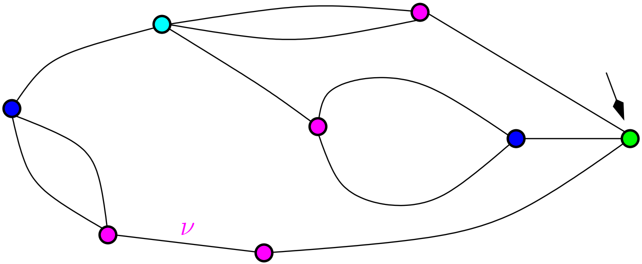
The Potts model on planar maps

Olivier Bernardi, Brandeis University, Boston
Mireille Bousquet-Mélou, CNRS, LaBRI, Bordeaux



http://www.labri.fr/∼bousquet

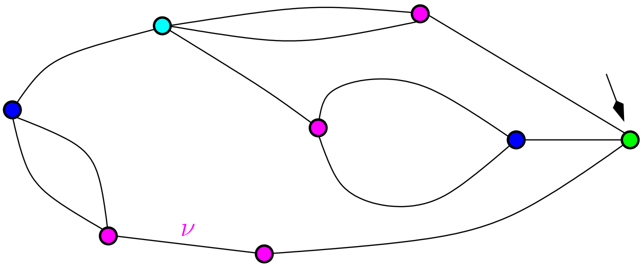
Outline

I. Planar maps and the Potts model

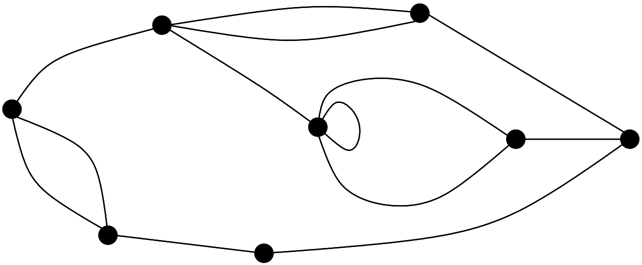
II. Main result

III. Where does it come from?

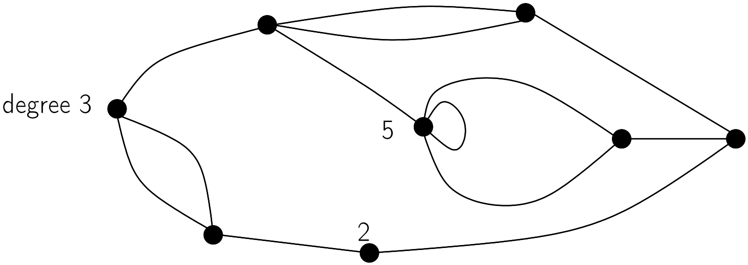
IV. Some special cases



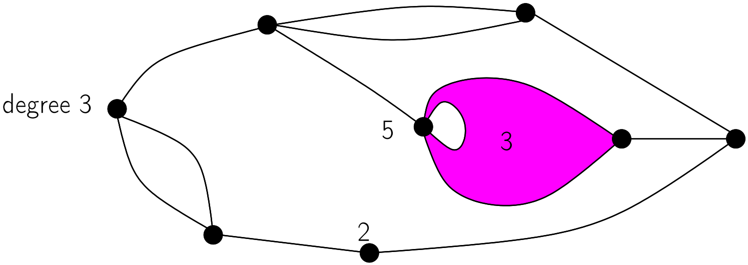
Planar maps



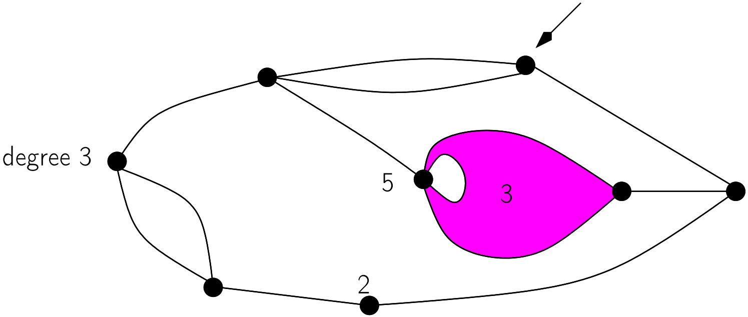
Planar maps



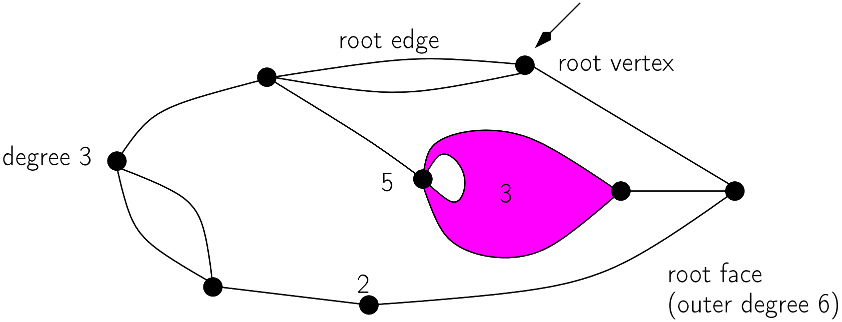
Planar maps



Planar maps: rooted version

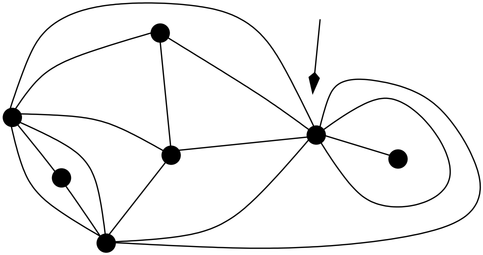


Planar maps: rooted version

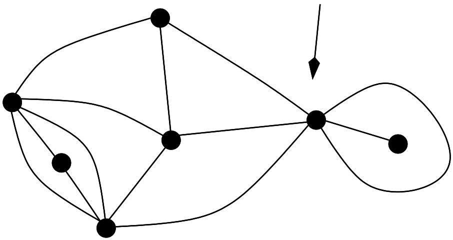


There is a finite number of maps with n edges

With degree constraints: rooted triangulations



With degree constraints: rooted triangulations



A near-triangulation with outer-degree 5

The Potts model on planar maps

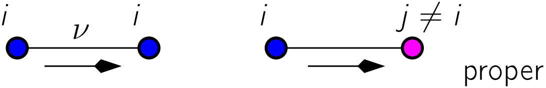
- The partition function of the q-state Potts model on a planar map M:

