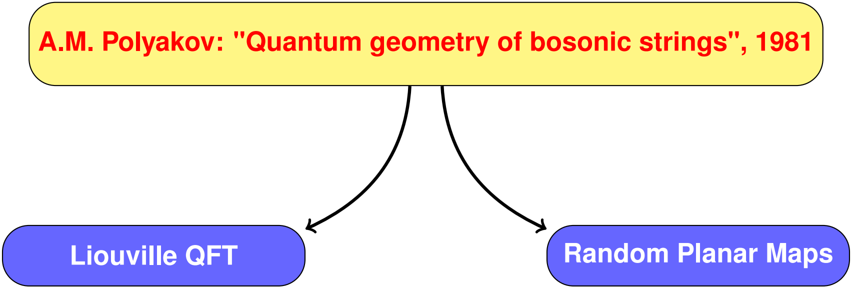
Liouville Quantum Gravity on the Riemann sphere

Rémi Rhodes
University Paris-Est Marne La Vallée

Joint work with F.David, A.Kupiainen, V.Vargas



Question:
What is the geometry of space under the coupled effect of Quantum Gravity interacting with Conformal Field Theories (CFT)?

Founding fathers: Polyakov, David, Distler, Kawai, Knizhnik, Zamolodchikov in the eighties

Plan of the talk

Constructing the Liouville quantum field theory on the Riemann sphere

Conjectures relating to random planar maps

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Uniformization and Liouville equation

Problem:
Given a $2d$ Riemann manifold $(M, \hat{g})$, find a metric g conformally equivalent to $\hat{g}$ with constant Ricci scalar curvature μ

$$R_g = -\mu$$

Liouville equation:
If $X : M \to \mathbb{R}$ solves the equation

$$\triangle_{\hat{g}} X - R_{\hat{g}} = \mu e^X$$

then the metric $g = e^X \hat{g}$ satisfies

$$R_g = -\mu$$

Liouville action:
Find such a X by minimizing the functional

$$S_L(X, \hat{g}) = \int_M \left(|\partial_{\hat{g}} X|^2 + 2R_{\hat{g}} X + \mu e^X \right) dV_{\hat{g}}$$

Notations: $\triangle_g = \textit{Laplacian}$, $\partial_g = \textit{gradient}$, R_g=Ricci curvature, V_g=volume form

Liouville Quantum Field Theory

Consider a $2d$ Riemann manifold $(M, \hat{g})$. Define a (probability) measure

$$e^{-\beta\, S_L(X,\hat{g})} DX$$

defined on the maps $X : M \to \mathbb{R}$, where S_L is the **Liouville action**

$$S_L(X, \hat{g}) = \frac{1}{4\pi} \int_M \left(|\partial_{\hat{g}} X|^2 + 2 R_{\hat{g}} X + \mu e^X \right) dV_{\hat{g}}$$

and DX is the "uniform measure" on maps

$$X : M \to \mathbb{R},$$

β fixed parameter and $\mu > 0$.

Get rid of the parameter β by making a change of variables $X \to \gamma X$.

Notations: $\triangle_g =$ *Laplacian*, $\partial_g =$ *gradient*, R_g=Ricci curvature, V_g=volume form

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Consider a $2d$ Riemann manifold $(M, \hat{g})$. Define a (probability) measure

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defined on the maps $X : M \to \mathbb{R}$, where S_L is the **Liouville action**

$$S_L(X, \hat{g}) = \frac{1}{4\pi} \int_M \left(|\partial_{\hat{g}} X|^2 + Q R_{\hat{g}} X + \mu e^{\gamma X} \right) dV_{\hat{g}}$$

and DX is the "uniform measure" on maps

$$X : M \to \mathbb{R},$$

and $\gamma \in [0, 2]$ and $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$ and $\mu > 0$.

Get rid of the parameter β by making a change of variables $X \to \gamma X$.

