IPhT, Saclay, June 11, 2015

Hurwitz numbers and matrix models

Leonid Chekhov (Steklov Math. Inst., Moscow, and QGM, Århus University)
(based on joint papers with Jan Ambjørn, NBI, Copenhagen)

- Grothendieck's dessins d'enfant: Belyi pairs and partitions of Riemann surfaces
- Complex matrix model for Grothendieck's dessins d'enfant
- Generalizations to hypergeometric Hurwitz numbers
- New matrix models of Toda chain type: spectral curves and topological recursion

Fat graph description for homotopy types of ramified mappings $\mathbb{C}P^1 \to C_g$
Hurwitz numbers: combinatorial classes of ramified mappings $f : \mathbb{C}P^1 \to C_g$ of the complex projective line onto a Riemann surface of genus g. Grothendieck's dessins d'enfant enumerate mappings ramified over exactly 3 points (0, 1, and ∞). At every point we have a ramification profile given by a Young tableaux: it fixes the set of ramification types at the given point.

Theorem

(Belyi) *A smooth complex algebraic curve C is defined over the field of algebraic numbers $\overline{\mathbb{Q}}$ if and only if it exists a nonconstant meromorphic function f on C ($f : C \to \mathbb{C}P^1$) ramified only over the points $0, 1, \infty \in \mathbb{C}P^1$.*

Lemma

(Grothendieck) *There is a one-to-one correspondence between the isomorphism classes of Belyi pairs and connected bipartite fat graphs.*

Grothendieck's dessins d'enfant and partitions of Riemann surfaces

Single and double Hurwitz numbers correspond to the cases in which ramification profiles (defines by the corresponding Young tableauxes λ or λ and μ) are respectively given at one (∞) or two (∞ and 1) distinct points and we take the sum over ramifications types at the remaining point(s).

"Original" Hurwitz numbers have only simple (square-root-type) ramifications at m points; [Goulden, Jackson, Okounkov, Pandharipande, Eynard, Borot,...]

Grothendieck's dessins d'enfant or Belyi pairs: exactly three ramification points with profiles λ, μ, and ν.

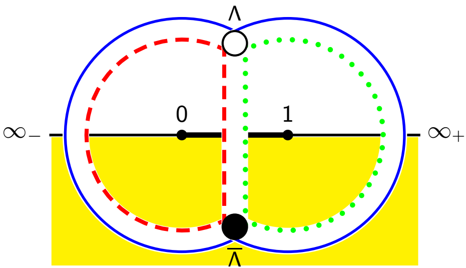
Clean Belyi pairs: exactly three ramification points with profiles λ, μ, and $\nu = (2, 2, \ldots, 2)$. (Only single Hurwitz numbers)

Hypergeometric (or generalized) Belyi pairs: exactly n ramification points with profiles λ, μ, and ν_i, $i = 2, \ldots, n-1$.

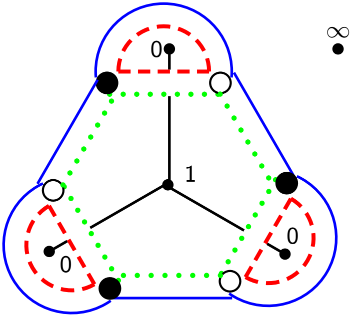
Integrable properties: generating functions of all of the above are KP hierarchy τ-functions for single and double Hurwitz numbers (A. Yu. Orlov and Shcherbin'02, Okounkov'00)

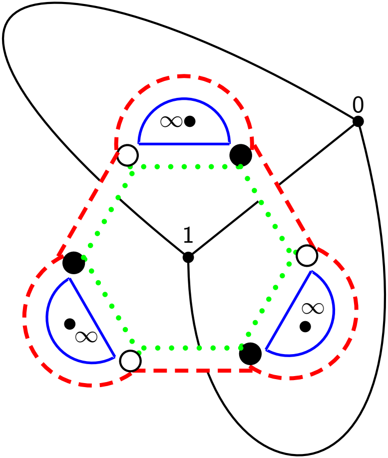
A **fat graph** corresponding to a dessin d'enfant is a 3-valent bipartite fat graph, which is a covering of a base graph (describing the nonramified map $\mathbb{C}P^1 \to \mathbb{C}P^1$) and describes a partition of C_g into sets of three-colored polygons (necessarily with even numbers of edges for every polygon).