$$\mathrm{Z}_M(q,\nu) = \sum_{c: V(M)\to\{1,2,\ldots,q\}} \nu^{m(c)}$$

where $m(c)$ of the number of monochromatic edges in the colouring c. In fact, $\mathrm{Z}_M(q,\nu)$ is a polynomial in q (and ν), divisible by q.

Example: When M has one edge and two vertices,

$$\mathrm{Z}_M(q,\nu) = q\nu + q(q-1)$$



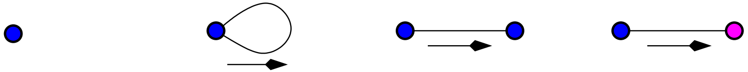
The Potts model on planar maps

- Generating function:

$$
\begin{aligned}
M_1(q,\nu,w,t) &= \frac{1}{q}\sum_{M} \mathsf{Z}_M(q,\nu) w^{\mathsf{v}(M)} t^{\mathsf{e}(M)} = \frac{1}{q}\sum_{M,\,c} w^{\mathsf{v}(M)} t^{\mathsf{e}(M)} \nu^{m(c)} \\
&= w + (\nu w + \nu w^2 + (q-1)w^2)t + O(t^2)
\end{aligned}
$$

"The Potts generating function of planar maps"

$\Rightarrow$ Enumeration of q-coloured planar maps, counted by vertices, edges, and monochromatic edges.



w $\qquad$ νw $\qquad$ νw^2 $\qquad$ $(q-1)w^2$

A possible answer [Tutte 68]

- Consider the refined Potts generating function:

$$M(x,y) \equiv M(q,\nu,w,t;x,y) = \frac{1}{q}\sum_{M} \mathrm{Z}_M(q,\nu) w^{\mathrm{v}(M)} t^{\mathrm{e}(M)} x^{\mathrm{dv}(M)} y^{\mathrm{df}(M)},$$

where $\mathrm{dv}(M)$ (resp. $\mathrm{df}(M)$) is the degree of the root-vertex (resp. root-face).

A possible answer [Tutte 68]

- Consider the refined Potts generating function:

$$M(x,y) \equiv M(q,\nu,w,t;x,y) = \frac{1}{q}\sum_M \mathrm{Z}_M(q,\nu) w^{\mathrm{v}(M)} t^{\mathrm{e}(M)} x^{\mathrm{dv}(M)} y^{\mathrm{df}(M)},$$

where $\mathrm{dv}(M)$ (resp. $\mathrm{df}(M)$) is the degree of the root-vertex (resp. root-face).

- By deletion/contraction of the root-edge:

$$\begin{aligned} M(x,y) \;=\; & w + xyt\left((\nu-1)(y-1)+qy\right) M(x,y)M(1,y) \\ & + xyt/w(x\nu-1)M(x,y)M(x,1) \\ & + xywt(\nu-1)\frac{xM(x,y)-M(1,y)}{x-1} + xyt\frac{yM(x,y)-M(x,1)}{y-1}. \end{aligned}$$

A discrete partial differential equation with two catalytic variables

Are we happy?

The refined Potts generating function satisfies:

$$\begin{aligned} M(x,y) &= w + xyt\left((\nu-1)(y-1)+qy\right) M(x,y) M(1,y) \\ &\quad + xyt/w(x\nu-1) M(x,y) M(x,1) \\ &\quad + xywt(\nu-1)\frac{xM(x,y)-M(1,y)}{x-1} + xyt\frac{yM(x,y)-M(x,1)}{y-1}. \end{aligned}$$

- compute coefficients efficiently (polynomial time)
- What is $M(1,1) \equiv M(q,\nu,w,t;1,1)$?
- asymptotics? phase transitions?

Are we happy?

The refined Potts generating function satisfies:

$$\begin{aligned} M(x,y) \;=\;& w + xyt\left((\nu-1)(y-1)+qy\right) M(x,y)M(1,y) \\ &+ xyt/w(x\nu-1)M(x,y)M(x,1) \\ &+ xywt(\nu-1)\frac{xM(x,y)-M(1,y)}{x-1} + xyt\frac{yM(x,y)-M(x,1)}{y-1}. \end{aligned}$$

- compute coefficients efficiently (polynomial time)
- What is $M(1,1) \equiv M(q,\nu,w,t;1,1)$?
- asymptotics? phase transitions?

Are we happy?

The refined Potts generating function satisfies:

$$
\begin{aligned}
M(x,y) \;=\; & w + xyt\left((\nu-1)(y-1)+qy\right) M(x,y)M(1,y) \\
& + xyt/w(x\nu-1)M(x,y)M(x,1) \\
& + xywt(\nu-1)\frac{xM(x,y)-M(1,y)}{x-1} + xyt\frac{yM(x,y)-M(x,1)}{y-1}.
\end{aligned}
$$

- compute coefficients efficiently (polynomial time)
- What is $M(1,1) \equiv M(q,\nu,w,t;1,1)$?
- asymptotics? phase transitions?