Notations: $\triangle_g = \mathit{Laplacian}$, $\partial_g = \mathit{gradient}$, R_g=Ricci curvature, V_g=volume form

Liouville Quantum Field Theory

Consider a $2d$ Riemann manifold $(M, \hat{g})$. Define a (probability) measure

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defined on the maps $X : M \to \mathbb{R}$, where S_L is the **Liouville action**

$$S_L(X,\hat{g}) = \frac{1}{4\pi}\int_M \Big(|\partial_{\hat{g}}X|^2 + QR_{\hat{g}}X + \mu e^{\gamma X}\Big)\, dV_{\hat{g}}$$

and DX is the "uniform measure" on maps

$$X : M \to \mathbb{R},$$

and $\gamma \in [0,2]$ and $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$ and $\mu > 0$.

Under this (probability) measure, one studies the "physical" random metric

$$e^{\gamma X}\hat{g}$$

through its volume form, Brownian motion, Riemann distance,...

Notations: $\triangle_g = \mathit{Laplacian}$, $\partial_g = \mathit{gradient}$, R_g=Ricci curvature, V_g=volume form

LQFT on the Riemann sphere

Sphere = $\mathbb{R}^2$ equipped with the round metric

$$\hat{g}(x)dx^2 = \frac{4}{(1+|x|^2)^2}dx^2.$$

Define the path integral by viewing it as a perturbation of the Gaussian measure

$$e^{-S_L(X,\hat{g})}DX = e^{-\frac{1}{4\pi}\int_{\mathbb{R}^2}\left(QR_{\hat{g}}X+\mu e^{\gamma X}\right)dV_{\hat{g}}} \underbrace{e^{-\frac{1}{4\pi}\int_{\mathbb{R}^2}|\partial_{\hat{g}}X|^2\,dV_{\hat{g}}}DX}_{\text{Gaussian measure}}$$

Notations: $\triangle_g = \mathit{Laplacian}$, $\partial_g = \mathit{gradient}$, R_g=Ricci curvature, V_g=volume form

Gaussian part

Focus on the "Gaussian measure"

$$e^{-\frac{1}{4\pi}\int_{\mathbb{R}^2}|\partial_{\hat{g}}X|^2\,dV_{\hat{g}}}DX$$

- gives equal weights to constant functions $\Rightarrow$ not a probability measure!
- independent of the choice of the metric g conformally equivalent to the round metric $\hat{g}$

$$\int_{\mathbb{R}^2}|\partial_g X|^2\,dV_g=\int_{\mathbb{R}^2}|\partial_{\hat{g}}X|^2\,dV_{\hat{g}}$$

- invariant under composition of X by Möbius transforms

$$\int_{\mathbb{R}^2}|\partial_g(X\circ\psi)|^2\,dV_g=\int_{\mathbb{R}^2}|\partial_g X|^2\,dV_g$$

Notations: $\triangle_g=$ *Laplacian*, $\partial_g=$ *gradient*, R_g=Ricci curvature, V_g=volume form

GFF with vanishing g-mean

▶ Denote by $m_g(f)$ the mean value of f in the metric g:

$$m_g(f) = \frac{1}{V_g(\mathbb{R}^2)} \int_{\mathbb{R}^2} f \, dV_g$$

If we restrict the Gaussian measure

$$e^{-\frac{1}{4\pi}\int_{\mathbb{R}^2} |\partial_{\hat{g}} X|^2 \, dV_{\hat{g}}} DX$$

to maps $X : \mathbb{R}^2 \to \mathbb{R}$ with vanishing mean in the metric g then it corresponds to the **Gaussian Free Field with vanishing g-mean X_g**.

▶ The GFF X_g is a Gaussian random "function" (rigorously a distribution) $X_g : \mathbb{R}^2 \to \mathbb{R}$ s.t.

$$\mathbb{E}[X_g(x)] = 0 \qquad \mathbb{E}[X_g(x)X_g(y)] = G_g(x, y)$$

where $G_g =$ Green function of the Laplacian $\triangle_g$ with 0-mean

$$-\triangle_g u = 2\pi f, \quad m_g(u) = 0.$$

Free GFF

- Rule for conformal change of metrics

$$X_g \overset{law}{=} X_{g'} - m_g(X_{g'})$$

Integrate with respect to Lebesgue to get rid of the metric dependency

$$X = c + X_g$$

where c is "distributed" as the Lebesgue measure.