n=3

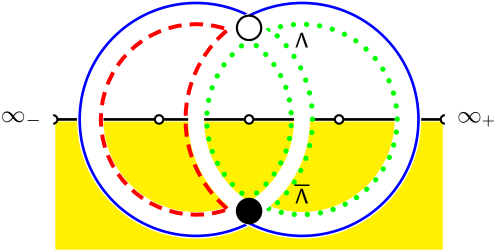


Grothendieck's dessins d'enfant and partitions of Riemann surfaces

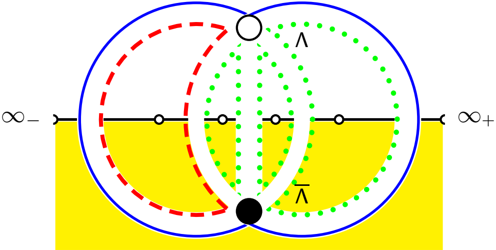




n=4

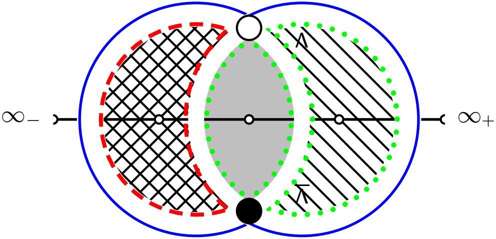

∞_-
∞_+
Λ
$\bar{\Lambda}$

n=5

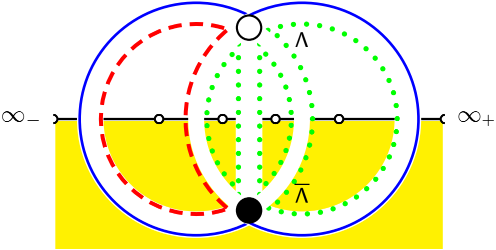

∞_-
∞_+
Λ
$\bar{\Lambda}$

Grothendieck's dessins d'enfant and partitions of Riemann surfaces

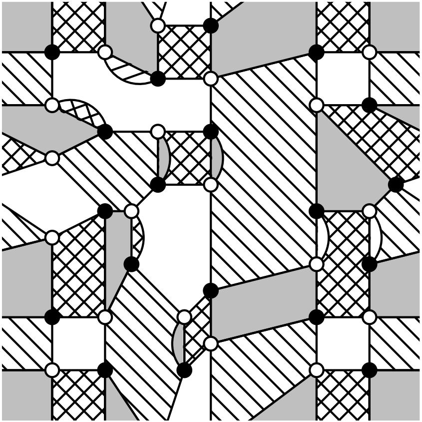
n=4



n=5



Maps $\mathcal{C}_1 \to \mathbb{C}P^1$ for $n = 4$



The matrix model

As was shown (Alexandrov, Mironov, Morozov, Natanzon; Harnad Orlov) the exponential of the generating function

$$\mathcal{F}\big[\{t_m\},\{\mathfrak{t}_r\},\gamma_2,\dots,\gamma_{n-1};N\big]=\sum_{\Gamma}\frac{1}{|\mathrm{Aut}\,\Gamma|}N^{2-2g}\prod_{r=1}^{\infty}\frac{t_r^{k_1^{(r)}}}{k_1^{(r)}!}\prod_{s=1}^{\infty}\frac{\mathfrak{t}_s^{k_n^{(s)}}}{k_n^{(s)}!}\prod_{j=2}^{n-1}\gamma_j^{k_j}$$

