Scaling laws for large deviations in Voronoi tessellations

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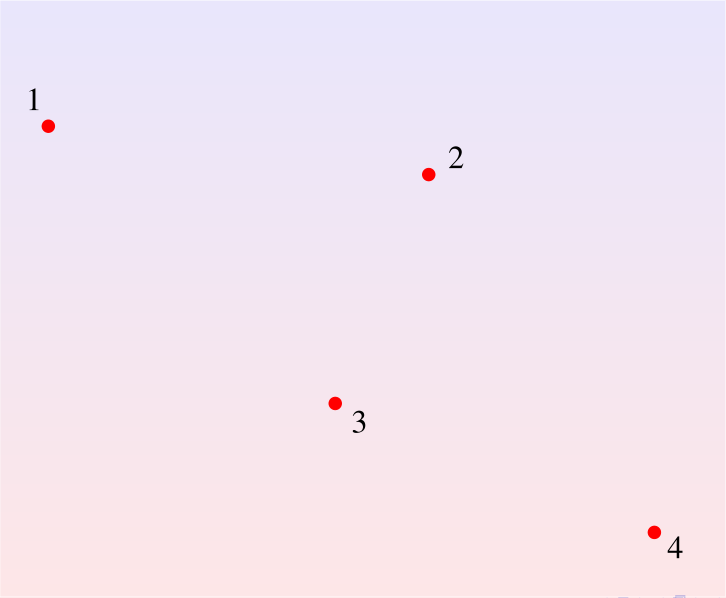
The Voronoi construction



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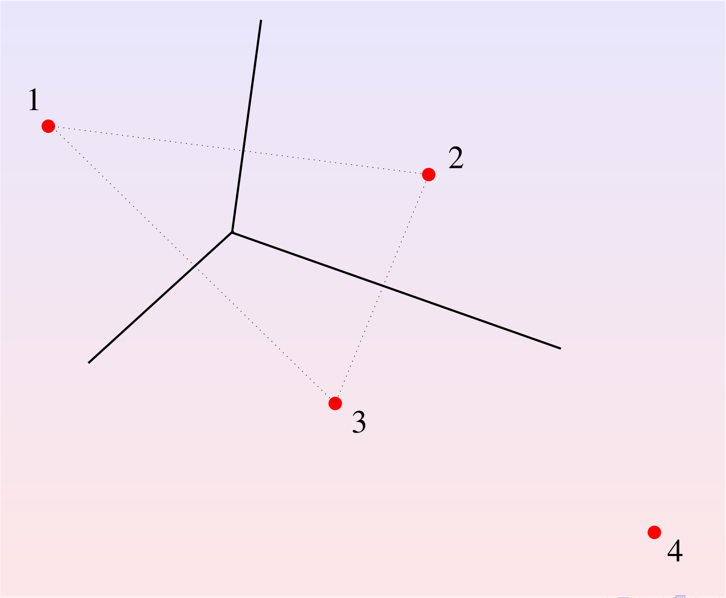

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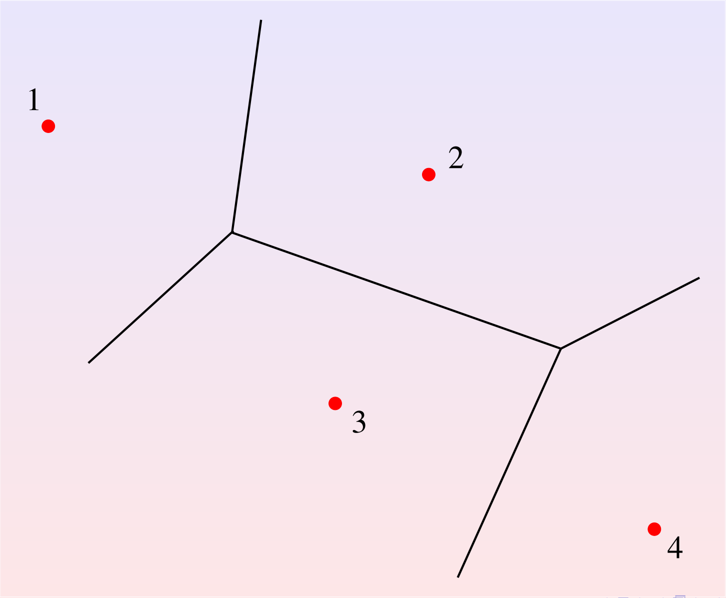