Experience makes us greedier: uncoloured maps ("pure gravity")

Experience makes us greedier: uncoloured maps ("pure gravity")

- Let

$$G(t;y) \equiv G(y) = \sum_M t^{\mathrm{e}(M)} y^{\mathrm{df}(M)}$$

where $\mathrm{e}(M)$ is the number of edges and $\mathrm{df}(M)$ the degree of the outer face. Then by deletion of the root-edge [Tutte 68]:

$$G(y) = 1 + ty^2 G(y)^2 + ty\,\frac{yG(y) - G(1)}{y-1}$$

A discrete differential equation with one catalytic variable, y.

Experience makes us greedier: uncoloured maps ("pure gravity")

- Let

$$G(t;y) \equiv G(y) = \sum_M t^{\mathrm{e}(M)} y^{\mathrm{df}(M)}$$

where $\mathrm{e}(M)$ is the number of edges and $\mathrm{df}(M)$ the degree of the outer face. Then by deletion of the root-edge [Tutte 68]:

$$G(y) = 1 + ty^2 G(y)^2 + ty\, \frac{yG(y) - G(1)}{y-1}$$

A discrete differential equation with one catalytic variable, y.

- The solution is an algebraic function!

$$G(1) = \sum_M t^{\mathrm{e}(M)} = \frac{(1-12t)^{3/2} - 1 + 18\,t}{54t^2}$$

Equivalently,

$$27\,t^2 G(1)^2 + (1 - 18\,t)\,G(1) + 16\,t - 1 = 0$$

Are we happy?

The refined Potts generating function satisfies:

$$\begin{aligned} M(x,y) \;=\;& w + xyt\left((\nu-1)(y-1)+qy\right) M(x,y)M(1,y) \\ & + xyt/w(x\nu-1)M(x,y)M(x,1) \\ & + xywt(\nu-1)\frac{xM(x,y)-M(1,y)}{x-1} + xyt\frac{yM(x,y)-M(x,1)}{y-1}. \end{aligned}$$

- compute coefficients efficiently (poly. time)
- What is $M(1,1) \equiv M(q,\nu,w,t;1,1)$?
- asymptotics? phase transitions?
- **Is $M(1,1)$ algebraic?**

An encouraging sign: the Ising model ($q = 2$)

Theorem

Let $v = (1+\nu)/(1-\nu)$. The Ising generating function of near-triangulations with outer degree 1 is

$$T_1(2,\nu,t) = \frac{(S-v)^2\,(S-2+v)\,\left(-2\,v+v^2-Sv-S^2v+3\,S^3\right)}{128t^4\,(1+v)^4\,S^2},$$

where S is the unique series in t with constant term v satisfying:

$$S = v + t^3\,\frac{64\,(1+v)^3\,S^2}{(S-2+v)\,(2\,v-v^2+2\,S+S^2-4\,S^3)}.$$

In particular, it is algebraic.

$\Rightarrow$ Asymptotics, exponents, transition at $\nu = 1 + 1/\sqrt{7}$
[Boulatov & Kazakov 87], [mbm & Schaeffer 02], [Bouttier, Di Francesco & Guitter 04]

Another special case: maps equipped with a spanning tree (Potts in the limit $q \to 0, \nu \to 1$) [Mullin 67]

Theorem

The GF of planar maps equipped with a spanning tree is

$$M_1(0,1,1,t) = \sum_{n \geq 0} \frac{1}{(n+1)(n+2)} \binom{2n}{n} \binom{2n+2}{n+1} t^n.$$

This series is transcendental (= non-algebraic), but D-finite (solution of a linear DE with polynomial coefficients).

$\Rightarrow$ Forget about algebraicity in general.

OK, but... is Potts D-finite?

A hierarchy of formal power series

- Rational series

$$A(t) = \frac{P(t)}{Q(t)}$$

- Algebraic series

$$P(t, A(t)) = 0$$

- Differentially finite series (D-finite)

$$\sum_{i=0}^{d} P_i(t) A^{(i)}(t) = 0$$

- D-algebraic series

$$P(t, A(t), A'(t), \dots, A^{(d)}(t)) = 0$$

Another special case: maps equipped with a connected subgraph (Potts in the limit $q \to 0$)

Theorem [mbm & Courtiel 13(a)]

The generating function of triangulations equipped with a connected subgraph, counted by edges and by the size of the subgraph, is not D-finite.

But... it is D-algebraic (2nd order non-linear DE).

$\Rightarrow$ Forget about D-finiteness in general

OK, but... is Potts D-algebraic?

II. Main result

The Potts generating function is D-algebraic

Theorem

The Potts generating function of planar maps:

$$M_1(q,\nu,w,t) = \frac{1}{q}\sum_M Z_M(q,\nu) w^{\mathsf{v}(M)} t^{\mathsf{e}(M)},$$

is

- algebraic if $q = 2 + 2\cos\frac{j\pi}{m}$, $q \neq 0, 4$ (includes $q = 2, 3$)
- D-algebraic (over $\mathbb{Q}(q,\nu,w,t)$) when q is an indeterminate

The same holds for triangulations.

[mbm-Bernardi 09(a)] Counting coloured planar maps: algebraicity results.
[mbm-Bernardi 15(a)] Counting coloured planar maps: differential equations

cf. [Eynard & Bonnet 99]: algebraicity w.r.t. the catalytic variable y (for near-triangulations)

An explicit differential system for Potts on planar maps

Let $D(t,u) = (q\nu + \beta^2)u^2 - q(\nu+1)u + \beta t(q-4)(wq+\beta) + q$, with $\beta = \nu - 1$.

An explicit differential system for Potts on planar maps

Let $D(t,u) = (q\nu + \beta^2)u^2 - q(\nu+1)u + \beta t(q-4)(wq+\beta) + q$, with $\beta = \nu - 1$.