Free GFF

Infinite measure on maps $\mathbb{R}^2 \to \mathbb{R}$, **conformally invariant**

$$\int F(X)\, e^{-\frac{1}{4\pi}\int |\partial_g X|^2\, dV_g}\, DX = \int_{\mathbb{R}} \mathbb{E}[F(c + X_g)]\, dc$$

The definition does not depend on g.

Remark: coincides with Lebesgue measure when restricted to constant functions (called **zero mode** in physics)

Gaussian multiplicative chaos

Recall the strategy

$$e^{-S_L(X,\hat{g})}DX = e^{-\frac{1}{4\pi}\int_{\mathbb{R}^2}\left(QR_{\hat{g}}X+\mu e^{\gamma X}\right)dV_{\hat{g}}}\underbrace{e^{-\frac{1}{4\pi}\int_{\mathbb{R}^2}|\partial_{\hat{g}}X|^2\,dV_{\hat{g}}}DX}_{\text{Gaussian measure}}$$

Recall that

$$X = c + X_{\hat{g}}$$

with c distributed as Lebesgue measure and $X_{\hat{g}}$ is the GFF with vanishing $\hat{g}$-mean.

$\Rightarrow$ construct the random measure

$$e^{\gamma X_{\hat{g}}(x)}\,dV_{\hat{g}}$$

- **Problem:** the GFF $X_{\hat{g}}$ is a distribution, not a fairly defined function

$$\mathbb{E}[X_{\hat{g}}(x)X_{\hat{g}}(y)] = \ln\frac{1}{|x-y|} + \text{smooth function}$$

Gaussian multiplicative chaos

Construct the random measure

$$e^{\gamma X_{\hat{g}}(x)}\, dV_{\hat{g}}$$

- **Regularization:**

$$X_{\hat{g}}^{\epsilon} = X_{\hat{g}} * \rho_{\epsilon}$$

where $\rho_{\epsilon} = \epsilon^{-2}\rho\big(\frac{\cdot}{\epsilon}\big)$ is a mollifying sequence.

- **Kahane '85:** for $\gamma \in [0, 2[$,

$$\lim_{\epsilon \to 0} e^{\gamma X_{\hat{g}}^{\epsilon}(x) - \frac{\gamma^2}{2}\mathbb{E}[(X_{\hat{g}}^{\epsilon}(x))^2]}\, V_{\hat{g}}(dx) = M_{\hat{g}}^{\gamma}(dx)$$

in probability. The random measure $M_{\hat{g}}^{\gamma}$ has finite mass, full support, is diffuse and has carrier with Hausdorff dimension $2 - \frac{\gamma^2}{2}$.

Path integral

Now we are in position to define the formal integral

$$\int F(X) e^{-\frac{1}{4\pi}\int_{\mathbb{R}^2}\left(QR_{\hat{g}}X + 4\pi\mu e^{\gamma X}\right) dV_{\hat{g}}} \underbrace{e^{-\frac{1}{4\pi}\int_{\mathbb{R}^2}|\partial_{\hat{g}}X|^2_{\hat{g}}\, dV_{\hat{g}}} DX}_{\text{Gaussian part}}$$

where $X = c + X_{\hat{g}}$. In the round metric, $R_{\hat{g}} = 2$, and the curvature term becomes

$$\frac{1}{4\pi}\int_{\mathbb{R}^2} QR_{\hat{g}}(c + X_{\hat{g}})\, dV_{\hat{g}} = 2Qc$$

so that the integration measure is

$$\mu_{\mathrm{L}}(dX) = e^{-2Qc - \mu e^{\gamma c} M^{\gamma}_{\hat{g}}(\mathbb{R}^2)}\, dc\, d\mathbb{P}_{X_{\hat{g}}}$$

Problem: μ_{L} is not a probability measure because

$$\int \mu_{\mathrm{L}}(dX) \geq \int_{\mathbb{R}} e^{-2Qc - \mu e^{\gamma c}}\, dc = +\infty$$

Law of X under μ_{L} invariant under the Möbius group $SL_2(\mathbb{C})$ non compact!

