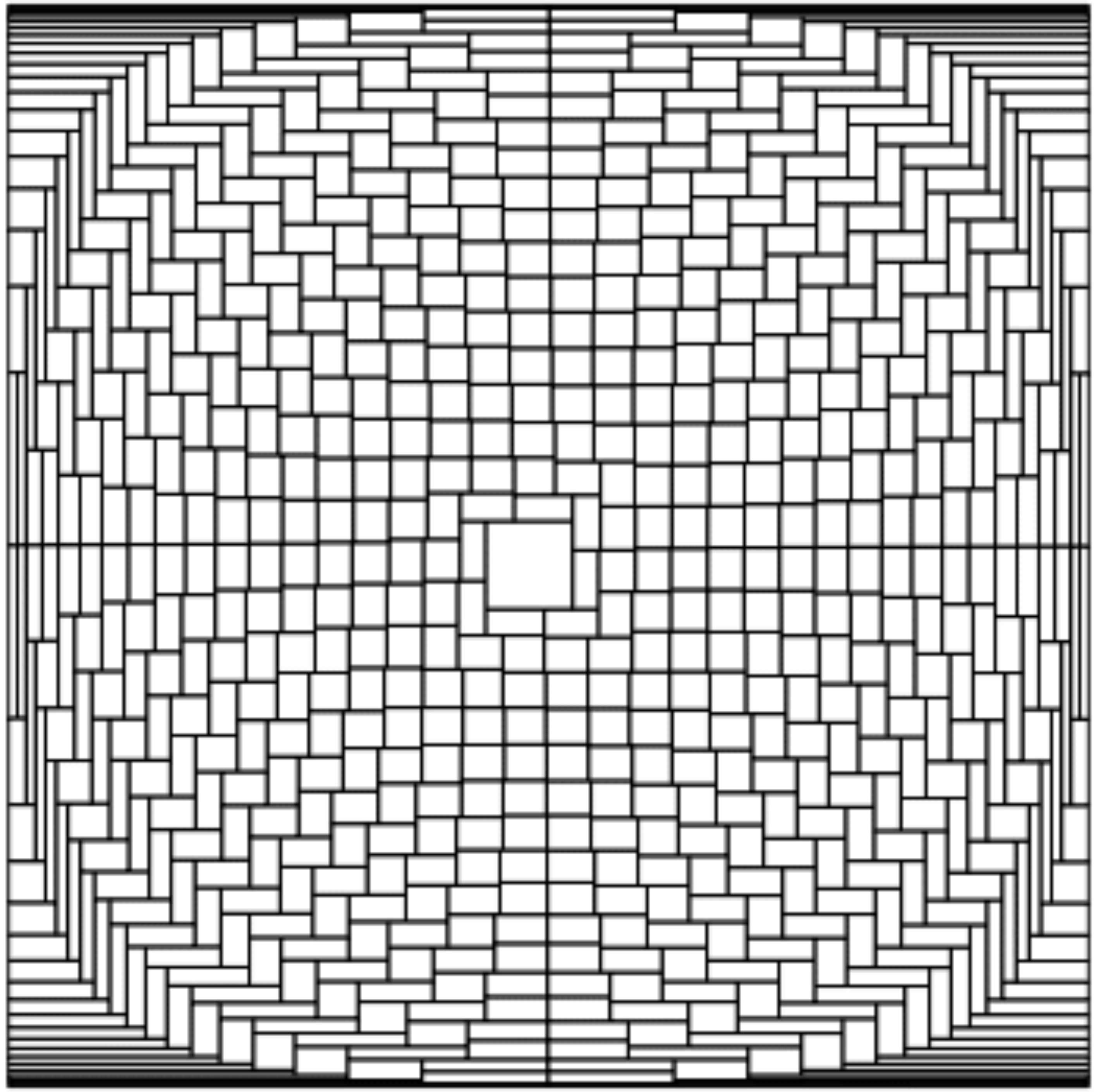
$\mathbb{Z}^2$, directed S&W








3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.2
0.4
0.6
0.8
1.0



The scaling limit of these mappings satisfy the “fixed-energy” Cauchy-Riemann equations

$$u_x v_y = 1$$

$$u_y v_x = -1.$$

Cauchy Riemann eqs

constant conductance:

$$u_x = v_y$$
$$u_y = -v_x$$

constant energy:

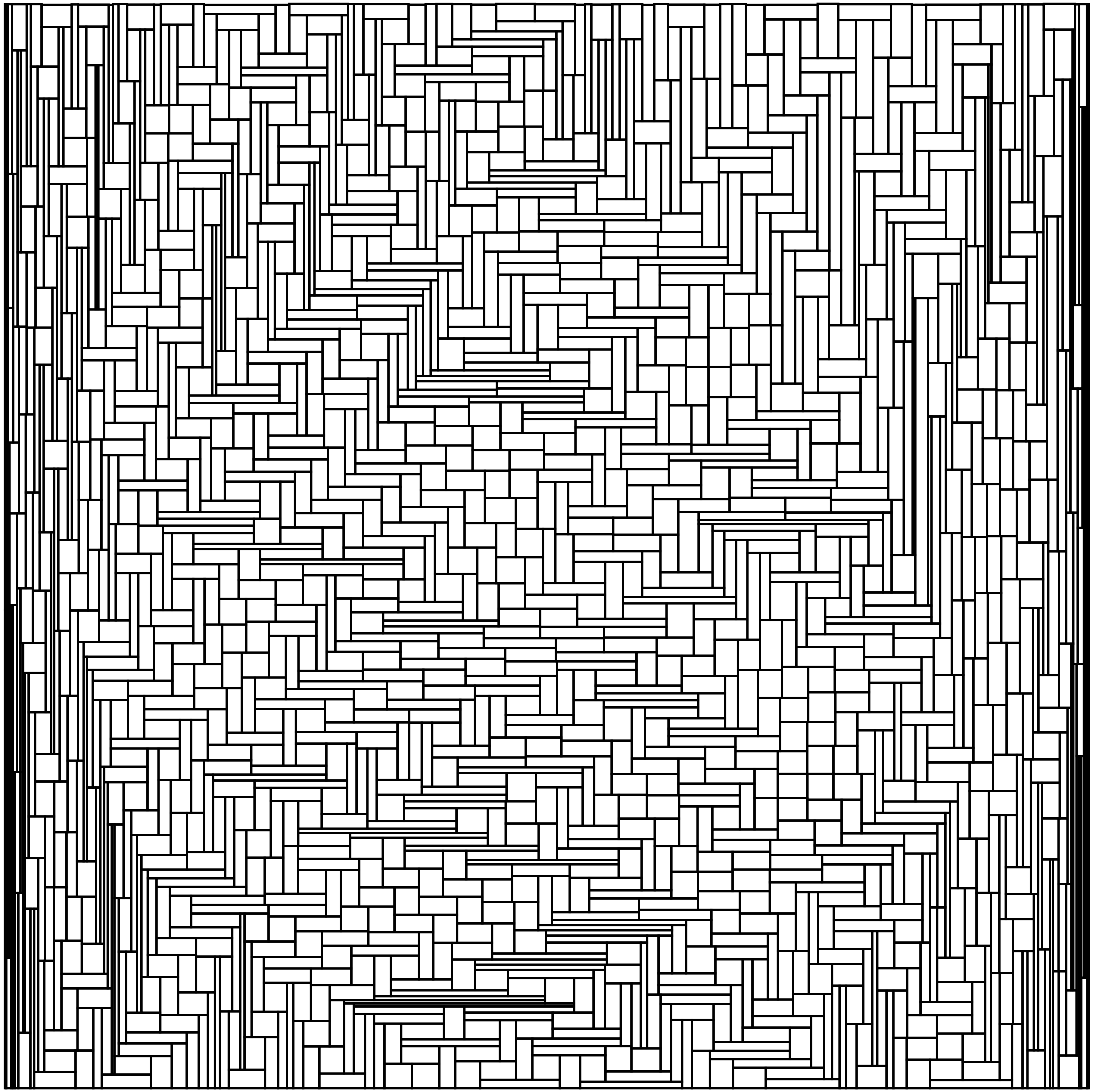
$$u_x v_y = 1$$
$$u_y v_x = -1$$

Associated laplacian:

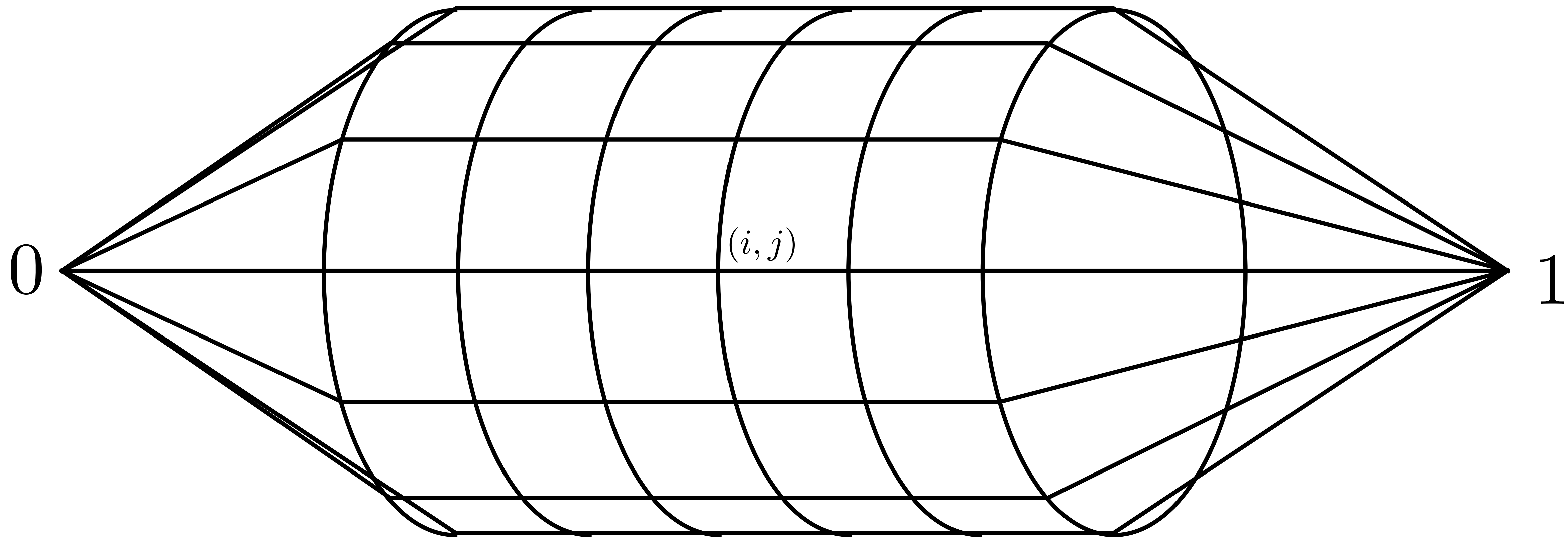
$$h_{xx} + h_{yy} = 0$$

$$\frac{h_{xx}}{h_x^2} + \frac{h_{yy}}{h_y^2} = 0.$$

What does a uniform random orientation look like?



Can we count bipolar orientations of $\mathbb{Z}^2$?



