Random Planar Geometry

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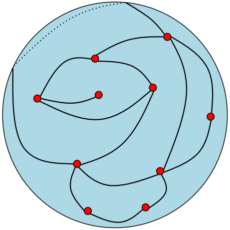

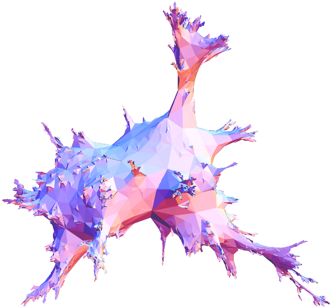

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(motivations from physics: 2D quantum gravity)

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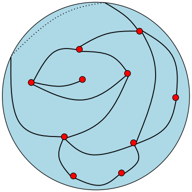

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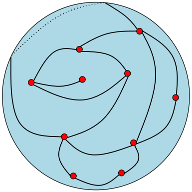
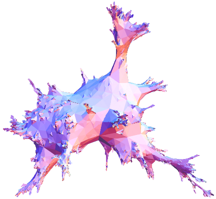


- Choose such a planar map uniformly at random in a suitable class and equip its vertex set with the graph distance.
- Let the size of the graph tend to infinity and pass to the limit after rescaling to get a random metric space: the Brownian map.



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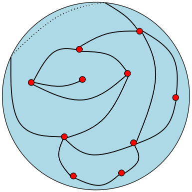

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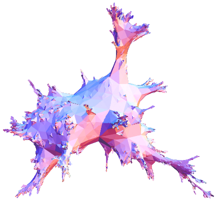

Strong analogy with Brownian motion, which is a canonical model for a random curve in space, obtained as the scaling limit of random walks on the lattice.

1. Convergence to the Brownian map

Definition

A planar map is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).

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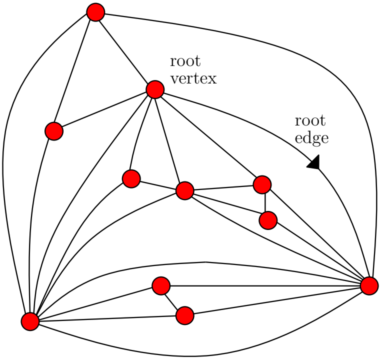


A rooted triangulation with 18 faces

Faces = connected components of the complement of edges

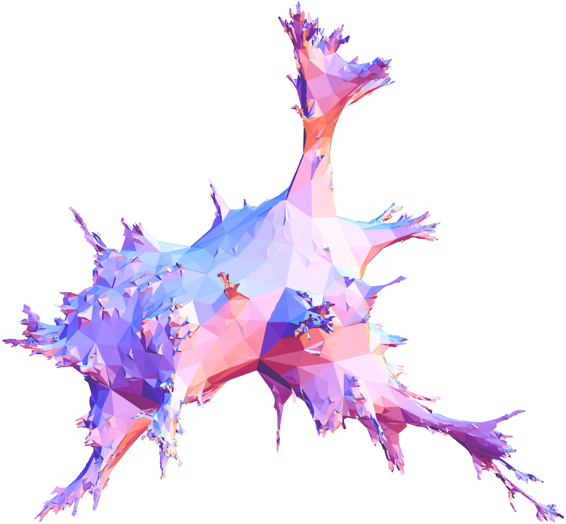
p-angulation:

- each face is bounded by p edges

$p = 3$: triangulation
$p = 4$: quadrangulation

Rooted map: distinguished oriented edge

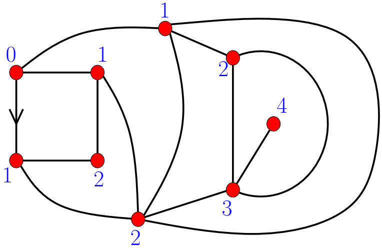
A large triangulation of the sphere (simulation: N. Curien)
Can we get a continuous model out of this ?



Planar maps as metric spaces

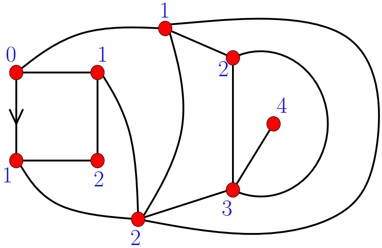
M planar map

- $V(M)$ = set of vertices of M
- d_{gr} graph distance on $V(M)$
- $(V(M), d_{\text{gr}})$ is a (finite) metric space



In blue : distances from root

Planar maps as metric spaces



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$\mathbb{M}_n^p = \{$rooted $p-$ angulations with n faces$\}$

$\mathbb{M}_n^p$ is a finite set (*finite number of possible "shapes"*)

Choose M_n uniformly at random in $\mathbb{M}_n^p$.