Back to classical uniformization theory

Liouville action

$$S_L(X,\hat{g}) = \frac{1}{4\pi}\int_M \left(|\partial_{\hat{g}} X|^2 + 2R_{\hat{g}}X + \mu e^X\right) dV_{\hat{g}}$$

Saddle point

$$\triangle_{\hat{g}}\varphi - R_{\hat{g}} = 2\pi\mu e^{\varphi}.$$

If $g = e^{\varphi}\hat{g}$ then

$$R_g = -2\pi\mu < 0$$

No such a metric on the sphere $\Rightarrow$ **Liouville action is not bounded from below on the sphere**

Back to classical uniformization theory

Remedy: Insert strong "bumps of positive curvature" at some fixed places to compensate for the lack of positive curvature

$$\triangle_{\hat{g}}\varphi - R_{\hat{g}} = 2\pi\mu e^{\varphi} - 2\pi\sum_{i=1}^{n}\alpha_i\delta_{z_i}$$

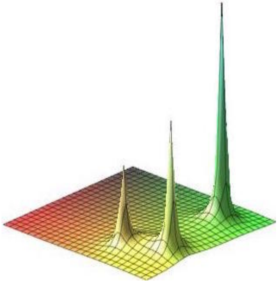
$z_i =$ location of the bump $\quad \alpha_i =$ "strength of the bump"

The metric $g = e^{\varphi}\hat{g}$ has curvature
$-2\pi\mu$ everywhere except at the places $(z_i)_i$ where

$$g(x) \sim |x - z_i|^{-\alpha_i} \quad \text{when} \quad x \to z_i$$

Troyanov '91: solvable if

$$\forall i,\ \ \alpha_i < 2 \quad \text{and} \quad \sum_i \alpha_i > 2$$



At least 3 insertions! Quantum analog?

Correlation functions

To get a probability measure, define "quantum insertions"

- Define the vertex operators

$$V_\alpha(z) = e^{\alpha X(z) - \frac{\alpha^2}{2}\mathbb{E}[X_{\hat{g}}(z)^2] + \frac{\alpha Q}{2}\ln \hat{g}}$$

- Correlation functions

$$\Pi_{\mu,\gamma}^{(z_i,\alpha_i)_i}(F) = \int F(X)\Big(\prod_{i=1}^{n} V_{\alpha_i}(z_i)\Big)\mu_{\mathrm{L}}(dX)$$

- Liouville partition function with n vertex operators on the sphere

$$\Pi_{\mu,\gamma}^{(z_i,\alpha_i)}(1) = C\big(\{z_i\}\big)\int_{\mathbb{R}} e^{(\sum_i \alpha_i - 2Q)c}\,\mathbb{E}\Big[e^{-\mu e^{\gamma c} Z_{(z_i,\alpha_i)}(\mathbb{R}^2)}\Big]\, dc$$

where

$$Z_{(z_i,\alpha_i)}(dx) = e^{\gamma\sum_i \alpha_i G_{\hat{g}}(x,z_i)} M_{\hat{g}}^{\gamma}(dx)$$

Seiberg's bounds

Liouville partition function with n vertex operators on the sphere

$$\Pi_{\mu,\gamma}^{(z_i,\alpha_i)}(1) = C\big((z_i)_i\big) \int_{\mathbb{R}} e^{(\sum_i \alpha_i - 2Q)c} \mathbb{E}\Big[e^{-\mu e^{\gamma c} Z_{(z_i,\alpha_i)}(\mathbb{R}^2)}\Big]\, dc$$

where

$$Z_{(z_i,\alpha_i)}(dx) = e^{\gamma \sum_i \alpha_i G_{\hat{g}}(x,z_i)} M_{\hat{g}}^{\gamma}(dx)$$

Theorem (DKRV '14)

The Liouville partition function is well defined, i.e. $0 < \Pi_{\mu,\gamma}^{(z_i,\alpha_i)}(1) < \infty$, iff

$$\sum_i \alpha_i > 2Q \quad \text{and} \quad \forall i, \quad \alpha_i < Q.$$

Remark: at least 3 vertex operators needed

Seiberg's bounds

Liouville partition function with n vertex operators on the sphere

$$\Pi_{\mu,\gamma}^{(z_i,\alpha_i)}(1) = C((z_i)_i) \int_{\mathbb{R}} e^{(\sum_i \alpha_i - 2Q)c} \mathbb{E}\left[e^{-\mu e^{\gamma c} Z_{(z_i,\alpha_i)}(\mathbb{R}^2)} \right] dc$$