- There exists a unique 11-tuple $(P_0(t), \dots, P_4(t), Q_0(t), \dots, Q_2(t), R_0(t), \dots, R_2(t))$ of series in t with coefficients in $\mathbb{Q}(q, \nu, w)$ such that

An explicit differential system for Potts on planar maps

Let $D(t, u) = (q\nu + \beta^2)u^2 - q(\nu + 1)u + \beta t(q - 4)(wq + \beta) + q$, with $\beta = \nu - 1$.

- There exists a unique 11-tuple $(P_0(t), \ldots, P_4(t), Q_0(t), \ldots, Q_2(t), R_0(t), \ldots, R_2(t))$ of series in t with coefficients in $\mathbb{Q}(q, \nu, w)$ such that

$$P_4(t) = Q_2(t) = 1, \qquad R_2(t) = \nu + 1 - w(q + 2\beta),$$

and

An explicit differential system for Potts on planar maps

Let $D(t,u) = (q\nu + \beta^2)u^2 - q(\nu+1)u + \beta t(q-4)(wq+\beta) + q$,
with $\beta = \nu - 1$.

- There exists a unique 11-tuple $(P_0(t), \ldots, P_4(t), Q_0(t), \ldots, Q_2(t),$ $R_0(t), \ldots, R_2(t))$ of series in t with coefficients in $\mathbb{Q}(q,\nu,w)$ such that

$$P_4(t) = Q_2(t) = 1, \qquad R_2(t) = \nu + 1 - w(q + 2\beta),$$

and

$$\frac{1}{Q}\,\frac{\partial}{\partial t}\left(\frac{Q^2}{PD^2}\right) = \frac{1}{R}\,\frac{\partial}{\partial u}\left(\frac{R^2}{PD^2}\right),$$

with $P \equiv P(t,u) = P_0(t) + P_1(t)u + \cdots + P_4(t)u^4$ and so on,

An explicit differential system for Potts on planar maps

Let $D(t,u) = (q\nu + \beta^2)u^2 - q(\nu+1)u + \beta t(q-4)(wq+\beta) + q$, with $\beta = \nu - 1$.

- There exists a unique 11-tuple $(P_0(t), \ldots, P_4(t), Q_0(t), \ldots, Q_2(t), R_0(t), \ldots, R_2(t))$ of series in t with coefficients in $\mathbb{Q}(q, \nu, w)$ such that

$$P_4(t) = Q_2(t) = 1, \qquad R_2(t) = \nu + 1 - w(q + 2\beta),$$

and

$$\frac{1}{Q}\,\frac{\partial}{\partial t}\left(\frac{Q^2}{PD^2}\right) = \frac{1}{R}\,\frac{\partial}{\partial u}\left(\frac{R^2}{PD^2}\right),$$

with $P \equiv P(t,u) = P_0(t) + P_1(t)u + \cdots + P_4(t)u^4$ and so on, and the initial conditions (at $t = 0$):

$$P(0,u) = u^2(u-1)^2 \quad \text{and} \quad Q(0,u) = u(u-1).$$

An explicit differential system for Potts on planar maps

Let $D(t,u) = (q\nu + \beta^2)u^2 - q(\nu+1)u + \beta t(q-4)(wq+\beta) + q$, with $\beta = \nu - 1$.

- There exists a unique 11-tuple $(P_0(t), \dots, P_4(t), Q_0(t), \dots, Q_2(t), R_0(t), \dots, R_2(t))$ of series in t with coefficients in $\mathbb{Q}(q,\nu,w)$ such that

$$P_4(t) = Q_2(t) = 1, \qquad R_2(t) = \nu + 1 - w(q+2\beta),$$

and

$$\frac{1}{Q}\,\frac{\partial}{\partial t}\left(\frac{Q^2}{PD^2}\right) = \frac{1}{R}\,\frac{\partial}{\partial u}\left(\frac{R^2}{PD^2}\right),$$

with $P \equiv P(t,u) = P_0(t) + P_1(t)u + \cdots + P_4(t)u^4$ and so on, and the initial conditions (at $t=0$):

$$P(0,u) = u^2(u-1)^2 \quad \text{and} \quad Q(0,u) = u(u-1).$$

- The Potts GF of planar maps M_1 is an explicit polynomial in the P_i's and Q_i's.

Is this a differential system?

... or a partial differential equation?

$$\frac{1}{Q}\ \frac{\partial}{\partial t}\left(\frac{Q^2}{PD^2}\right) = \frac{1}{R}\ \frac{\partial}{\partial u}\left(\frac{R^2}{PD^2}\right),$$

with $P \equiv P(t,u) = P_0(t) + P_1(t)u + \cdots + P_4(t)u^4$ and so on.

Is this a differential system?

... or a partial differential equation?

$$\frac{1}{Q}\ \frac{\partial}{\partial t}\left(\frac{Q^2}{PD^2}\right) = \frac{1}{R}\ \frac{\partial}{\partial u}\left(\frac{R^2}{PD^2}\right),$$

with $P \equiv P(t,u) = P_0(t) + P_1(t)u + \cdots + P_4(t)u^4$ and so on.

Both: The equation reads

$$2Q_tPD - QP_tD - 2QPD_t = 2R_uPD - RP_uD - 2RPD_u.$$

Extracting the coefficient of $u^0, \ldots, u^7$ gives a system of 8 DEs (in t) between the 8 unknowns series. For u^7 for instance, one finds:

$$P_3'(t) - 2Q_1'(t) + 4(1+\nu) - 4w(2\beta + q) = 0$$

Is this a differential system?

... or a partial differential equation?

$$\frac{1}{Q}\,\frac{\partial}{\partial t}\left(\frac{Q^2}{PD^2}\right) = \frac{1}{R}\,\frac{\partial}{\partial u}\left(\frac{R^2}{PD^2}\right),$$

with $P \equiv P(t,u) = P_0(t) + P_1(t)u + \cdots + P_4(t)u^4$ and so on.