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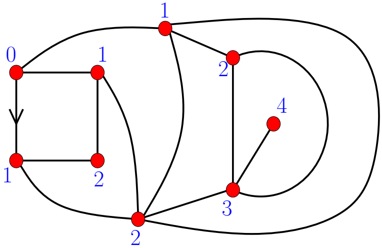

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View $(V(M_n), d_{\text{gr}})$ as a random variable with values in

$$\mathbb{K} = \{\text{compact metric spaces, modulo isometries}\}$$

which is equipped with the Gromov-Hausdorff distance. (A sequence (E_n) of compact metric spaces converges if one can embed all E_n's isometrically in the same big space E so that they converge for the Hausdorff metric on compact subsets of E.)

Main result: The Brownian map

$\mathbb{M}_n^p = \{$rooted $p-$ angulations with n faces$\}$

M_n uniform over $\mathbb{M}_n^p$, $V(M_n)$ vertex set of M_n, $d_{\rm gr}$ graph distance

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Theorem (The scaling limit of *p*-angulations)

Suppose that either $p = 3$ (triangulations) or $p \geq 4$ is even. Set

$$c_3 = 6^{1/4} \quad , \quad c_p = \left(\frac{9}{p(p-2)}\right)^{1/4} \quad \text{if } p \text{ is even.}$$

Then,

$$(V(M_n), c_p\, n^{-1/4}\, d_{\mathrm{gr}}) \xrightarrow[n\to\infty]{\mathrm{(d)}} (\mathbf{m}_\infty, D^*)$$

in the Gromov-Hausdorff sense. The limit $(\mathbf{m}_\infty, D^*)$ is a random compact metric space that does not depend on p (universality) and is called the Brownian map (after Marckert-Mokkadem).

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Remarks. The case $p = 4$ was obtained independently by Miermont. Extensions to other classes of random planar maps: Abraham, Addario-Berry-Albenque, Beltran-LG, Bettinelli-Jacob-Miermont, etc.

Two properties of the Brownian map

Theorem (Hausdorff dimension)

$$\dim(\mathbf{m}_\infty, D^*) = 4 \qquad \textit{a.s.}$$

(Already known in the physics literature.)

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Theorem (topological type, LG-Paulin)

Almost surely, $(\mathbf{m}_\infty, D^*)$ is homeomorphic to the 2-sphere $\mathbb{S}^2$.

Why study planar maps and their continuous limits ?

- combinatorics Many papers since Tutte's work in the 60-70s (recently Bousquet-Mélou, Bouttier-di Francesco-Guitter, Fusy, Noy, Schaeffer, etc.)
- theoretical physics
  - enumeration of maps related to matrix integrals ['t Hooft 74, Brézin, Itzykson, Parisi, Zuber 78, etc.]
    more recent work: Eynard, etc.
  - large random planar maps as models of random geometry
    2D-quantum gravity, cf Ambjørn, Durhuus, Jonsson 95,
    recent papers of Bouttier-Guitter, Ambjørn, Budd, etc.
    work of Duplantier-Sheffield (Gaussian free field approach),
    also David-Kupiainen-Rhodes-Vargas, etc.
    higher dimensional extensions: Rivasseau, Gurau, etc.
- probability theory: model for a Brownian surface
  - analogy with Brownian motion as continuous limit of discrete paths
  - universality of the limit
  - asymptotic properties of "typical" large planar graphs
  - connections with recent work of Duplantier-Miller-Sheffield (QG as a mating of trees, Quantum Loewner Evolution, etc.)

Construction of the Brownian map

The Brownian map $(\mathbf{m}_\infty, D^*)$ is constructed by identifying certain pairs of points in Aldous' Brownian continuum random tree (CRT).