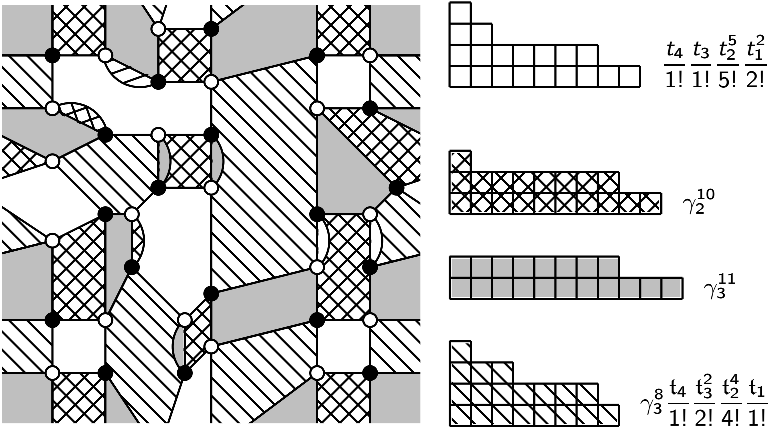
is a tau function of the KP hierarchy in times t or $\mathfrak{t}$. A matrix model "solvable" in terms of the topological recursion method (Ch-Eynard-Orantin) was proposed by Ambjørn and LCh in the case where $\gamma_3=\gamma_4=\cdots=\gamma_{n-1}$ leaving $\gamma_2>\gamma_3$ arbitrary:

$$\mathcal{F}\big[\{t_m\},\{\mathfrak{t}_r\},\gamma_2,\gamma_3;N\big]=\sum_{\Gamma}\frac{1}{|\mathrm{Aut}\,\Gamma|}N^{2-2g}\prod_{r=1}^{\infty}\frac{t_r^{k_1^{(r)}}}{k_1^{(r)}!}\prod_{s=1}^{\infty}\frac{\mathfrak{t}_s^{k_n^{(s)}}}{k_n^{(s)}!}\gamma_2^{k_2}\gamma_3^{k_3+\cdots+k_{n-1}},$$

where N, γ_2, γ_3, t_r, and $\mathfrak{t}_r$ are formal independent parameters and the sum ranges all (connected) generalized Belyi fat graphs. We encode the second time dependence through the *external matrix field* $\Lambda=\mathrm{diag}\,(\lambda_1,\dots,\lambda_{\gamma_3 N})$, the corresponding times are

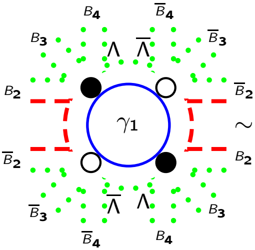
$$\mathfrak{t}_r=tr\big[(\Lambda\overline{\Lambda})^r\big].$$

Maps $\mathcal{C}_1 \to \mathbb{C}P^1$ for $n = 4$



The matrix model

- First step is to contract cycles corresponding to t-variables (white polygons in the figure).



$$\sim \frac{1}{2} N t_2 \, tr\left[\left(B_2 B_3 B_4 \Lambda \overline{\Lambda}\, \overline{B}_4 \overline{B}_3 \overline{B}_2\right)^2\right]$$

Here B_2 is a rectangular complex matrix of size $\gamma_2 N \times \gamma_3 N$ (we assume $\gamma_2 \geq \gamma_3$) and all other B_i are quadratic matrices of size $\gamma_3 N \times \gamma_3 N$.

The matrix model

Note that the first model of this sort was proposed by Itzykson and P. Di Francesco in hepth/9212108: for clean Belyi numbers (when one of the partitions is $(2, 2, \ldots, 2)$). Then, contracting these cycles (all of order four) and denoting $t_k = tr\left[(\Lambda'\overline{\Lambda}')^k\right]$ and $\mathrm{t}_r = tr\left[(\Lambda\overline{\Lambda})^r\right]$ we obtain the matrix model with complex matrices:

$$\int dB d\overline{B} e^{-\, tr\, B\overline{B} + \frac{1}{2}\, tr\left[(B\Lambda\overline{\Lambda B \Lambda}'\Lambda')^2\right]}$$

[This model however is not a KP tau-function]

The generating function reads:

$$\int DB_2 \cdots DB_{n-1} e^{N \sum_{r=\mathbf{1}}^{\infty} \frac{t_r}{r}\, tr\left[\left(B_{\mathbf{2}} \cdots B_{n-\mathbf{1}} \Lambda \overline{\Lambda}\, \overline{B}_{n-\mathbf{1}} \cdots \overline{B}_{\mathbf{2}}\right)^r\right] - \sum_{j=\mathbf{2}}^{n-\mathbf{1}} N\, tr(B_j \overline{B}_j)}$$