Both: The equation reads

$$2Q_tPD - QP_tD - 2QPD_t = 2R_uPD - RP_uD - 2RPD_u.$$

Extracting the coefficient of $u^0, \ldots, u^7$ gives a system of 8 DEs (in t) between the 8 unknowns series. For u^7 for instance, one finds:

$$P_3'(t) - 2Q_1'(t) + 4(1+\nu) - 4w(2\beta + q) = 0$$

Elimination $\Rightarrow$ existence of a DE of order 5 for M_1.

An explicit differential system for Potts on planar maps

Let $D(t,u) = (q\nu+\beta^2)u^2 - q(\nu+1)u + \beta t(q-4)(wq+\beta) + q$, with $\beta = \nu - 1$.

- There exists a unique 11-tuple $(P_0(t), \dots, P_4(t), Q_0(t), \dots, Q_2(t), R_0(t), \dots, R_2(t))$ of series in t with coefficients in $\mathbb{Q}(q,\nu,w)$ such that

$$P_4(t) = Q_2(t) = 1, \qquad R_2(t) = \nu + 1 - w(q+2\beta),$$

and

$$\frac{1}{Q}\,\frac{\partial}{\partial t}\left(\frac{Q^2}{PD^2}\right) = \frac{1}{R}\,\frac{\partial}{\partial u}\left(\frac{R^2}{PD^2}\right),$$

with $P \equiv P(t,u) = P_0(t) + P_1(t)u + \cdots + P_4(t)u^4$ and so on, and the initial conditions (at $t=0$):

$$P(0,u) = u^2(u-1)^2 \quad \text{and} \quad Q(0,u) = u(u-1).$$

- The Potts GF of planar maps M_1 is an explicit polynomial in the P_i's and Q_i's $\Rightarrow$ DE of order 5

An explicit differential system for Potts on triangulations

Let $D(t, u) = q\nu^2 u^2 + \beta(4\beta + q)u + q\beta\nu(q-4)t + \beta^2$,
with $\beta = \nu - 1$.

- There exists a unique 9-tuple $(P_0(t), \dots, P_3(t), Q_0(t), \dots, Q_2(t), R_0(t), R_1(t))$ of series in t with coefficients in $\mathbb{Q}(q, \nu)$ such that

$$P_3(t) = 1, \qquad Q_2(t) = 2\nu,$$

and

$$\frac{1}{Q}\,\frac{\partial}{\partial t}\left(\frac{Q^2}{PD^2}\right) = \frac{1}{R}\,\frac{\partial}{\partial u}\left(\frac{R^2}{PD^2}\right),$$

with $P \equiv P(t, u) = P_0(t) + P_1(t)u + \cdots + P_3(t)u^3$ and so on, and the initial conditions (at $t = 0$):

$$P(0, u) = u^2(u + 1/4) \quad \text{and} \quad Q(0, u) = u(2\nu u + 1).$$

- The Potts GF T_1 of near-triangulations (outer degree 1) T_1 is an explicit polynomial in the P_i's and Q_i's $\Rightarrow$ DE of order 4

III. Where does it come from?

$$\frac{1}{Q}\,\frac{\partial}{\partial t}\left(\frac{Q^2}{PD^2}\right)=\frac{1}{R}\,\frac{\partial}{\partial u}\left(\frac{R^2}{PD^2}\right)$$

A (vague) idea of the proof

In the footsteps of W. Tutte

- For the GF $T(x,y)$ of properly q-coloured triangulations:

$$T(x,y) = x(q-1) + xyz\,T(x,y)T(1,y) + xz\frac{T(x,y)-T(x,0)}{y} - x^2yz\frac{T(x,y)-T(1,y)}{x-1}$$

[Tutte 73] Chromatic sums for rooted planar triangulations: the cases $\lambda = 1$ and $\lambda = 2$
[Tutte 73] Chromatic sums for rooted planar triangulations, II: the case $\lambda = \tau + 1$
[Tutte 73] Chromatic sums for rooted planar triangulations, III: the case $\lambda = 3$
[Tutte 73] Chromatic sums for rooted planar triangulations, IV: the case $\lambda = \infty$
[Tutte 74] Chromatic sums for rooted planar triangulations, V: special equations
[Tutte 78] On a pair of functional equations of combinatorial interest
[Tutte 82] Chromatic solutions
[Tutte 82] Chromatic solutions II
[Tutte 84] Map-colourings and differential equations

$\triangleleft \quad \triangleleft \quad \diamond \quad \triangleright \quad \triangleright$

[Tutte 95]: Chromatic sums revisited

Structure of the proof

1. Tutte's equation with two catalytic variables:

$$\begin{aligned} M(x,y) &= w + xyt\left((\nu-1)(y-1)+qy\right)M(x,y)M(1,y) \\ &\quad + xyt/w(x\nu-1)M(x,y)M(x,1) \\ &\quad + xywt(\nu-1)\frac{xM(x,y)-M(1,y)}{x-1} + xyt\frac{yM(x,y)-M(x,1)}{y-1} \end{aligned}$$