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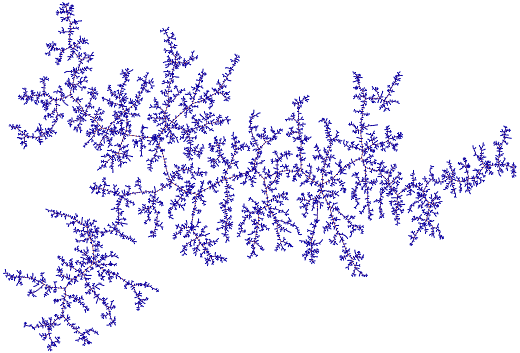
Constructions of the CRT (Aldous, 1991-1993):

- As the scaling limit of many classes of discrete trees
- As the random real tree $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ coded by a Brownian excursion.



A simulation of the CRT

If one explores the tree in clockwise order from a vertex ρ chosen as random, the distance from ρ evolves like a Brownian excursion.

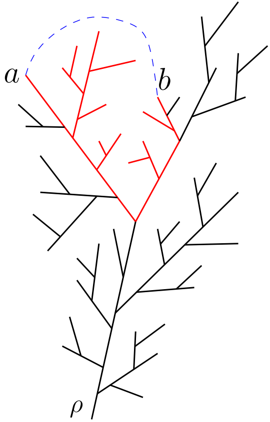
Constructing the Brownian map

First step. Equip the CRT $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ with Brownian labels $(Z_a)_{a\in\mathcal{T}_{\mathbf{e}}}$:
conditionally on $\mathcal{T}_{\mathbf{e}}$, $(Z_a)_{a\in\mathcal{T}_{\mathbf{e}}}$ is the centered Gaussian process such that

- $Z_\rho = 0$ (where ρ is the root)
- $E[(Z_a - Z_b)^2] = d_{\mathbf{e}}(a,b)$, $a, b \in \mathcal{T}_{\mathbf{e}}$

Second step. Identify two vertices $a, b \in \mathcal{T}_{\mathbf{e}}$ if:

- they have the same label $Z_a = Z_b$,
- one can go from a to b around the tree (in the clockwise or counterclockwise cyclic exploration) visiting only vertices with label greater than or equal to $Z_a = Z_b$.



for any red vertex c,
$Z_c \geq Z_a = Z_b$

The Brownian map $\mathbf{m}_\infty$ is the quotient space resulting from these identifications (also need to define the distance D^* on $\mathbf{m}_\infty$).

2. The UIPT and the Brownian plane

Let Δ_n be unif. distributed over $\{$rooted triangulations with n faces$\}$.

One can prove (Angel-Schramm 2003, Stephenson 2014) that

$$\Delta_n \xrightarrow[n\to\infty]{\text{(d)}} \Delta_\infty$$

where Δ_∞ is a (rooted) infinite random triangulation called the UIPT for Uniform Infinite Planar Triangulation.

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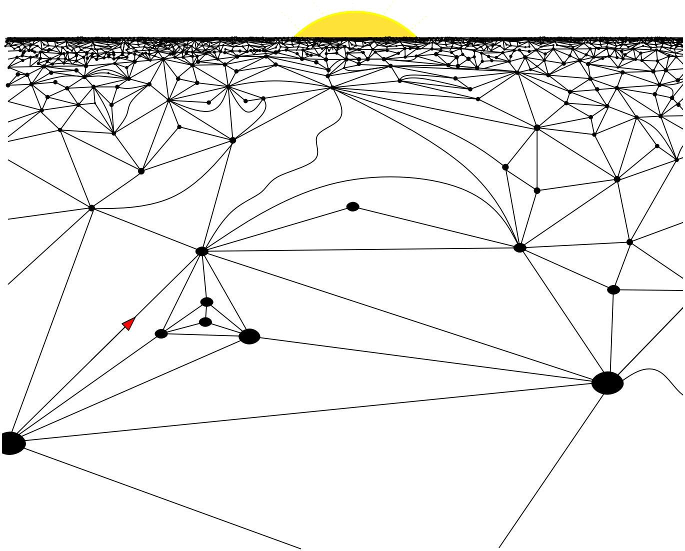
The convergence holds in the sense of local limits: if $B_r(\Delta_n)$ denotes the ball of radius r in Δ_n, defined as the union of all triangles having a vertex at distance $< r$ from the root vertex ρ, then for every fixed planar map M,

$$\mathbb{P}(B_r(\Delta_n) = M) \xrightarrow[n\to\infty]{} \mathbb{P}(B_r(\Delta_\infty) = M).$$

This is very different from the Gromov-Hausdorff convergence: Here we do no rescaling and thus the limit is a non-compact (infinite) random lattice.