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Remark: at least 3 vertex operators needed

- Zero mode contribution

$$\int_{\mathbb{R}} e^{(\sum_i \alpha_i - 2Q)c} e^{-\mu e^{\gamma c}} dc < +\infty \quad \Longleftrightarrow \quad \sum_i \alpha_i > 2Q$$

Seiberg's bounds

Liouville partition function with n vertex operators on the sphere

$$\Pi_{\mu,\gamma}^{(z_i,\alpha_i)}(1) = C\big((z_i)_i\big) \int_{\mathbb{R}} e^{(\sum_i \alpha_i - 2Q)c} \mathbb{E}\Big[e^{-\mu e^{\gamma c} Z_{(z_i,\alpha_i)}(\mathbb{R}^2)}\Big]\, dc$$

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Remark: at least 3 vertex operators needed

- the random measure $Z_{(z_i,\alpha_i)}(dx)$ is of finite total mass if and only if

$$\forall i, \quad \alpha_i < Q.$$

KPZ

Theorem (DKRV '14))

(a) **KPZ scaling laws:** scaling w.r.t the cosmological constant μ

$$\Pi_{\mu,\gamma}^{(z_i,\alpha_i)}(\hat{g},1) = \mu^{\frac{2Q-\sum_i \alpha_i}{\gamma}} \Pi_{1,\gamma}^{(z_i,\alpha_i)}(\hat{g},1).$$

(b) **Conformal covariance:** let ψ be Möbius. Then

$$\Pi_{\mu,\gamma}^{(\psi(z_i),\alpha_i)_i}(\hat{g},1) = \Big(\prod_i |\psi'(z_i)|^{-2\Delta_{\alpha_i}}\Big)\Pi_{\mu,\gamma}^{(z_i,\alpha_i)_i}(\hat{g},1)$$

where the conformal weights are defined by $\triangle_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$.

(c) **Weyl invariance:** Let $g = e^h \hat{g}$

$$\ln \frac{\Pi_{\mu,\gamma}^{(z_i,\alpha_i)_i}(e^h\hat{g},1)}{\Pi_{\mu,\gamma}^{(z_i,\alpha_i)_i}(\hat{g},1)} = \frac{c_L}{96\pi}\Big(\int_{\mathbb{R}^2} |\partial_{\hat{g}} h|^2 \, dV_{\hat{g}} + \int_{\mathbb{R}^2} 2R_{\hat{g}} h \, dV_{\hat{g}}\Big)$$

where the **central charge** of the Liouville theory is

$$c_L = 1 + 6Q^2$$

Liouville measure

It is the "volume form" Z associated to the (physical) metric

$$e^{\gamma X}\, dV_{\hat g},$$

where the law of X is ruled by $\Pi_{\mu,\gamma}^{(z_i,\alpha_i)_i}(\hat g,\cdot)$.

Explicit expression

$$\mathbb{E}^{\gamma,\mu}_{(z_i,\alpha_i)}[F(Z(A))] = C\mathbb{E}\Big[F\big(\xi\frac{Z_{(z_i,\alpha_i)}(A)}{Z_{(z_i,\alpha_i)}(\mathbb{R}^2)}\big)\frac{1}{Z_{(z_i,\alpha_i)}(\mathbb{R}^2)^{\frac{\sum_i \alpha_i - 2Q}{\gamma}}}\Big]$$

where C is a renormalization constant and

- ξ has Gamma law $\Gamma(\frac{\sum_i \alpha_i - 2Q}{\gamma},\mu)$ independent of $Z_{(z_i,\alpha_i)}(dx)$
- (reminder)

$$Z_{(z_i,\alpha_i)}(dx) = e^{\gamma\sum_{i=1}\alpha_i G_{\hat g}(x,z_i)} M^{\gamma}_{\hat g}(dx)$$

- Conformally invariant and metric independent.