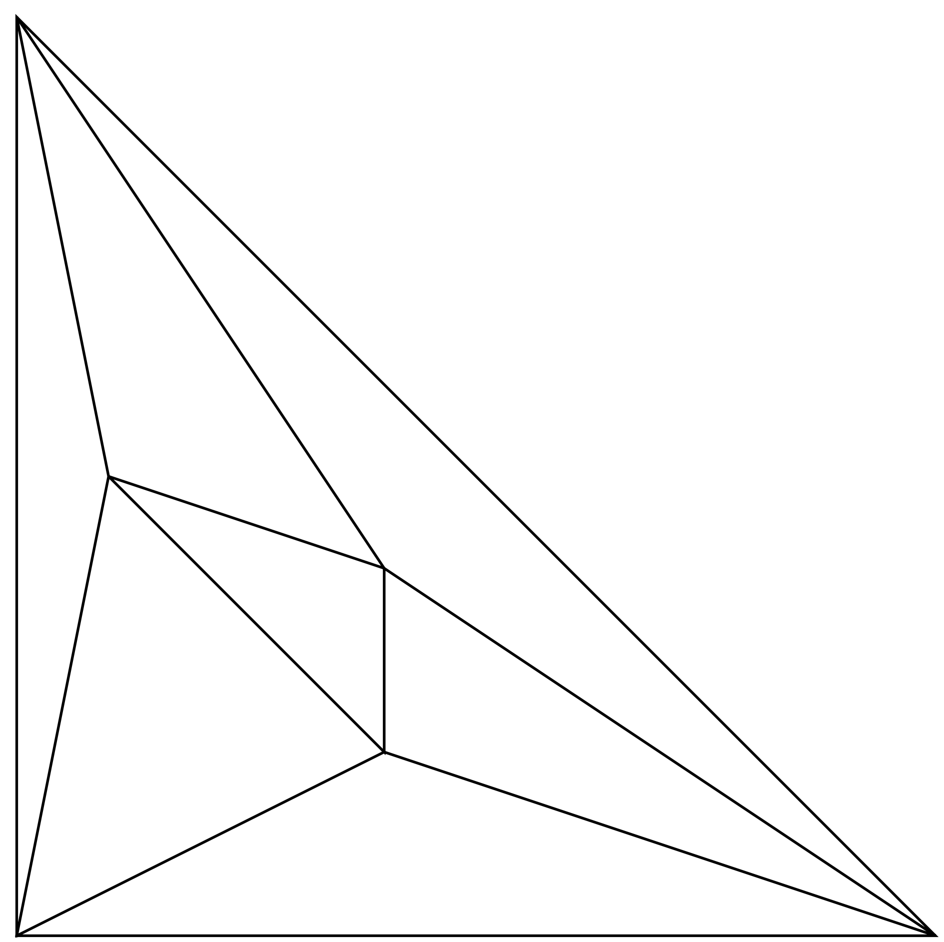
A recurrence:

$$X_{i+1,j} = X_{i,j} + \frac{1}{\frac{1}{X_{i,j}-X_{i,j+1}} + \frac{1}{X_{i,j}-X_{i,j-1}} + \frac{1}{X_{i,j}-X_{i-1,j+1}}}$$