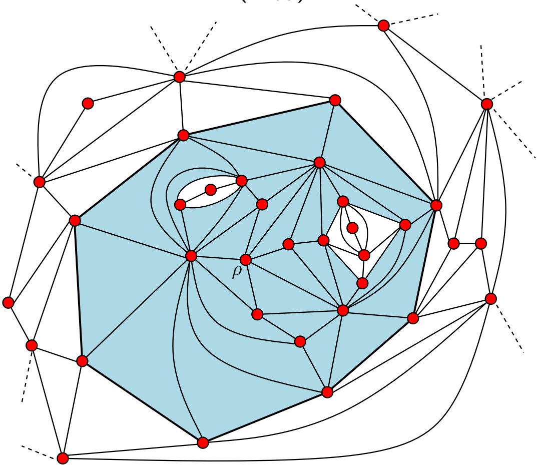


An artistic representation of the UIPT (artist: N. Curien)

Recurrence of random walk on UIPT: Gurel-Gurevich and Nachmias.

Balls and hulls in the UIPT

The hull of radius r, denoted by $B_r^\bullet(\Delta_\infty)$, is obtained by filling in the "holes" in the ball $B_r(\Delta_\infty)$.



The shaded part is the ball $B_2(\Delta_\infty)$ (all triangles that contain a vertex at distance ≤ 1 from ρ)

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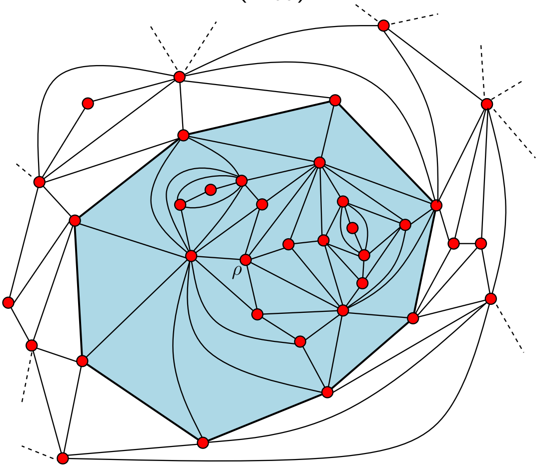

The hull $B_2^\bullet(\Delta_\infty)$ is the union of $B_2(\Delta_\infty)$ and the two holes.

Asymptotics for volumes and perimeters of hulls

P_r perimeter of the hull $B_r^\bullet(\Delta_\infty)$ (number of edges in boundary)
V_r volume of the hull $B_r^\bullet(\Delta_\infty)$ (number of triangles)

Theorem (Scaling limit of the hull process, Curien-LG)

We have the following convergence in distribution

$$\Big(n^{-2}P_{[nt]}, n^{-4}V_{[nt]}\Big)_{t\geq 0} \xrightarrow[r\to\infty]{(d)} \Big(X_t, Y_t\Big)_{t\geq 0}.$$

where

- $(X_t)_{t\geq 0}$ *is a time-reversed continuous-state branching process with branching mechanism* $\psi(u) = c\, u^{3/2}$,
- $Y_t = \sum_{s\leq t} \xi_s\,(\Delta X_s)^2$, *where the random variables* ξ_s *are i.i.d. with density*

$$\frac{1}{\sqrt{2\pi}} x^{-5/2}\, e^{-1/2x}\, \mathbf{1}_{\{x>0\}}.$$

Asymptotic formulas for laws of perimeters and volumes of hulls

We have

$$\lim_{n\to\infty} \mathbb{E}\Big[e^{-\lambda n^{-2} P_{[nt]}}\Big] = (1 + c\lambda t^2)^{-3/2}$$

$$\lim_{n\to\infty} \mathbb{E}\Big[e^{-\lambda n^{-4} V_{[nt]}}\Big] = 3^{3/2} \cosh(c'\lambda^{1/4} t)\Big(\cosh^2(c'\lambda^{1/4} t) + 2\Big)^{-3/2}.$$

Also explicit formula for the (asymptotic) conditional distribution of the volume knowing the perimeter.

The limiting distributions are universal (only the constants c, c' depend on the random lattice that is considered).