Structure of the proof

1. Tutte's equation with two catalytic variables:

$$\mathrm{Pol}(M(x,y), M(x,1), M(1,y), x, y) = 0$$

Structure of the proof

1. Tutte's equation with two catalytic variables:

$$\mathrm{Pol}(M(x,y), M(x,1), M(1,y), x, y) = 0$$

2. For $q = 2 + 2\cos(j\pi/m)$, there also exists an equation with only one catalytic variable defining $M(1,y)$: [B-mbm 09(a)]

$$\mathrm{Pol}_{j,m}(M(1,y), M(1,1), y) = 0$$

Structure of the proof

1. Tutte's equation with two catalytic variables:

$$\mathrm{Pol}(M(x,y), M(x,1), M(1,y), x, y) = 0$$

2. For $q = 2 + 2\cos(j\pi/m)$, there also exists an equation with only one catalytic variable defining $M(1,y)$: [B-mbm 09(a)]

$$\mathrm{Pol}_{j,m}(M(1,y), M(1,1), y) = 0$$

3. Derive from it a polynomial equation for the Potts generating function $M_1 = M(1,1)$. [mbm-Jehanne 06]

Structure of the proof

1. Tutte's equation with two catalytic variables:

$$\mathrm{Pol}(M(x,y), M(x,1), M(1,y), x, y) = 0$$

2. For $q = 2 + 2\cos(j\pi/m)$, there also exists an equation with only one catalytic variable defining $M(1,y)$: [B-mbm 09(a)]

$$\mathrm{Pol}_{j,m}(M(1,y), M(1,1), y) = 0$$

3. Derive from it a differential system for the Potts generating function $M_1 = M(1,1)$.

Structure of the proof

1. Tutte's equation with two catalytic variables:

$$\mathrm{Pol}(M(x,y), M(x,1), M(1,y), x, y) = 0$$

2. For $q = 2 + 2\cos(j\pi/m)$, there also exists an equation with only one catalytic variable defining $M(1,y)$: [B-mbm 09(a)]

$$\mathrm{Pol}_{j,m}(M(1,y), M(1,1), y) = 0$$

3. Derive from it a differential system for the Potts generating function $M_1 = M(1,1)$.

4. It depends polynomially on q, and is thus valid for any q.

Structure of the proof

1. Tutte's equation with two catalytic variables:

$$\mathrm{Pol}(M(x,y), M(x,1), M(1,y), x, y) = 0$$

2. For $q = 2 + 2\cos(j\pi/m)$, there also exists an equation with only one catalytic variable defining $M(1,y)$: [B-mbm 09(a)]

$$\mathrm{Pol}_{j,m}(M(1,y), M(1,1), y) = 0$$

3. Derive from it a polynomial equation for the Potts generating function $M_1 = M(1,1)$.

4. It depends polynomially on q, and is thus valid for any q.

Structure of the proof

1. Tutte's equation with two catalytic variables:

$$\mathrm{Pol}(M(x,y), M(x,1), M(1,y), x, y) = 0$$

2. For $q = 2 + 2\cos(j\pi/m)$, there also exists an equation with only one catalytic variable defining $M(1,y)$: [B-mbm 09(a)]

$$\mathrm{Pol}_{j,m}(M(1,y), M(1,1), y) = 0$$

3. Derive from it a differential system for the Potts generating function $M_1 = M(1,1)$.

4. It depends polynomially on q, and is thus valid for any q.

Planar maps: algebraic solution

- The functional equation

$$G(y) = 1 + ty^2 G(y)^2 + ty\,\frac{yG(y) - G(1)}{y-1}$$

Planar maps: algebraic solution

- The functional equation, written with a square:

$$\left(2ty^2(y-1)G(y)+ty^2-y+1\right)^2$$
$$=(y-1-y^2t)^2-4ty^2(y-1)^2+4t^2y^3(y-1)G_1:=\Delta(y)$$

(a polynomial in y)

Planar maps: algebraic solution

- The functional equation, written with a square:

$$\left(2ty^2(y-1)G(y)+ty^2-y+1\right)^2$$
$$= (y-1-y^2t)^2-4ty^2(y-1)^2+4t^2y^3(y-1)G_1 := \Delta(y)$$

(a polynomial in y)

- There exists a (unique) series $Y \equiv Y(t)$ that cancels the LHS:

$$Y = 1 + tY^2 + 2tY^2(Y-1)G(Y).$$

$\Rightarrow$ characterizes inductively the coefficient of t^n

Planar maps: algebraic solution

- The functional equation, written with a square:

$$\left(2ty^2(y-1)G(y) + ty^2 - y + 1\right)^2$$
$$= (y - 1 - y^2 t)^2 - 4ty^2(y-1)^2 + 4t^2y^3(y-1)G_1 := \Delta(y)$$

(a polynomial in y)

- There exists a (unique) series $Y \equiv Y(t)$ that cancels the LHS:

$$Y = 1 + tY^2 + 2tY^2(Y-1)G(Y).$$

$\Rightarrow$ characterizes inductively the coefficient of t^n

- This series Y must be a root of $\Delta(y)$, and in fact a double root.

Planar maps: algebraic solution

- The functional equation, written with a square:

$$\left(2ty^2(y-1)G(y)+ty^2-y+1\right)^2$$
$$= (y-1-y^2t)^2-4ty^2(y-1)^2+4t^2y^3(y-1)G_1 := \Delta(y)$$

(a polynomial in y)

- There exists a (unique) series $Y \equiv Y(t)$ that cancels the LHS:

$$Y = 1+tY^2+2tY^2(Y-1)G(Y).$$

$\Rightarrow$ characterizes inductively the coefficient of t^n

- This series Y must be a root of $\Delta(y)$, and in fact a double root.
- Algebraic consequence: The discriminant of $\Delta(y)$ w.r.t. y is zero:

$$27\,t^2G_1^2+(1-18\,t)\,G_1+16\,t-1=0$$

Planar maps: differential solution

- The polynomial

$$\Delta(y) = (y - 1 - y^2 t)^2 - 4ty^2(y-1)^2 + 4t^2y^3(y-1)G_1$$

has degree 4 in y, and admits a double root $Y(t)$:

$$\Delta(t;y) \quad = \quad P(t;y)(y - Y(t))^2, \qquad\qquad P \text{ of degree 2 in } y$$