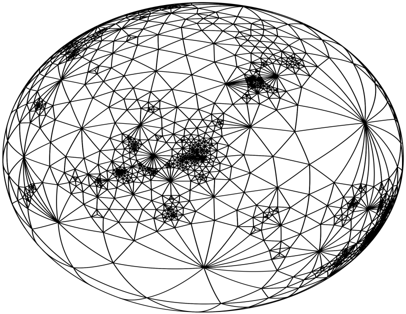
Plan of the talk

Constructing the Liouville quantum field theory on the Riemann sphere

Conjectures relating to random planar maps

Triangulations of the sphere

Triangulations of the sphere:
glue N triangles together along their edges so as to get a shape with the topology of a sphere.

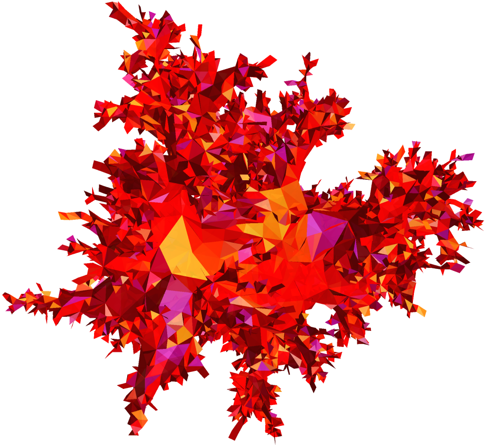


Combinatorics: finite number of such objects up to angle preserving homeomorphisms.

Discrete LQG

Conformal structure of triangulations

- see each face of the triangulation as an equilateral triangle and glue them along their edges so as to get a conformal structure with the topology of a sphere.



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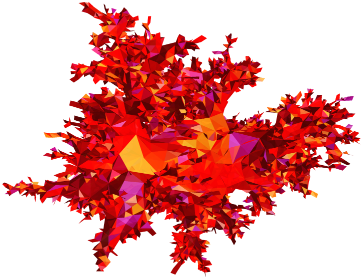
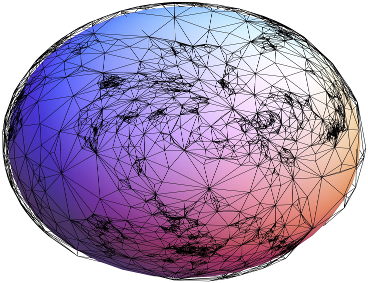


Conformal map



- $N \to \infty$ scaling limit of random triangulations conformally embedded onto the sphere $\mathbb{S}^2$?
- In which sense? Which notion of randomness?

Random Planar Maps (Pure discrete gravity)

- Let $\mathcal{T}_N^3$ be the set of N-triangulations with **3 marked faces**

$$\text{Partition function:} \qquad Z_{\bar{\mu}} = \sum_N e^{-\bar{\mu}N}\mathrm{Card}(\mathcal{T}_N^3)$$

- Asymptotically (Tutte)

$$\mathrm{Card}(\mathcal{T}_N^3) \sim N^{-\frac{1}{2}} e^{\mu_c N}$$

- for $\bar{\mu} > \mu_c$, define a probability measure on $\bigcup_N \mathcal{T}_N^3$

$$\mathbb{E}_{\bar{\mu}}[F(T)] = \frac{1}{Z_{\bar{\mu}}} \sum_N e^{-\bar{\mu}N} \sum_{T \in \mathcal{T}_N^3} F(T)$$

Remark: If $\bar{\mu} \downarrow \mu_c$, triangulations with large number of faces are more likely to be sampled.

Random Planar Maps+Matter

- Consider a lattice model (percolation, Ising,...) on triangulations at its critical point

- Let $Z(T)$ be the partition function of the model on the triangulation T. Conjecturally, for some $\gamma > 0$ depending on the model,

$$Z_N = \sum_{T \in \mathcal{T}_N^3} Z(T) \sim N^{1-\frac{4}{\gamma^2}} e^{\mu_c N}$$

- for $\bar{\mu} > \mu_c$, define a probability measure on $\bigcup_N \mathcal{T}_N^3$

$$\mathbb{E}_{\bar{\mu}}[F(T)] = \frac{1}{Z_{\bar{\mu}}} \sum_N e^{-\bar{\mu} N} \sum_{T \in \mathcal{T}_N^3} Z(T) F(T)$$