These distributions can be interpreted in terms of the continuous object called the Brownian plane (infinite volume version of the Brownian map)

Convergence to the Brownian plane

Relations between quadrangulations and the Brownian plane $\mathcal{P}$.

Uniform Quadrangulations

Brownian Map

$$\begin{tikzcd}[column sep=huge, row sep=huge]
(Q_n, \mathrm{d_{gr}}) \arrow[r, "\text{Scaling } \cdot n^{-1/4}"] \arrow[d, "\text{Local}"'] \arrow[dr, dotted, "\cdot k_n^{-1} >> n^{-1/4}"] & (\mathbf{m}_\infty, D^*) \arrow[d, "\text{Local}"] \\
(Q_\infty, \mathrm{d_{gr}}) \arrow[r, "\text{Scaling } \cdot \lambda \to 0"'] & (\mathcal{P}, D_\infty)
\end{tikzcd}$$

UIPQ

Brownian Plane

Should also hold for triangulations instead of quadrangulations (the arrow at the bottom is still missing!).

The Brownian plane is scale invariant: $(\mathcal{P}, \lambda D_\infty) \overset{(d)}{=} (\mathcal{P}, D_\infty)$.

3. First passage percolation on random planar maps

(work in progress with Nicolas Curien)

Idea: Assign i.i.d. random weights (lengths) w_e to the edges of a (random) planar map M.

Define the weight $w(\gamma)$ of a path γ as the sum of the weights of the edges it contains.

The first passage percolation distance $d_{\rm FPP}$ is defined on the vertex set $V(M)$ by

$$d_{\rm FPP}(v,v') = \inf\{w(\gamma) : \gamma \text{ path from } v \text{ to } v'\}.$$

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Goal: In large scales, $d_{\rm FPP}$ behaves like the graph distance $d_{\rm gr}$ (asymptotically, balls for $d_{\rm FPP}$ are close to balls for $d_{\rm gr}$).

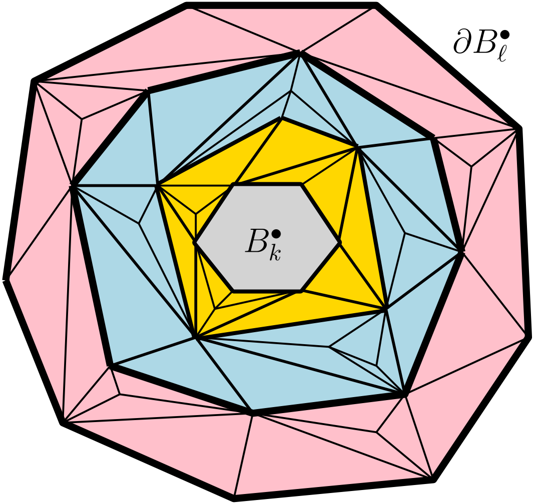
This is not expected to be true in deterministic lattices such as $\mathbb{Z}^d$, but random planar maps are in a sense more isotropic.

Consequence: The scaling limit of the metric space associated with $d_{\rm FPP}$ will again be the Brownian map!

Method: Discuss first the UIPT.

Layers in the UIPT

In view of studying the first-passage percolation distance on the UIPT, one needs more information about its geometry. Set $B_r^\bullet = B_r^\bullet(\Delta_\infty)$.



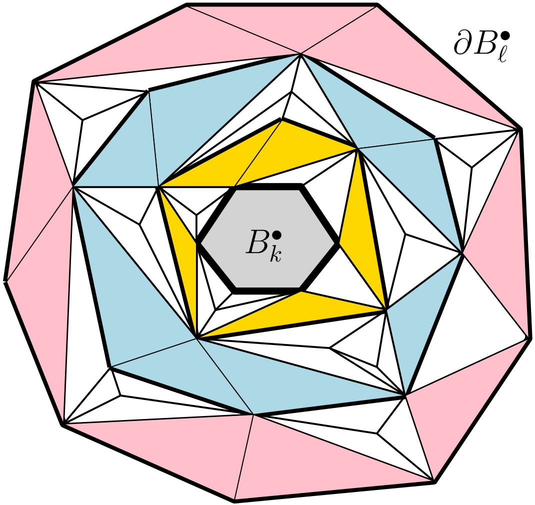
For $k < \ell$, the successive layers between $B_k^\bullet$ and $B_\ell^\bullet$ are the sets

$$B_j^\bullet \backslash B_{j-1}^\bullet$$

for $k < j \leq \ell$.