Performing the variable changing

$$\begin{aligned} \mathfrak{B}_2 &= B_2 B_3 \cdots B_{n-1} \\ \mathfrak{B}_3 &= B_3 \cdots B_{n-1} \\ &\vdots \\ \mathfrak{B}_{n-1} &= B_{n-1} \end{aligned}$$

and assuming that all matrices $\mathfrak{B}_3, \ldots, \mathfrak{B}_{n-1}$ *are invertible* (the matrix $\mathfrak{B}_2$ remains rectangular) we obtain

$$\int D\mathfrak{B}_2 \cdots D\mathfrak{B}_{n-1} \exp\Big\{ -\gamma_2 N\, tr \log(\mathfrak{B}_3 \overline{\mathfrak{B}}_3) - \sum_{j=4}^{n-1} \gamma_3 N\, tr \log(\mathfrak{B}_j \overline{\mathfrak{B}}_j)$$

$$+ \sum_{r=1}^{\infty} N \frac{t_r}{r}\, tr\left[(\mathfrak{B}_2 |\Lambda|^2 \overline{\mathfrak{B}}_2)^r\right] - N\, tr\left[\mathfrak{B}_2 \mathfrak{B}_3^{-1} \overline{\mathfrak{B}}_3^{-1} \overline{\mathfrak{B}}_2\right]$$

$$-N\, tr\left[\mathfrak{B}_3 \mathfrak{B}_4^{-1} \overline{\mathfrak{B}}_4^{-1} \overline{\mathfrak{B}}_3\right] - \cdots - N\, tr\left[\mathfrak{B}_{n-2} \mathfrak{B}_{n-1}^{-1} \overline{\mathfrak{B}}_{n-1}^{-1} \overline{\mathfrak{B}}_{n-2}\right] - N\, tr\left[\mathfrak{B}_{n-1} \overline{\mathfrak{B}}_{n-1}\right]\Big\}$$

The matrix model

An integral over general complex matrices $\mathfrak{B}_i$ can be written (Ambjørn, Kristjansen, Makeenko) in terms of positive definite Hermitian matrices X_i upon the variable changing

$$X_i := \overline{\mathfrak{B}}_i \mathfrak{B}_i, \quad i = 2, \dots, n-1.$$

All the matrices X_i ($i = 2, \dots, n-1$) are of the same size $\gamma_3 N \times \gamma_3 N$. Changing the integration measure for rectangular complex matrices is governed by the Marchenko–Pastur law and introduces a simple logarithmic term.

Theorem

[Marchenko, Pastur, 1967] *Upon eliminating unitary group degrees of freedom, for $\gamma_2 \geq \gamma_3$, we have the measure transformation*

$$\prod_{i=1}^{\gamma_2 N} \prod_{j=1}^{\gamma_3 N} d\, Re\, B_{ij}\, d\, Im\, B_{ij} = \prod_{j_1 < j_2} (x_{j_1} - x_{j_2})^2 \prod_{j=1}^{\gamma_3 N} x_j^{(\gamma_2 - \gamma_3)N} \prod_{j=1}^{\gamma_3 N} dx_j \frac{dU_{\gamma_3 N}\, dU_{\gamma_2 N}}{dU_{(\gamma_2 - \gamma_3)N}}$$

where $x_j \geq 0$ are nonnegative eigenvalues of the Hermitian $\gamma_3 N \times \gamma_3 N$ matrix $X := B^\dagger B$.

The matrix model

Performing the scaling $X_i \to X_i|\Lambda|^{-2}$, we obtain an integral over a chain of matrices:

$$\int DX_{2\geq 0}\cdots DX_{n-1\geq 0}\exp\Big\{N\sum_{r=1}^{\infty}\frac{t_r}{r}\,tr(X_2^r) - N\,tr(X_2X_3^{-1}) - N\,tr(X_3X_4^{-1})$$

$$-\cdots - N\,tr(X_{n-2}X_{n-1}^{-1}) - N\,tr\big(X_{n-1}|\Lambda|^{-2}\big)$$

$$+(\gamma_2-\gamma_3)N\,tr\log X_2 - \gamma_2 N\,tr\log X_3 - \gamma_3 N\,tr\log(X_4\cdots X_{n-1})\Big\}$$

The logarithmic term in X_2 stabilizes the equilibrium distribution of eigenvalues of this matrix in the domain of positive real numbers; if $\gamma_2=\gamma_3$, we lose this term and must use the technique of matrix models with hard walls.