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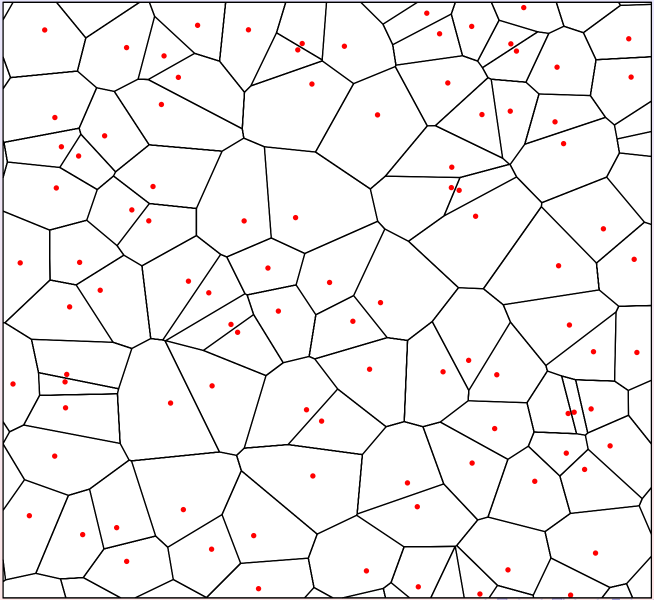
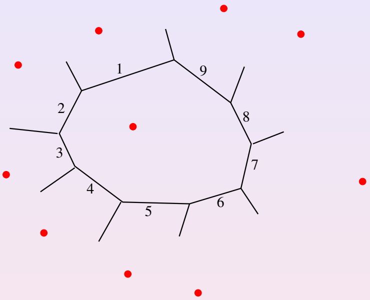


The sidedness probability p_n



Question:

What is the probability p_n *that an arbitrarily picked cell have exactly* n *sides?*

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RANDOM GEOMETRY AND THE STATISTICS OF TWO-DIMENSIONAL CELLS

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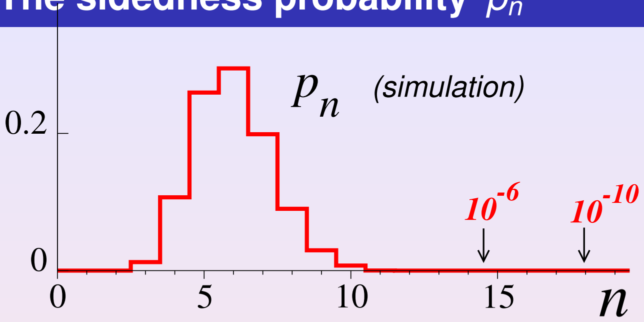
We obtain accurate values of p_n, the probability of finding an n-sided Voronoï cell in a two-dimensional Poisson random lattice up to large values of $n(\sim 50)$. Numerical as well as analytical evidence points to a n^{-an} behavior for large n with a of order 2. We also study the mean area and the mean distance to a neighbor as functions of n, and check accurately several sum rules.

1. Introduction

As part of a program for studying field theory on random lattices pioneered by Christ, Friedberg and Lee [1, 2], we present here some numerical results on two-dimensional random geometry. More precisely we study the statistics of Voronoï cells. Neither the methods [3, 4] nor the results for small numbers of sides n are new.

However, we have pushed the investigation to a rather large number of sides down to very small probabilities. For the largest values $n \sim 50$, the corresponding probability is of order 10^{-75}! This allows a study of the asymptotic behavior of p_n, the probability of finding a cell with n sides which we find behaves as $p_n \sim n^{-an}$. The numerical study favors $a \simeq 2$, while we are able to prove that $1 \leq a \leq 2$.