Planar maps: differential solution

- The polynomial

$$\Delta(y) = (y - 1 - y^2 t)^2 - 4ty^2(y-1)^2 + 4t^2y^3(y-1)G_1$$

has degree 4 in y, and admits a double root $Y(t)$:

$$\begin{array}{rcll} \Delta(t;y) & = & P(t;y)(y - Y(t))^2, & \quad P \text{ of degree 2 in } y \\ \Delta_y(t;y) & = & Q(t;y)(y - Y(t)), & \quad Q \text{ of degree 2 in } y \end{array}$$

Planar maps: differential solution

- The polynomial

$$\Delta(y) = (y - 1 - y^2 t)^2 - 4ty^2(y-1)^2 + 4t^2y^3(y-1)G_1$$

has degree 4 in y, and admits a double root $Y(t)$:

$$\begin{array}{rcll} \Delta(t;y) & = & P(t;y)(y - Y(t))^2, & \quad P \text{ of degree 2 in } y \\ \Delta_y(t;y) & = & Q(t;y)(y - Y(t)), & \quad Q \text{ of degree 2 in } y \\ \Delta_t(t;y) & = & R(t;y)(y - Y(t)), & \quad R \text{ of degree 3 in } y \end{array}$$

Planar maps: differential solution

- The polynomial

$$\Delta(y) = (y - 1 - y^2 t)^2 - 4ty^2(y-1)^2 + 4t^2y^3(y-1)G_1$$

has degree 4 in y, and admits a double root $Y(t)$:

$$\begin{array}{rclr}
\Delta(t;y) & = & P(t;y)(y - Y(t))^2, & P \text{ of degree 2 in } y \\
\Delta_y(t;y) & = & Q(t;y)(y - Y(t)), & Q \text{ of degree 2 in } y \\
\Delta_t(t;y) & = & R(t;y)(y - Y(t)), & R \text{ of degree 3 in } y
\end{array}$$

- Elimination of Y and Δ:

$$\frac{1}{Q}\frac{\partial}{\partial t}\left(\frac{Q^2}{P}\right) = \frac{1}{R}\frac{\partial}{\partial y}\left(\frac{R^2}{P}\right).$$

Planar maps: differential solution

- The polynomial

$$\Delta(y) = (y - 1 - y^2 t)^2 - 4ty^2(y-1)^2 + 4t^2y^3(y-1)G_1$$

has degree 4 in y, and admits a double root $Y(t)$:

$$\begin{array}{rcll} \Delta(t;y) &=& P(t;y)(y-Y(t))^2, & P \text{ of degree 2 in } y \\ \Delta_y(t;y) &=& Q(t;y)(y-Y(t)), & Q \text{ of degree 2 in } y \\ \Delta_t(t;y) &=& R(t;y)(y-Y(t)), & R \text{ of degree 3 in } y \end{array}$$

- Elimination of Y and Δ:

$$\frac{1}{Q}\frac{\partial}{\partial t}\left(\frac{Q^2}{P}\right) = \frac{1}{R}\frac{\partial}{\partial y}\left(\frac{R^2}{P}\right).$$

- Writing $P(t;y) = P_0(t) + yP_1(t) + y^2P_2(t)$ and so on for Q and R, this gives a system of differential equations in t relating the series P_i, Q_i and R_i.

Planar maps: differential solution

- The polynomial

$$\Delta(y) = (y - 1 - y^2 t)^2 - 4ty^2(y-1)^2 + 4t^2y^3(y-1)G_1$$

has degree 4 in y, and admits a double root $Y(t)$:

$$\begin{array}{rcll} \Delta(t;y) & = & P(t;y)(y - Y(t))^2, & P \text{ of degree 2 in } y \\ \Delta_y(t;y) & = & Q(t;y)(y - Y(t)), & Q \text{ of degree 2 in } y \\ \Delta_t(t;y) & = & R(t;y)(y - Y(t)), & R \text{ of degree 3 in } y \end{array}$$

- Elimination of Y and Δ:

$$\frac{1}{Q}\frac{\partial}{\partial t}\left(\frac{Q^2}{P}\right) = \frac{1}{R}\frac{\partial}{\partial y}\left(\frac{R^2}{P}\right).$$

- Writing $P(t;y) = P_0(t) + yP_1(t) + y^2P_2(t)$ and so on for Q and R, this gives a system of differential equations in t relating the series P_i, Q_i and R_i.
- Finally, the GF G_1 is a polynomial in the series P_i, Q_i, R_i.

IV. Special values of q and ν

Specializations: explicit DE for the Potts generating function

Specialization	general maps	triangulations
$q = 2, 3$	algebraic	algebraic
Proper colourings ($\nu = 0$)		$D = (4 - q)x + 1$ order 2
Four colours ($q = 4$)	$D = ((\nu + 1)x - 2)^2$ order 3	$D = (2\nu x + \beta)^2$ order 2
Connected subgraphs (spanning forests) ($q = 0$)		$D = \beta^2(1 + 4x)$ order 2
Self-dual model ($q = \beta^2, w = 1/\beta$)	$R_2 = 0$ order 3	

Specializations: explicit DE for the Potts generating function

Specialization	general maps	triangulations
$q = 2, 3$	algebraic	algebraic
Proper colourings ($\nu = 0$)		$D = (4 - q)x + 1$ order 2
Four colours ($q = 4$)	$D = ((\nu + 1)x - 2)^2$ order 3	$D = (2\nu x + \beta)^2$ order 2
Connected subgraphs (spanning forests) ($q = 0$)		$D = \beta^2(1 + 4x)$ order 2
Self-dual model ($q = \beta^2, w = 1/\beta$)	$R_2 = 0$ order 3	

Two combinatorial problems:
1. Properly 3-coloured planar maps [Bernardi-mbm 09(a)]

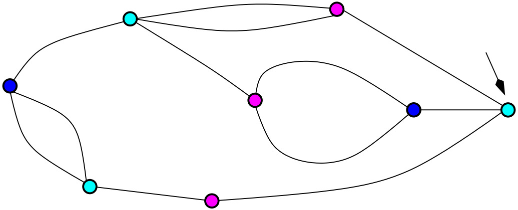
Let A be the quartic series in t defined by

$$A = t\ \frac{(1+2A)^3}{1-2A^3}.$$

Then the generating function of properly 3-coloured planar maps is

$$M_1(3,0,1,t) = \frac{(1+2A)(1-2A^2-4A^3-4A^4)}{(1-2A^3)^2}$$

The proof is at the moment   h o r r i b l e ...