The matrix model for Grothendieck's dessins d'enfant

For $n = 3$, we have only one matrix X_2 and the generating function for double Hurwitz numbers becomes the Brezín–Gross–Witten integral

$$\int DX_{2\geq 0} \exp\left\{ N \sum_{r=1}^{\infty} \frac{t_r}{r} tr(X_2^r) - N\, tr\left(X_2 |\Lambda|^{-2}\right) + (\gamma_2 - \gamma_3) N\, tr \log X_2 \right\}$$

(which was known (Mironov–Morozov–Semenoff'94) to be a KP tau-function).

For simple Hurwitz numbers $\Lambda = \mathbb{E}$, and we obtain a mere Hermitian one-matrix model integral for the corresponding generating function:

$$\int DX_{\geq 0} \exp\left\{ N\, tr \left[\sum_{r=1}^{\infty} \frac{t_r}{r} X^r - X + (\gamma_2 - \gamma_3) \log X \right] \right\} \quad \text{[De Mello Koch, Ramgoolam]}$$

For clean Belyi morphisms $t_i = -\delta_{i,2}$ we have KPMM [Ch, Makeenko'91] equivalent to 1MM with matrices of size $(\gamma_2 - \gamma_3)N \times (\gamma_2 - \gamma_3)N$:

$$\int DX_{\geq 0} \exp\left\{ N\, tr \left[-X^2/2 - X |\Lambda|^{-2} + (\gamma_2 - \gamma_3) \log X \right] \right\}$$

Performing the scaling $X_i \to X_i|\Lambda|^{-2}$, we obtain an integral over a chain of matrices:

$$\int DX_{2\geq 0} \cdots DX_{n-1\geq 0} \exp\Big\{N \sum_{r=1}^{\infty} \frac{t_r}{r} tr(X_2^r) - N\,tr(X_2 X_3^{-1}) - N\,tr(X_3 X_4^{-1})$$
$$- \cdots - N\,tr(X_{n-2}X_{n-1}^{-1}) - N\,tr\big(X_{n-1}|\Lambda|^{-2}\big)$$
$$+(\gamma_2 - \gamma_3)N\,tr\log X_2 - \gamma_2 N\,tr\log X_3 - \gamma_3 N\,tr\log(X_4 \cdots X_{n-1})\Big\}$$

The logarithmic term in X_2 stabilizes the equilibrium distribution of eigenvalues of this matrix in the domain of positive real numbers; if $\gamma_2 = \gamma_3$, we lose this term and must use the technique of matrix models with hard walls.

Performing the scaling $X_i \to X_i|\Lambda|^{-2}$, we obtain an integral over a chain of matrices:

$$\int DX_{2_{\geq 0}} \cdots DX_{n-1_{\geq 0}} \exp\Big\{ N \sum_{r=1}^{\infty} \frac{t_r}{r} \, tr(X_2^r) - N\, tr(X_2 X_3^{-1}) - N\, tr(X_3 X_4^{-1})$$
$$- \cdots - N\, tr(X_{n-2} X_{n-1}^{-1}) - N\, tr\big(X_{n-1}|\Lambda|^{-2}\big)$$
$$+(\gamma_2 - \gamma_3) N\, tr \log X_2 - \gamma_2 N\, tr \log X_3 - \gamma_3 N\, tr \log(X_4 \cdots X_{n-1}) \Big\}$$

The logarithmic term in X_2 stabilizes the equilibrium distribution of eigenvalues of this matrix in the domain of positive real numbers; if $\gamma_2 = \gamma_3$, we lose this term and must use the technique of matrix models with hard walls.

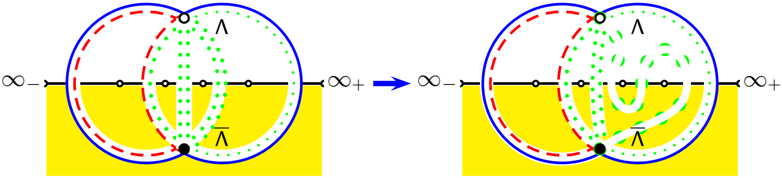
This model is the main object of study.