The sidedness probability p_n



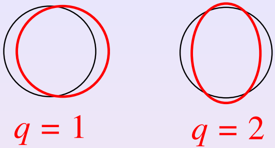
Drouffe and Itzykson (1984):
Asymptotic guess: $p_n \sim n^{-\alpha n}$ $\quad (n \to \infty)$

Analytically: $1 \leq \alpha \leq 2$
Fit to the data: $\alpha \approx 2$ $\quad (n \lesssim 50)$

An exact asymptotic result for p_n

$$
\begin{aligned}
p_n &= \int \mathrm{d}R_1 \ldots \mathrm{d}R_n \; \underbrace{\chi_n(R_1, \ldots, R_n)} \; \underbrace{\mathrm{e}^{-\rho A_n(R_1,\ldots,R_n)}} \\
&= \ldots \\
&= \ldots \quad \text{(HJH, 2005)} \\
&= \ldots \\
&= \frac{(8\pi^2)^n}{(2n)!} \, C_2 \qquad\qquad (n \to \infty)
\end{aligned}
$$

The quantity C_2



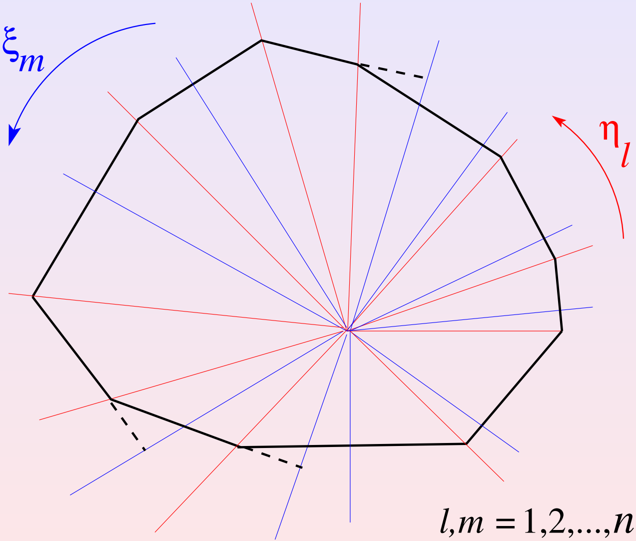
Expansion in powers of $1/n$:

$$C_2 = \frac{1}{4\pi^2} \prod_{q=1}^{\infty} \left(1 - \frac{1}{q^2} + \frac{4}{q^4}\right)^{-1} + O\left(\frac{1}{n}\right)$$