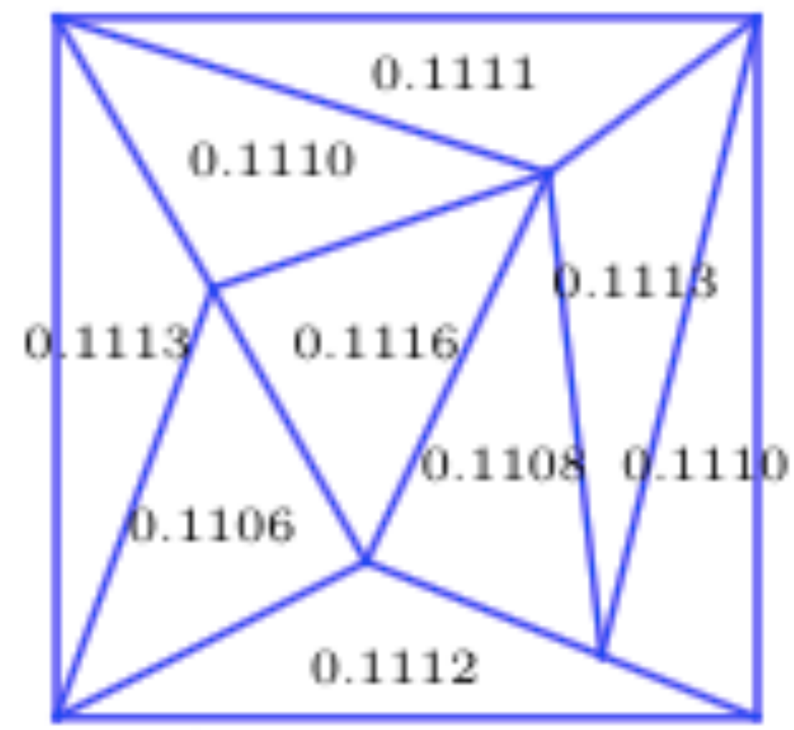
$$\deg(X_{i,j}) \approx 4^i$$

triangular dissections

given a triangular dissection of a triangle is there a combinatorially equivalent one with prescribed areas?



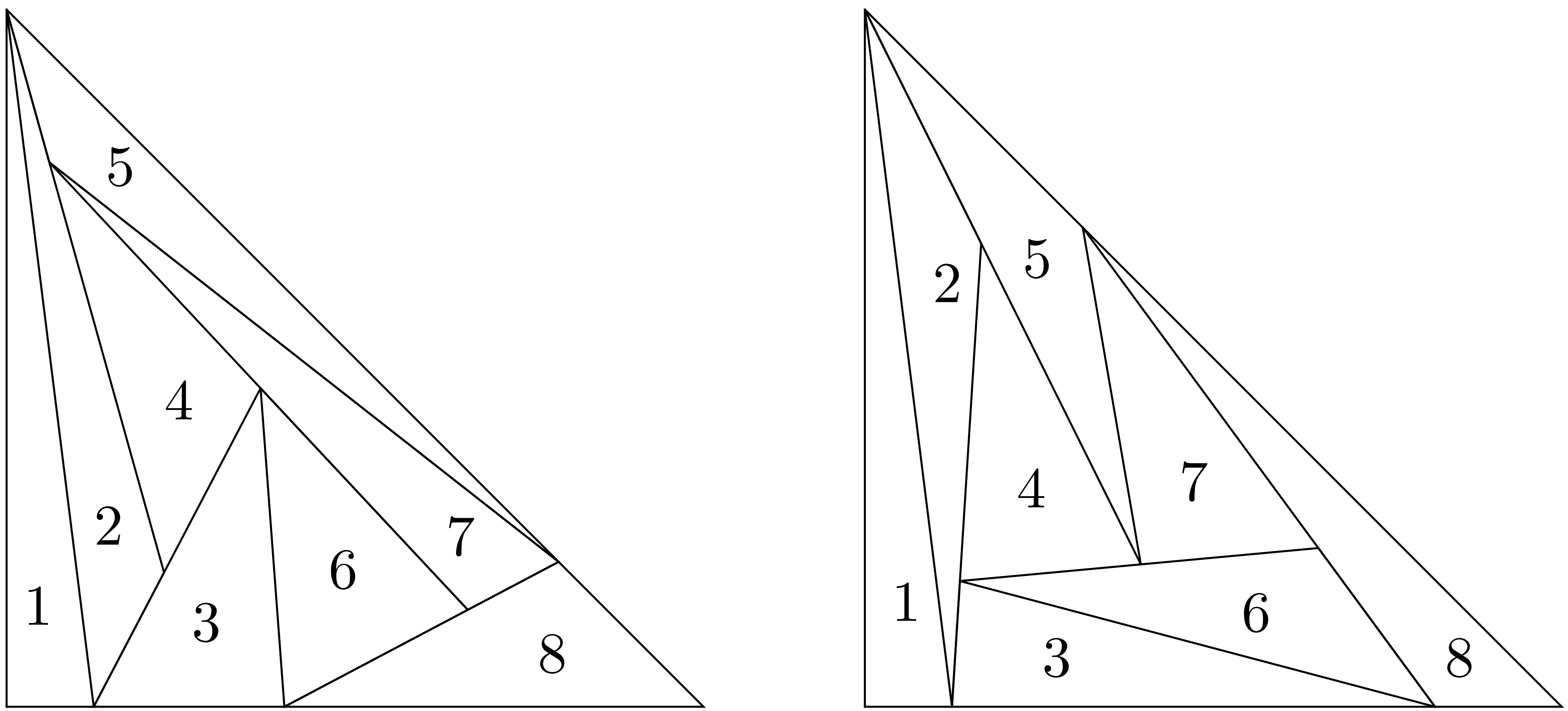
no, but...