Braid-group action

The above matrix chains admit the braid-group action resulted from that an order of ramification points is not fixed *a priori*.

$$\beta_i: \ \left\{X_i \to X_{i-1}X_i^{-1}X_{i+1};\ X_j \to X_j,\ j \neq i\right\}.$$

It is easy to see that the action of each such generator with $3 \geq i \geq n-2$ leaves the matrix chain action invariant.



We apply the Harish-Chandra–Itzykson–Zuber integration formula to every term in the chain of matrices. Taking into account that, for instance, the integral over the unitary group for the term $e^{-N\,tr\,X_k X_{k+1}^{-1}}$ gives

$$\int DU e^{-N\sum_{i,j=1}^{\gamma_3 N} U_{ij}x_i^{(k)}U_{ij}^*[x_j^{(k+1)}]^{-1}} = \frac{\det_{i,j}[e^{-Nx_i^{(k)}/x_j^{(k+1)}}]}{\Delta(x^{(k)})\Delta(1/x^{(k+1)})}$$

and that $1/\Delta(1/x^{(k+1)}) = \prod_{i=1}^{\gamma_3 N}[x_i^{(k+1)}]^{\gamma_3 N-1}/\Delta(x^{(k+1)})$ we write the expression in terms of eigenvalues of the matrices X_k. For

$$\varphi_i^{(r)} = \log x_i^{(r)}, \quad r = 3, \dots, n-1,$$

the model integral takes a Toda chain-like form:

$$\prod_{i=1}^{\gamma_3 N}\int_0^\infty dx_i^{(2)}\frac{\Delta(x^{(2)})}{\Delta(|\Lambda|^{-2})}\prod_{i=1}^{\gamma_3 N}\Bigg[\int_{-\infty}^{\infty}\prod_{k=3}^{n-1}d\varphi_i^{(k)}\times$$

$$\times\exp\Bigg[N\sum_{r=1}^{\infty}\frac{t_r}{r}(x_i^{(2)})^r + (\gamma_2-\gamma_3)N\log x_i^{(2)} - (\gamma_2-\gamma_3)N\varphi_i^{(3)}$$

$$-Nx_i^{(2)}e^{-\varphi_i^{(3)}} - Ne^{\varphi_i^{(3)}-\varphi_i^{(4)}} - \dots - Ne^{\varphi_i^{(n-2)}-\varphi_i^{(n-1)}} - Ne^{\varphi_i^{(n-1)}}|\Lambda|_i^{-2}\Bigg]\Bigg].$$

In this form it is clear that all integrals w.r.t. $\varphi_i^{(k)}$ are convergent.

We consider the following variations of the matrix fields X_i:

$$\begin{aligned} \delta X_1 &= \frac{1}{x - X_1}\xi([\widehat{X}_1]), \\ \delta X_i &= X_i \frac{1}{x - X_1}\eta_i([\widehat{X}_i]), \ 2 \le i \le n-2 \\ \delta X_{n-1} &= \frac{1}{x - X_1}\chi([\widehat{X}_{n-1}]), \end{aligned}$$

where ξ, η_i, and χ are Laurent polynomials in all but one of arguments indicated by the symbol $[\widehat{X}_i]$.
We introduce the standard notation for the leading term of the $1/N^2$-expansion of the one-loop mean of the matrix field X_1:

$$\omega_1(x) := \frac{1}{N}\left\langle tr\, \frac{1}{x - X_1}\right\rangle_0 .$$

The exact loop equations obtained upon the above variations read ([BIPZ])