(Here 3 layers)

Downward triangles in the layers

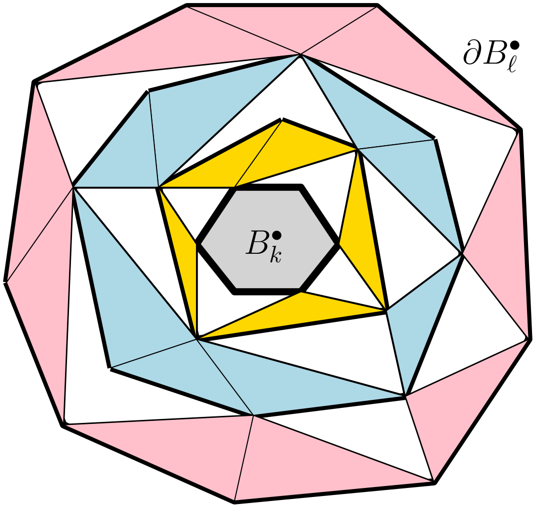


$k < \ell$ fixed

For each layer $B^\bullet_j \backslash B^\bullet_{j-1}$ with $k < j \leq \ell$ the downward triangles are all triangles contained in the layer $B^\bullet_j \backslash B^\bullet_{j-1}$ that have an edge in $\partial B^\bullet_j$ (their third vertex is on $\partial B^\bullet_{j-1}$).

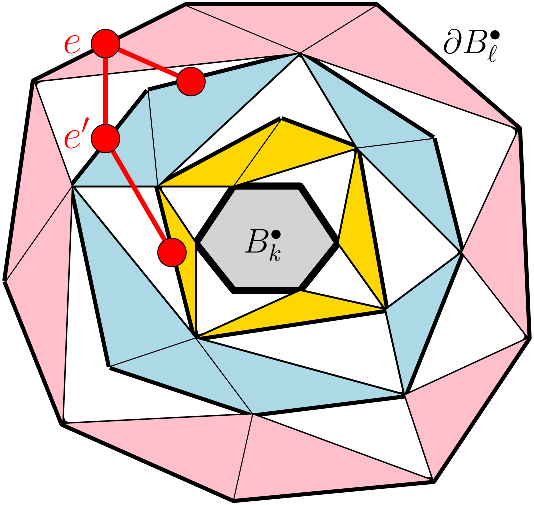
(Note that we do not get all triangles in the layer $B^\bullet_j \backslash B^\bullet_{j-1}$, only those that have an edge in the exterior boundary of the layer)

Downward triangles in the layers



Remove the edges not on the downward triangles.
This creates “white” holes.

The forest coding downward triangles

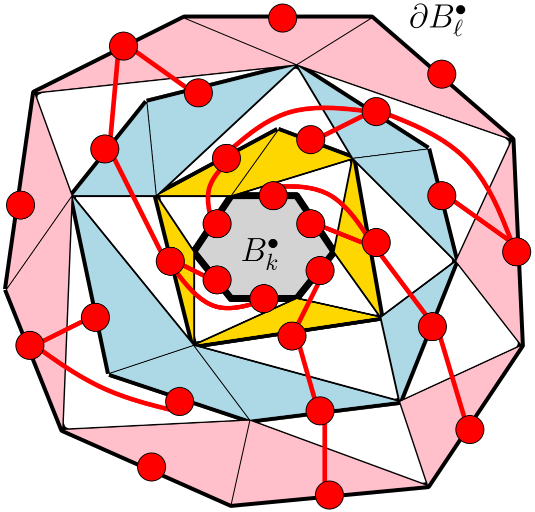


Can represent the configuration by a forest of trees whose vertices are the edges of $\partial B^\bullet_j$ for all $k \le j \le \ell$.

An edge e of $\partial B^\bullet_j$ is the parent of an edge e' of $\partial B^\bullet_{j-1}$ if the white hole whose boundary contains e' is bounded on its right by the downward triangle associated with e.

Trees grow from the boundary $\partial B^\bullet_\ell$ of the "big" hull to the boundary $\partial B^\bullet_k$ of the small hull.

The forest coding downward triangles



The forest representing the structure of layers between $B^\bullet_k$ and $B^\bullet_\ell$. The roots of trees in the forest are all edges of $\partial B^\bullet_\ell$.

