Exponential Bounds on the Number of 3-dimensional Causal Triangulations

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June 10 2014

20th Itzykson Conference
Random Surfaces and Random Geometry
IPhT Saclay

Outline

- The 2-dimensional case
- Locally constructible 3-dimensional triangulations
- 3-dimensional causal triangulations
- Generalisations to higher dimensions

A CENSUS OF PLANAR TRIANGULATIONS

W. T. TUTTE

1. Triangulations. Let P be a closed region in the plane bounded by a simple closed curve, and let S be a simplicial dissection of P. We may say that S is a dissection of P into a finite number α of triangles so that no vertex of any one triangle is an interior point of an edge of another. The triangles are "topological" triangles and their edges are closed arcs which need not be straight segments. No two distinct edges of the dissection join the same two vertices, and no two triangles have more than two vertices in common.

There are $k \geqslant 3$ vertices of S in the boundary of P, and they subdivide this boundary into k edges of S. We call these edges *external* and the remaining edges of S, if any, *internal*. If r is the number of internal edges we have

$$3\alpha = 2r + k, \tag{1.1}$$

$$r \equiv k \pmod 3. \tag{1.2}$$

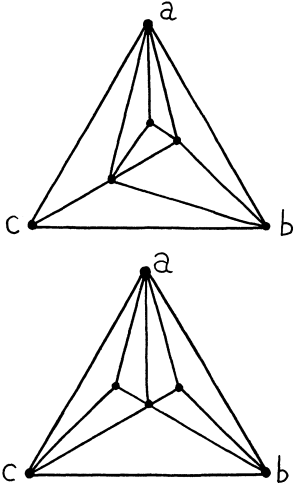


FIGURE 3

Two dimensions

- Tutte 1962

$$N_{0,1} \sim n^{-5/2} c^n$$

- Bender and Canfield 1986

$$N_{g,b}(n) \sim n^{5(g-1)/2+b-1} c^n$$

- No restriction on topology

$$N(n) = \sum_{g=0}^{\infty} N_{g,1}(n) \sim (3n/2)!$$

- 1980s 2d simplicial gravity models: Ambjørn, Boulatov, David, Durhuus, Kazakov, Kostov, Migdal etc. etc.

3 dimensions

- Ambjørn, Durhuus, Jonsson 1991

$$S_{EH}(T) = \kappa|T| + \lambda \ell_1(T)$$

- In order for

$$Z = \sum_{T \in \mathcal{T}} e^{-S_{EH}}$$

to converge for some κ we need

$$\#\{T \in \mathcal{T} : |T| = n\} \leq C^n$$

for some constant C.

Locally Constructible Triangulations

A 3d simplicial manifold M has a local construction if there is a sequence of simplicial manifolds $M_1, M_2, \ldots, M_k$ such that

(i) M_1 is a tetrahedron

(ii) M_{i+1} is obtained from M_i by either gluing a tetrahedron to M_i along a triangle or by identifying two triangles in ∂M_i which already share a triangle

(iii) $M_k = M$

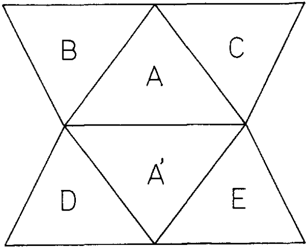


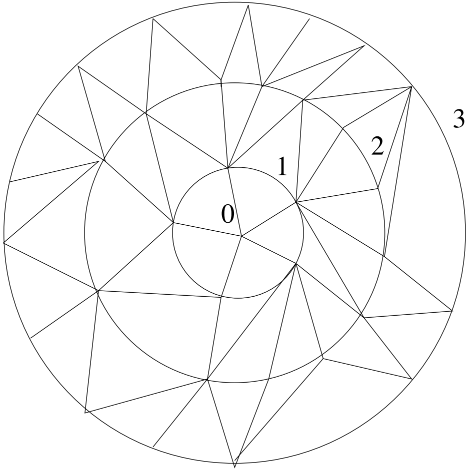
Fig. 1. After identifying the triangles A and A' the triangles B and D become nearest neighbours and the same applies to the triangles C and E. They may therefore be identified once A and A' have been glued together.