$$\frac{1}{N^2}\left\langle tr\,\frac{1}{x-X_1}\,tr\,\frac{1}{x-X_1}\xi([\widehat{X}_1])\right\rangle^c+\left[\omega_1(x)+V'(x)\right]\left\langle tr\,\frac{1}{x-X_1}\xi([\widehat{X}_1])\right\rangle$$
$$+\left\langle tr\,\frac{V'(X_1)-V'(x)}{x-X_1}\xi([\widehat{X}_1])\right\rangle+\left\langle tr\,X_2^{-1}\frac{1}{x-X_1}\xi([\widehat{X}_1])\right\rangle=0;$$
$$\left\langle tr\,\frac{-1}{x-X_1}\eta([\widehat{X}_2])X_2^{-1}X_1\right\rangle+\left\langle tr\,X_3^{-1}X_2\frac{1}{x-X_1}\eta([\widehat{X}_2])\right\rangle$$
$$+(\gamma_2-\gamma_3)\left\langle tr\,\frac{1}{x-X_1}\eta([\widehat{X}_2])\right\rangle=0;$$
$$\left\langle tr\,\frac{1}{x-X_1}\rho([\widehat{X}_i])X_i^{-1}X_{i-1}\right\rangle=\left\langle tr\,\frac{1}{x-X_1}\rho([\widehat{X}_i])X_{i+1}X_i\right\rangle;$$
$$\left\langle tr\,X_{n-2}\frac{1}{x-X_1}\chi([\widehat{X}_{n-1}])\right\rangle+\left\langle tr\,U'(X_{n-1})\frac{1}{x-X_1}\chi([\widehat{X}_{n-1}])\right\rangle=0,$$

where $U(X_{n-1})$ is the potential obtained from the external field $\Lambda\overline{\Lambda}$ by the replica method.

Finding the spectral curve (n=5)

$$\mathbf{a} := \left\langle tr\, \frac{1}{x-M_1} \frac{U'(M_4)-U'(z)}{M_4-z} \right\rangle_0,$$
$$\mathbf{c} := \left\langle tr\, M_3 \frac{1}{x-M_1} \frac{U'(M_4)-U'(z)}{M_4-z} \right\rangle_0,$$
$$\mathbf{d} := \left\langle tr\, M_2 \frac{1}{x-M_1} \frac{U'(M_4)-U'(z)}{M_4-z} \right\rangle_0.$$

After some algebra (≥ 6 pages in A4) we come to the system of equations that holds for **any** z:

$$\Big[x\big[\omega_1(x)+V'(x)\big]+(\gamma_2-\gamma_3)\Big]\mathbf{a}+z\mathbf{c}$$
$$= -xP_{n,m}(x,z)-\widehat{Q}_m(z)+\left\langle tr\, \frac{1}{x-M_1}[M_3^2+M_3U'(z)]\right\rangle_0$$
$$\Big[x\big[\omega_1(x)+V'(x)\big]+(\gamma_2-\gamma_3)\Big]\mathbf{c}+\mathbf{d} = -x\widehat{P}_{n,m}(x,z)-\widehat{\widehat{Q}}_m(z)$$
$$\mathbf{a}+\big[\omega_1(x)+V'(x)\big]\mathbf{d} = -\widehat{\widehat{P}}_{n,m}(x,z).$$

Finding the spectral curve (n=5)

Degeneracy conditions expresses z: $z = -r^2(x)y(x)$, where

$$r(x) := xy(x) + (\gamma_2 - \gamma_3), \qquad y(x) := \omega_1(x) + V'(x)$$

and the spectral curve is the condition of solvability of inhomogeneous system:

$$-xP_{n,m}(x,z) - \widehat{Q}_m(z) + \left\langle tr\, \frac{1}{x - M_1}[M_3^2 + M_3 U'(z)] \right\rangle_0$$
$$+ r(x)\widehat{\widehat{P}}_{n,m}(x,z) - y(x)r(x)\left[x\widehat{P}_{n,m}(x,z) + \widehat{\widehat{Q}}_m(z)\right] = 0,$$

For the terms $\left\langle tr\, \frac{1}{x-M_1} M_3^k M_2^l \right\rangle_0$ we have a recurrent procedure expressing them through polynomials and rational functions of $y(x)$.

Given an (algebraic) spectral curve $S(x,y) = 0$ and two differentials, dx and $y(x)dx$ on it, we have the machinery of topological recursion (CEO) that produces correlation functions and free energy terms for **all genera**.

Conclusion

I remember the time ((in)famous 1984) the Russian translation of the FAMOUS Quantum Field Theory textbook by Claude Itzykson and Jean-Bernard Zuber has appeared (those days I was a PhD student in quantum field theory at the Steklov Mathematical Institute, Moscow). This textbook immediately became very popular among students and researchers; one of just a handful of cases when a Russian translation of a Foreign textbook got a great acclaim, not vice versa!

Merci de votre patience!