To reconstruct $B^\bullet_\ell \setminus B^\bullet_k$ one only needs

- the forest coding the layers,
- the triangulations (with boundaries) filling in the holes.

The Galton-Watson structure

Let $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_{P_\ell}$ be the forest coding the configuration of downward triangles between $\partial B_k^\bullet$ and $\partial B_\ell^\bullet$. Here $k < \ell$, and P_ℓ is the size of $\partial B_\ell^\bullet$.

$\tau_1, \ldots, \tau_p$ deterministic forest with height $\ell - k$ and q vertices at height $\ell - k$. Write $V_*(\tau_i)$ for all vertices of τ_i except those at height $\ell - k$.

Proposition (related to Krikun (2005))

$$P\Big((\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_{P_\ell}) = (\tau_1, \ldots, \tau_p) \,\Big|\, P_k = q\Big) = \frac{h(p)}{h(q)} \prod_{v \in V_*(\tau_1) \cup \cdots \cup V_*(\tau_p)} \theta(c_v)$$

where

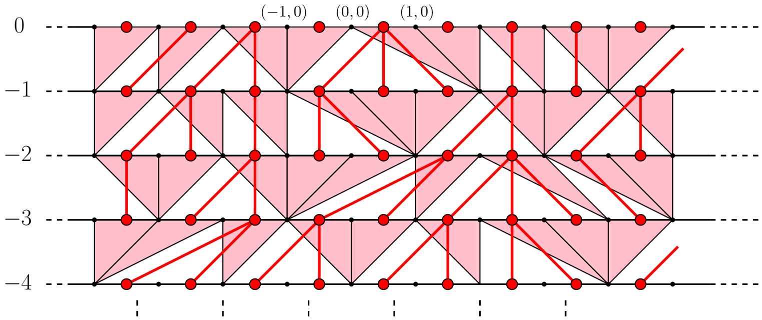
- c_v *is the number of children of* v*;*
- $(\theta(n))_{n \geq 0}$ *determined by:* $\sum \theta(n)\, x^n = 1 - \Big(1 + \frac{1}{\sqrt{1-x}}\Big)^{-2}$;
- $h(p) = 4^{-p} \frac{(2p)!}{(p!)^2}$ *(Stationary distribution for the* θ*-GW process).*

Consequence: The trees $\mathcal{T}_1, \ldots, \mathcal{T}_{P_\ell}$ are "almost" independent Galton-Watson trees with offspring distribution θ (genealogical trees for a population where each individual has n children with probab. $\theta(n)$).

The half-plane model

Construct a triangulation $\mathcal{H}$ of the lower half-plane as follows.

- Each horizontal edge on the line $\mathbb{Z} \times \{-k\}$ belongs to a downward triangle whose third vertex is on the line $\mathbb{Z} \times \{-k-1\}$.



- The trees characterizing the configuration of downward triangles are independent Galton-Watson trees with offspring distribution θ.
- Holes are filled in with "free triangulations" with a boundary (probab. of a given triangul. with n inner vertices is $C\,(12\sqrt{3})^{-n}$).

First-passage percolation in the half-plane model

Assign i.i.d. weights w_e to the edges of $\mathcal{H}$, with common distribution ν such that $0 < c \leq w_e \leq C < \infty$. Consider the associated first-passage percolation distance d_{FPP}.

Proposition

Let $\rho = (0,0)$ be the root and for every $k \geq 0$, let L_k be the horizontal line at vertical coordinate $-k$. Then

$$\frac{1}{k}\, d_{\mathrm{FPP}}(\rho, L_k) \xrightarrow[k\to\infty]{\text{a.s.}} c_0 \in [c, C].$$

First-passage percolation in the half-plane model

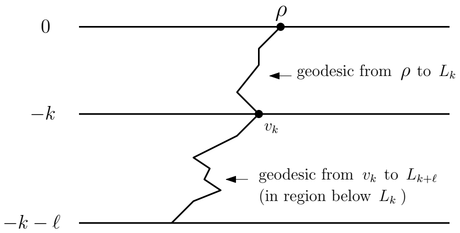
Assign i.i.d. weights w_e to the edges of $\mathcal{H}$, with common distribution ν such that $0 < c \leq w_e \leq C < \infty$. Consider the associated first-passage percolation distance $d_{\rm FPP}$.