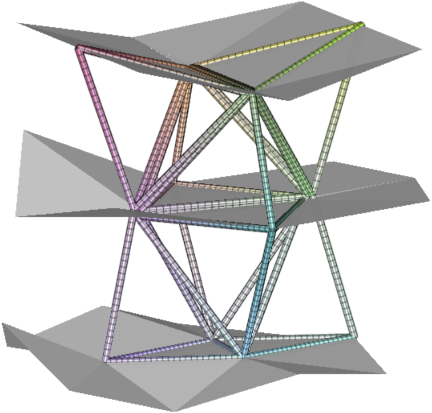
Results about LC triangulations

- **Theorem** (Durhuus and T.J. 1995) *There is a* $C > 1$ *such that the number of locally constructible triangulations of* S^3 *of volume* V *is bounded by* C^V.
- There exists a simplicial 3-ball whose boundary triangles can be identified pairwise such that no triangle is identified with any of its neighbours but the resulting 3-manifold is simply connected.
- **Theorem** (Bendetti and Ziegler 2011) *Not all triangulations of* S^3 *are locally constructible.*

Causal Triangulations

- Ambjørn, Loll, Jurkiewicz, ... 1998, 2001
- A causal slice is a triangulation of $[0,1] \times S^2$ such that all the vertices lie on the boundary and every tetrahedron has at least one vertex in each boundary component.
- A causal slice K has two boundary components Σ_{in} and Σ_{out} each of which is a triangulation of S^2.
- A causal triangulation K is a sequence of causal slices $K_1, \ldots, K_n$, $\partial K_i = \Sigma^i_{in} \cup \Sigma^i_{out}$, which are disjoint except K_j and K_{j+1} intersect in $\Sigma^j_{out} = \Sigma^{j+1}_{in}$, $j = 1, \ldots, n-1$. We have $\partial K = \Sigma^1_{in} \cup \Sigma^n_{out}$.


0
1
2
3



http://www.thephysicsmill.com/2013/10/13/causal-dynamical-triangulations/

Main results

Work with B. Durhuus

- **Theorem 1** *The number $N_3(V)$ of 3-dimensional causal triangulations of volume V satisfies*

$$N_3(V) \leq C^V$$

for a suitable constant C.

- **Theorem 2** *All 3-dimensional causal triangulations have a local construction.*

- **Theorem 3** *If $N(V, \Sigma_{in}, \Sigma_{out})$ is the number of causal triangulations with boundary components Σ_{in} and Σ_{out} and volume V, then there is a constant C_0 independent of Σ_{in} and Σ_{out} such that*

$$N(V, \Sigma_{in}, \Sigma_{out}) = C_0^{V+o(V)}$$

The Structure of Causal Slices

- A realisation of a triangulation K (simplicial complex) is a mapping of its vertex set K^0

$$\phi : K^0 \mapsto \mathbf{R}^n$$

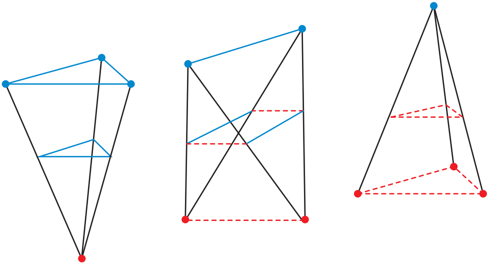
  such that $\phi(\sigma)$ is an affinely independent set for any simplex σ in K and

$$\mathrm{conv}\phi(\sigma) \cap \mathrm{conv}\phi(\sigma') = \mathrm{conv}\phi(\sigma \cap \sigma')$$

  for all simplicies σ, σ' in K.
- A D-dimensional simplicial complex has a realisation for $n = 2D + 1$.
- We will not distinguish between a causal slice and its realisations all of which are equivalent.

The Building Blocks

- Let K be a causal slice and colour the vertices in Σ_{in} red and the ones in Σ_{out} blue.
- Then there are 3 types of tetrahedra in K:



- Types: $(1,3)$, $(2,2)$, $(3,1)$

The Hight Function

- Let $x \in K$. Then x is contained in some tetrahedron in K with red vertices r_i and blue vertices b_j. There is a unique way of expressing x:

$$x = \sum_i \mu_i r_i + \sum_j \lambda_j b_j$$

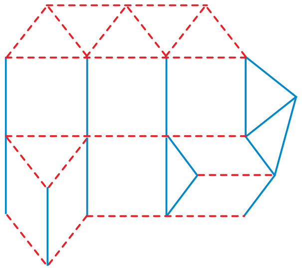
  $\mu_i, \lambda_j \geq 0$, $\sum_i \mu_i + \sum_j \lambda_j = 1$.
- Define the hight function $h : K \mapsto [0,1]$ by

$$h(x) = \sum_j \lambda_j$$

  Then $h(x) = 0$ if and only if $x \in \partial K_{red}$ and $h(x) = 0$ if and only if $x \in \partial K_{blue}$.