- If $\gamma \in [\sqrt{2}, 2]$, then triangulations with large number of faces are more likely to be sampled when $\bar{\mu} \downarrow \mu_c$.

$$\gamma = \sqrt{8/3} \text{ Percolation}, \quad \gamma = \sqrt{3} \text{ Ising}, \quad \gamma = 2 \text{ GFF}, \ldots$$

Scaling limit of random measures

- Fix the conformal map so that the three marked faces are sent to three fixed points z_1, z_2, z_3 on the sphere $\mathbb{S}^2$.
- Give a mass a to each face of the triangulation and push to **a measure** $\nu_{T,a}$ on the sphere
- Sample T according to $\mathbb{P}_{\bar{\mu}}$, hence

$$\text{Law of } \nu_{T,a}: \quad \mathbb{E}_{\bar{\mu}}[F(\nu_{\bar{\mu},a}(dx))] = \frac{1}{Z_{\bar{\mu}}} \sum_N e^{-\bar{\mu}N} \sum_{T \in \mathcal{T}_N^3} Z(T) F(\nu_{T,a}(dx))$$

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Conjecture (DKRV)

Fix $\mu > 0$. In the regime

$$a \to 0 \quad \text{and} \quad \bar{\mu} \sim \mu_c + a\mu,$$

$\nu_{\bar{\mu},a}$ converges in law towards the Liouville measure Z of Liouville QFT with parameters (γ, μ) and three vertex insertions $(z_i, \gamma)_{i=1,2,3}$.

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The Liouville measure (for $(z_i, \gamma)_{i=1,2,3}$):

$$\mathbb{E}^{\gamma,\mu}_{(z_i,\gamma)}[F(Z(A))] = C\mathbb{E}\Big[F\big(\xi \frac{Z_{(z_i,\gamma)}(A)}{Z_{(z_i,\gamma)}(\mathbb{R}^2)}\big) Z_{(z_i,\gamma)}(\mathbb{R}^2)^{-(3-\frac{2Q}{\gamma})}\Big]$$

where ξ has Gamma law $\Gamma(3 - \frac{2Q}{\gamma}, \mu)$ and

$$Z_{(z_i,\gamma)}(dx) = e^{\gamma \sum_{i=1}^{3} \alpha_i G_{\hat{g}}(x,z_i)} M^{\gamma}_{\hat{g}}(dx)$$

Open question: DOZZ formula

Fix 3 points z_1, z_2, z_3 and consider the 3-point function

$$\Pi_{\mu,\gamma}^{(z_i,\alpha_i)}(1) = \mu^{\frac{2Q-\sum_i \alpha_i}{\gamma}} |z_1 - z_2|^{2\triangle_{12}} |z_2 - z_3|^{2\triangle_{23}} |z_1 - z_3|^{2\triangle_{13}} C_\gamma(\alpha_1, \alpha_2, \alpha_3)$$

with $\triangle_{12} = \triangle_{\alpha_3} - \triangle_{\alpha_1} - \triangle_{\alpha_2}$,...

Believed to determine the whole Liouville theory.

Conjectural **explicit analytic expression for** $C_\gamma(\alpha_1, \alpha_2, \alpha_3)$. Prove it!

Similars results on the Random Planar Maps (without matter):

- for 2-point distances (Ambjørn-Watabiki, Di Francesco-Guitter).
- for 3-point distances (Bouttier-Guitter '08)

Connection with 2-point quantum sphere

► Can one give sense to the 2-point correlation functions (Seiberg '92)? Makes sense only for equal weights

$$(z_1, \gamma) \quad (z_2, \gamma)$$

The resulting Liouville measure must be **invariant under dilation** $\Rightarrow$

Dilemma: not a true probability measure **or** not conformally invariant

► Recent rigorous construction by Duplantier-Miller-Sheffield 2014

► Possible to recover the 2 point function from the three point functions: take three vertex operators

$$(0, \gamma) \quad (\infty, \gamma) \quad (z, \epsilon)$$

and send $\epsilon \to 0$. Convergence modulo multiplicative constants of the unit volume Liouville measure

Thanks!