(Monsky)

No dissection of a square into an odd number of equal-area triangles.

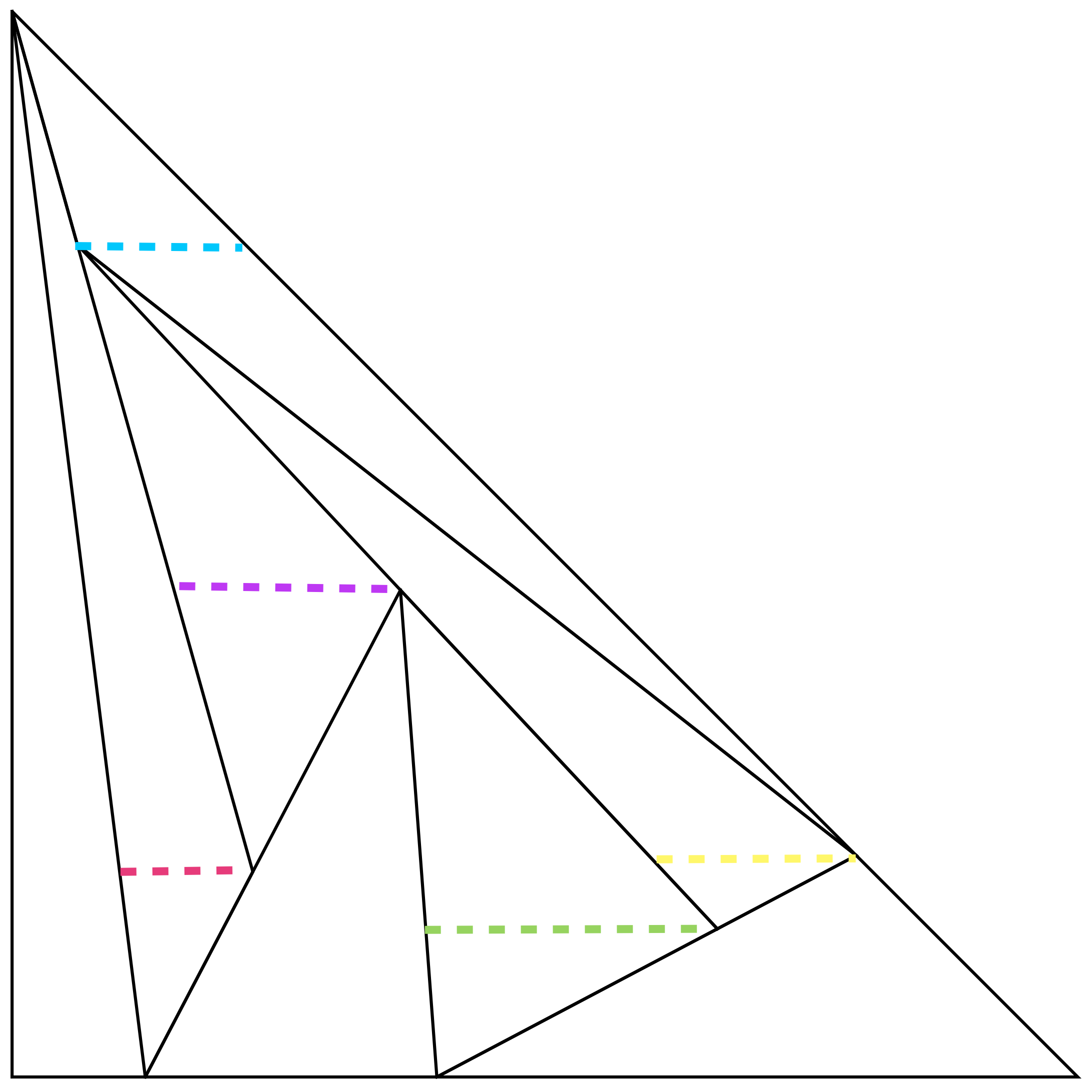
we can write an explicit rational map from “conductances” to areas