Proposition

Let $\rho = (0,0)$ be the root and for every $k \geq 0$, let L_k be the horizontal line at vertical coordinate $-k$. Then

$$\frac{1}{k}\, d_{\rm FPP}(\rho, L_k) \xrightarrow[k\to\infty]{\text{a.s.}} c_0 \in [c, C].$$

Proof: Kingman's subadditive ergodic theorem.



$$d_{\rm FPP}(\rho, L_{k+\ell}) \leq d_{\rm FPP}(\rho, L_k) + Z_{k,\ell}$$

where

$$Z_{k,\ell} \stackrel{(d)}{=} d_{\rm FPP}(\rho, L_\ell)$$

and $Z_{k,\ell}$ is independent of $d_{\rm FPP}(\rho, L_k)$.

First-passage percolation in the UIPT

Assign i.i.d. weights w_e with common distribution ν to the edges of the UIPT Δ_∞ and consider the associated first-passage percolation distance d_{FPP}.
For every real $r \geq 0$, let $B_r^{\mathrm{FPP}}(\Delta_\infty)$ be the ball of radius r for d_{FPP}.
Let c_0 be as in the half-plane model.

Theorem

For every $\varepsilon > 0$, we have

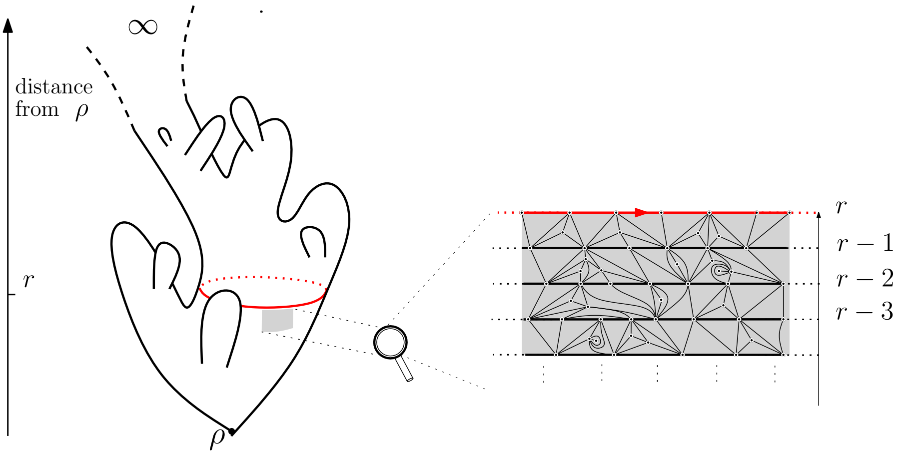
$$B_{(1-\varepsilon)r/c_0}(\Delta_\infty) \subset B_r^{\mathrm{FPP}}(\Delta_\infty) \subset B_{(1+\varepsilon)r/c_0}(\Delta_\infty)$$

with probability tending to 1 as $r \to \infty$.

The ball of radius r for the FPP distance is asymptotically close to the ball of radius r/c_0 for the graph distance.

Idea of the proof

Locally (below the boundary of the hull of radius r), the UIPT looks like the half-plane model.



Can use the result in the half-plane model to estimate the FPP distance between a typical point of $\partial B^\bullet_r(\Delta_\infty)$ and $\partial B^\bullet_{(1-\varepsilon)r}(\Delta_\infty)$.

First-passage percolation in finite triangulations

Δ_n is uniformly distributed over $\{$triangulations with n faces$\}$
d_{FPP} first-passage percolation distance on $V(\Delta_n)$ defined using weights i.i.d. according to ν.

Theorem

$$(V(\Delta_n), 6^{1/4}\, n^{-1/4}\, d_{\mathrm{FPP}}) \xrightarrow[n\to\infty]{\mathrm{(d)}} (\mathbf{m}_\infty, c_0\, D^*)$$

in the Gromov-Hausdorff sense. Here $(\mathbf{m}_\infty, D^*)$ is the Brownian map.

Idea of the proof: Use absolute continuity arguments to relate large (finite) triangulations to the UIPT, and then apply the theorem about the UIPT.

Remark. In general one cannot calculate the constant c_0, except in special cases (e.g. Eden model, corresponding to exponential edge weights on the dual graph of the UIPT). See however Budd (2015).