The Midsection

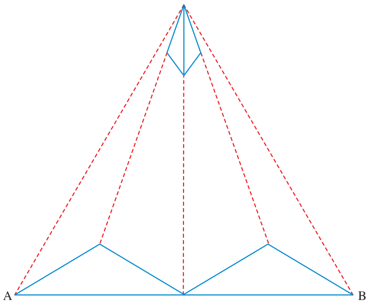
- The set $S_K = \{x \in K : h(x) = \frac{1}{2}\}$ is called the midsection.
- The intersection of a tetrahedron in K with the midsection is a blue triangle, a red triangle or a square with opposite edges of different colours:



- S_K is a cell complex with coloured edges.

Properties of The Midsection

- The midsection is homeomorphic to S^2 and isomorphic causal slices give rise to isomorphic midsections.
- Different causal slices give rise to different midsections.
- Not all coloured cell complexes, as we have described, can arise as midsections:



Outline of The Proof of Theorem 1

- It is enough to bound the number of causal slices made up of V tetrahedra.
- It is enough to bound the number of possible midsections made up of V cells.
- Subdivide each square in the midsection into two triangles by a new black edge.
- Then we get a triangulation of S^2 with $\leq 2V$ triangles and each edge has 3 possible colours.
- The number of such triangulations is bounded by an exponential function of V by Tutte's 1962 result.

Outline of The Proofs of Theorems 2 and 3

- The existence of the midsection and the fact that it determines a causal slice uniquely gives rise to a local construction of any causal slice.
- Local construction for causal slices gives a local construction for any causal triangulation.
- For any Σ_{in} and Σ_{out} there is a constant V_0 such that

$$N(V_1, \Sigma_{in}, \Sigma_{out}) N(V_2, \Sigma_{in}, \Sigma_{out}) \leq N(V_1+V_2+V_0, \Sigma_{in}, \Sigma_{out}).$$

- By standard arguments it follows that

$$\lim_{V \to \infty} \frac{\log N(V, \Sigma_{in}, \Sigma_{out})}{V}$$

exists and is independent of the boundary triangulations.

Generalised Causal Triangulations

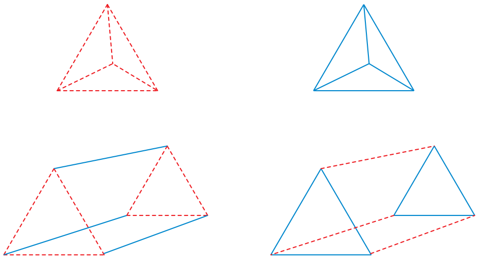
- A generalised causal slice is a simplicial 3-manifold K with two boundary components ∂K_{red} and ∂K_{blue} such that all mono-coloured simplicies belong to the boundary.
- The midsection of a generalised causal slice (defined as before) is a closed simplicial 2-manifold homeomorphic to both ∂K_{red} and ∂K_{blue}.
- The number $N_{3,g}(V)$ of generalised causal triangulations with volume V and midsection of genus g satisfies

$$N_{3,g}(V) \leq C_g^V$$

  for a suitable constant C_g.
- In case $g = 0$ generalised causal triangulations are the same as causal triangulations.

Generalisation to 4 dimensions

- One can generalise the definition of a causal triangulation to any dimension.
- One can generalise the construction of a midsection to 4-dimensional causal slices.
- There are 4 types of 4-simplicies that arise: (1,4), (2,3), (3,2) and (4,1).
- The midsection is a 3-dimensional cell complex made up of coloured tetrahedra and prisms:



- Let $M_D(V)$ be the number of triangulations of S^D made up of V D-simplicies.
- Let $N_D(V)$ be the number of D-dimensional causal triangulations made up of V D-simplicies.
- **Theorem 4** If $M_3(V) \leq C^V$ for some C then the exists a constant $\tilde{C}$ such that

$$N_4(V) \leq \tilde{C}^V.$$

- The proof is similar to the 3-dimensional case.
- **Conjecture** If $M_{D-1}(V) \leq C_D^V$, then there exists $\tilde{C}$ such that

$$N_D(V) \leq \tilde{C}^V.$$

Final Remarks

- Proving that $M_3(V) \leq C^V$ is most likely a hard problem but an important one.
- Recent work by Collet, Eckmann and Younan gives sufficient conditions for the bound to hold
- Many interesting questions can be asked ....