Byproduct: cell becomes circular

$$R \simeq \left(\frac{n}{4\pi}\right)^{1/2}$$

The $2n$ angles



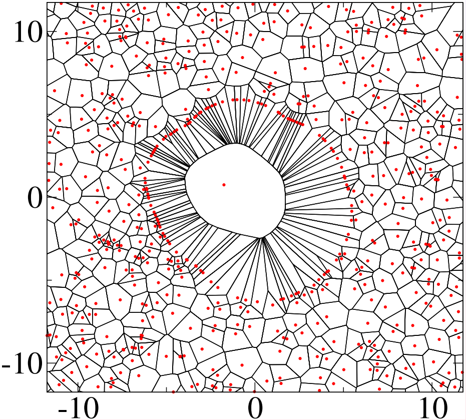
$$l, m = 1, 2, \ldots, n$$

A very-many-sided cell

Another byproduct:
an algorithm without attrition

$n = 96$

$p_n \approx 10^{-177}$



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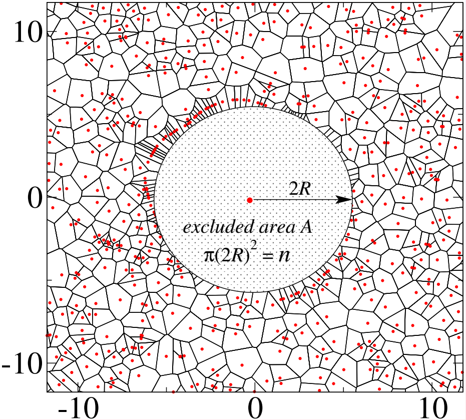
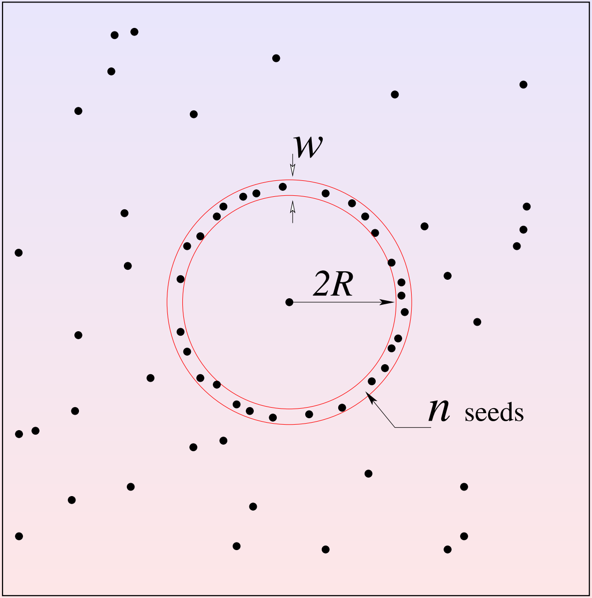


An entropy argument



Heuristically

$$w \sim R^{-3/2}$$

Maximize entropy under constraints:

$$R^* \simeq \left(\frac{n}{4\pi}\right)^{1/2},$$

the exact result.

3D cell, n faces

$p_n = ?$

2009 Entropy argument

Spherical excluded volume equal to n

$$\Longrightarrow \quad R \simeq \frac{1}{2}\left(\frac{3n}{4\pi}\right)^{1/3}$$

2013-2014: Monte Carlo data by E. Lazar

– statistics of n-faced cells, very good agreement for R

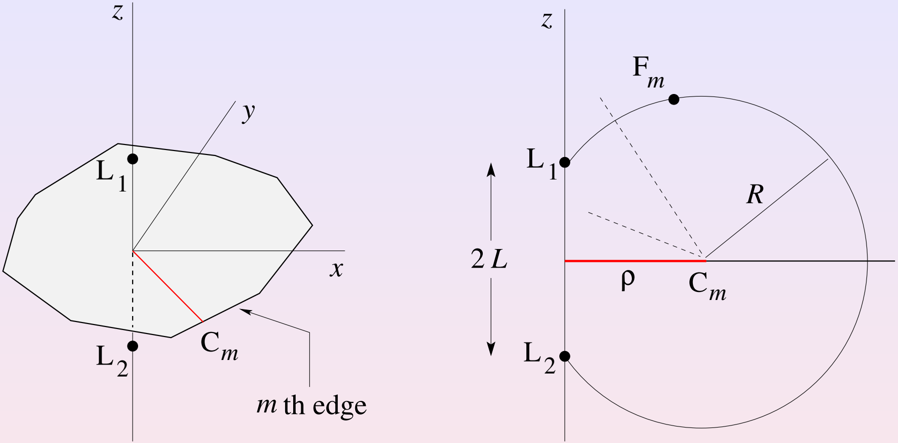
But also:

– statistics of n-edged faces

–...

–...

A face shared by two 3D cells



$P_n(L)$ *for* $n \to \infty$?

Two 3D cells sharing an n-edged face, $n \to \infty$

(1) The n-edged face becomes circular

(2) Probability of an n-edged face

$$p_n = \frac{(12\pi^2)^n}{(2n)!} C_3$$

(3) The excluded volume becomes a torus of equal radii

$$R \simeq \rho \simeq \left(\frac{n}{2\pi^2}\right)^{1/3}$$

(4) Average of L given n

$$\langle L \rangle = \frac{\sqrt{3}}{\pi}\left(\frac{2}{\pi n}\right)^{1/6}$$

entropic attraction between seeds!

(5) Conditional distribution of L

$$Q_n(n^{1/6}L) \simeq Q_\infty(n^{1/6}L), \qquad Q_\infty(y) = c_0 y^2 \mathrm{e}^{-c_1 y^2}$$

Distribution of L (E. Lazar and HJH, 2014)