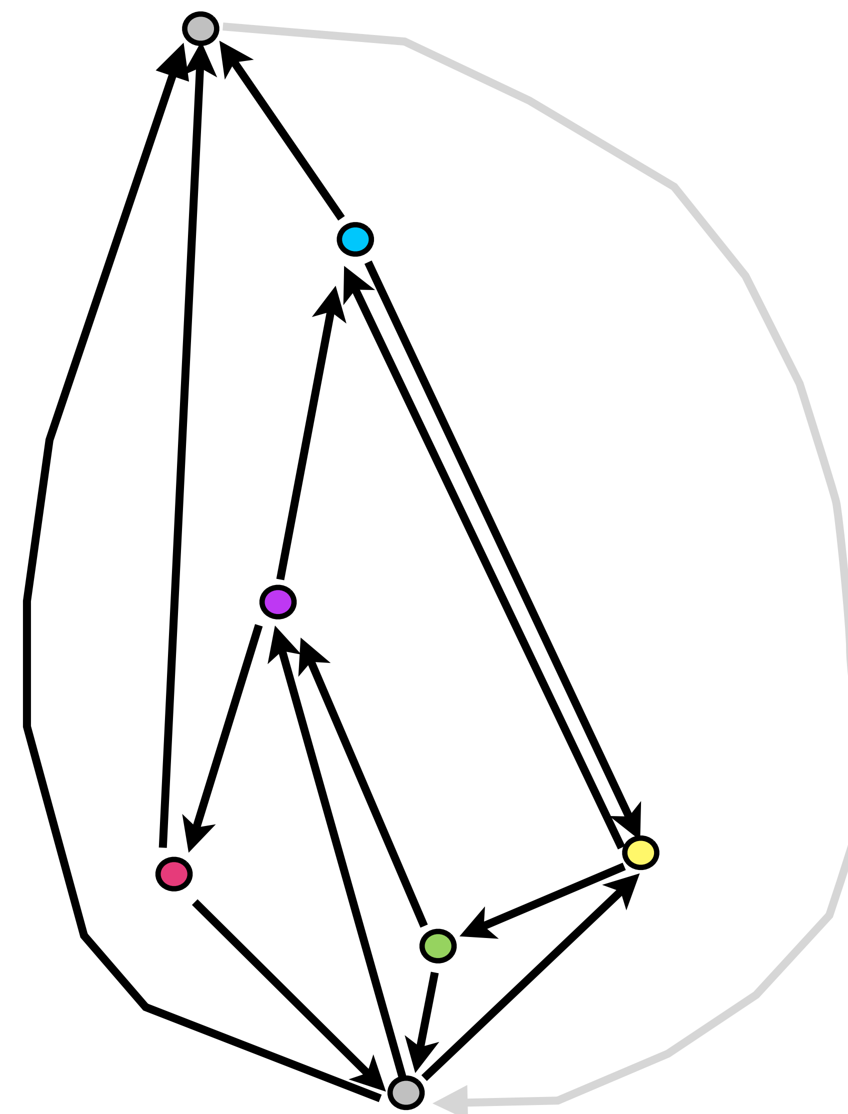


Two combinatorially equivalent solutions

triangulation and planar Markov chain

(with two outgoing edges from each vertex)





triangle = vertex

y-coord = harmonic function

$1/\text{slope}$ = winding number

width/height = stationary msr on edges

area = energy

THANK YOU