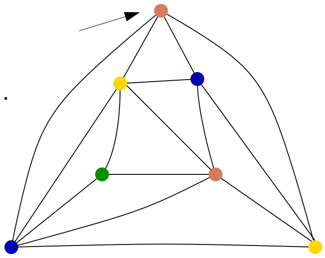
Two combinatorial problems:
2. Properly q-coloured triangulations [Tutte 84]

The number $c(n)$ of properly q-coloured triangulations having n vertices satisfies:

$$(n+1)(n+2)c(n+2) = (q-4)(3n-1)(3n-2)c(n+1) + 2\sum_{i=1}^{n} i(i+1)(3n-3i+1)c(i+1)c(n+2-i),$$

with the initial condition $c(2) = q - 1$.

The proof is at the moment h o r r i b l e ...



More questions are left...

A. The differential system

Simpler, and/or more combinatorial derivation? Done for

- the Ising model ($q = 2$)
  [mbm & Schaeffer 02], [Bouttier, Di Francesco & Guitter 04]
- spanning forests/connected subgraphs ($q \to 0$)
  [Bouttier et al. 07], [mbm-Courtiel 13(a)])

B. Asymptotics, phase transitions

- Asymptotic number of properly q-coloured triangulations when $q \in (28/11, 4] \cup [5, \infty)$ [Odlyzko-Richmond 83])
- Critical point of the Potts model on planar maps when $q \in (0, 4)$ (for the outer degree) [Borot et al. 12])
- Critical exponents for near-triangulations (for the outer degree)
  [Eynard & Bonnet 99]

Maps equipped with an additional structure

How many maps equipped with...

– a spanning tree?
[Mullin 67]

– a spanning forest?
[Bouttier et al., Sportiello et al., mbm-Courtiel 13]

– a self-avoiding walk?
[Duplantier-Kostov 88]

– a proper q-colouring?
[Tutte 74, Bouttier et al. 02]

What is the expected partition function of...

– the Ising model?
[Boulatov, Kazakov, mbm, Schaeffer, Bouttier et al.]

– the hard-particle model?
[mbm, Schaeffer, Jehanne, Bouttier et al. 02, 07]

– the Potts model?
[Eynard-Bonnet 99, Baxter 01, mbm-Bernardi 09, Guionnet et al. Borot et al. 12]

$q = 2$: The Ising model on planar maps

Let A be the series in t, with polynomial coefficients in ν, defined by

$$A = t\,\frac{\left(1 + 3\,\nu\,A - 3\,\nu\,A^2 - \nu^2 A^3\right)^2}{1 - 2\,A + 2\,\nu^2 A^3 - \nu^2 A^4}.$$

Then the Ising generating function of planar maps is

$$M(2, \nu, t; 1, 1) = \frac{1 + 3\,\nu\,A - 3\,\nu\,A^2 - \nu^2 A^3}{\left(1 - 2\,A + 2\,\nu^2 A^3 - \nu^2 A^4\right)^2}\;P(\nu, A)$$

where

$$\begin{aligned} P(\nu, A) = \nu^3 A^6 &+ 2\,\nu^2(1-\nu)A^5 + \nu\,(1 - 6\,\nu)A^4 \\ &- \nu\,(1 - 5\,\nu)A^3 + (1 + 2\,\nu)A^2 - (3 + \nu)A + 1. \end{aligned}$$

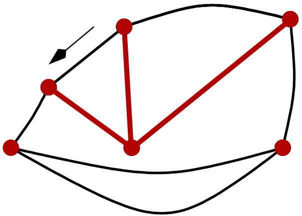
$\rightsquigarrow$ Asymptotics: Phase transition at $\nu_c = \frac{3+\sqrt{5}}{2}$, critical exponents...

$q = 0$: Cubic maps equipped with a spanning forest

$D(u) = 4u + 1$ has degree 1 only and is independent of t

$$\frac{1}{Q}\,\frac{\partial}{\partial t}\left(\frac{Q^2}{PD^2}\right) = \frac{1}{R}\,\frac{\partial}{\partial u}\left(\frac{R^2}{PD^2}\right)$$

$\Rightarrow$ A DE of order 2 for the generating function of planar cubic maps equipped with a spanning forest, by edges and number of trees in the forest.



Combinatorial solution: [Bouttier, Di Francesco & Guitter 07], [mbm-Courtiel 13]

$\nu = 0$: Properly q-coloured triangulations (Tutte's problem)

$D(u) = 1 + u(4 - q)$ has degree 1 only and is independent of t

$$\frac{1}{Q} \frac{\partial}{\partial t} \left(\frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial u} \left(\frac{R^2}{PD^2} \right)$$

$\Rightarrow$ A DE of order 2 for $T(q, 0, t; 1, 1)$, which counts properly coloured planar triangulations.

$q = 4$: Four-coloured triangulations

$D(u) = (2u\nu + \nu - 1)^2$ is a square and is independent of t

$$\frac{1}{Q}\ \frac{\partial}{\partial t}\left(\frac{Q^2}{PD^2}\right) = \frac{1}{R}\ \frac{\partial}{\partial u}\left(\frac{R^2}{PD^2}\right),$$

$\Rightarrow$ A DE of order 2 for $T(4, \nu, t; 1, 1)$, which counts 4-coloured planar triangulations (the 4-state Potts model).

$+$ A similar result for 4-coloured planar maps (DE of order 3)