

GENERALIZED SLE MULTIFRACTALITY

Bertrand Duplantier

Institut de Physique Théorique, Saclay, France

RANDOM SURFACES & RANDOM GEOMETRY

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Joint works with

- C. Nguyen, N. Nguyen, M. Zinsmeister (Orléans)
- H. Ho, B. Le, M. Zinsmeister (Orléans)

Caius Fellow Prof Sir Sam Edwards (1928-2015)



Polymer Physics:

P. J. Flory (1910-1985) Nobel in Chemistry

P.-G. de Gennes (1932-2007) Nobel in Physics

J. des Cloizeaux

Serendipity

Let $f(z) = \sum_{n \geq 0} a_n z^n$ be a holomorphic function in the unit disc \mathbb{D} .

Further assume that f is injective. Then $a_1 \neq 0$ and **Bieberbach** proved in 1916 that $|a_2| \leq 2|a_1|$. In the same paper, he famously conjectured that $\forall n \geq 2, |a_n| \leq n|a_1|$, guided by the intuition that the **Koebe function**

$$\mathcal{K}(z) := - \sum_{n \geq 1} n(-z)^n = \frac{z}{(1+z)^2},$$

which is a holomorphic bijection between \mathbb{D} and $\mathbb{C} \setminus [1/4, +\infty)$, should be **extremal**. This conjecture was finally proven in 1984 by **de Branges**. The earliest important contribution to the proof of Bieberbach's conjecture is that by **Loewner** in 1923 that $|a_3| \leq 3|a_1|$. **Oded Schramm** revived Loewner's method in 1999, introducing **randomness** into it, as driven by **standard Brownian motion**.

Whole-Plane SLE & LLE

$$\frac{\partial f_t}{\partial t} = z \frac{\partial f_t}{\partial z} \frac{\lambda(t) + z}{\lambda(t) - z}, \quad z \in \mathbb{D},$$

$$\lambda(t) = e^{i\sqrt{\kappa}B_t} [e^{i\xi L_t}].$$

The characteristic function of a Lévy process L_t has the form

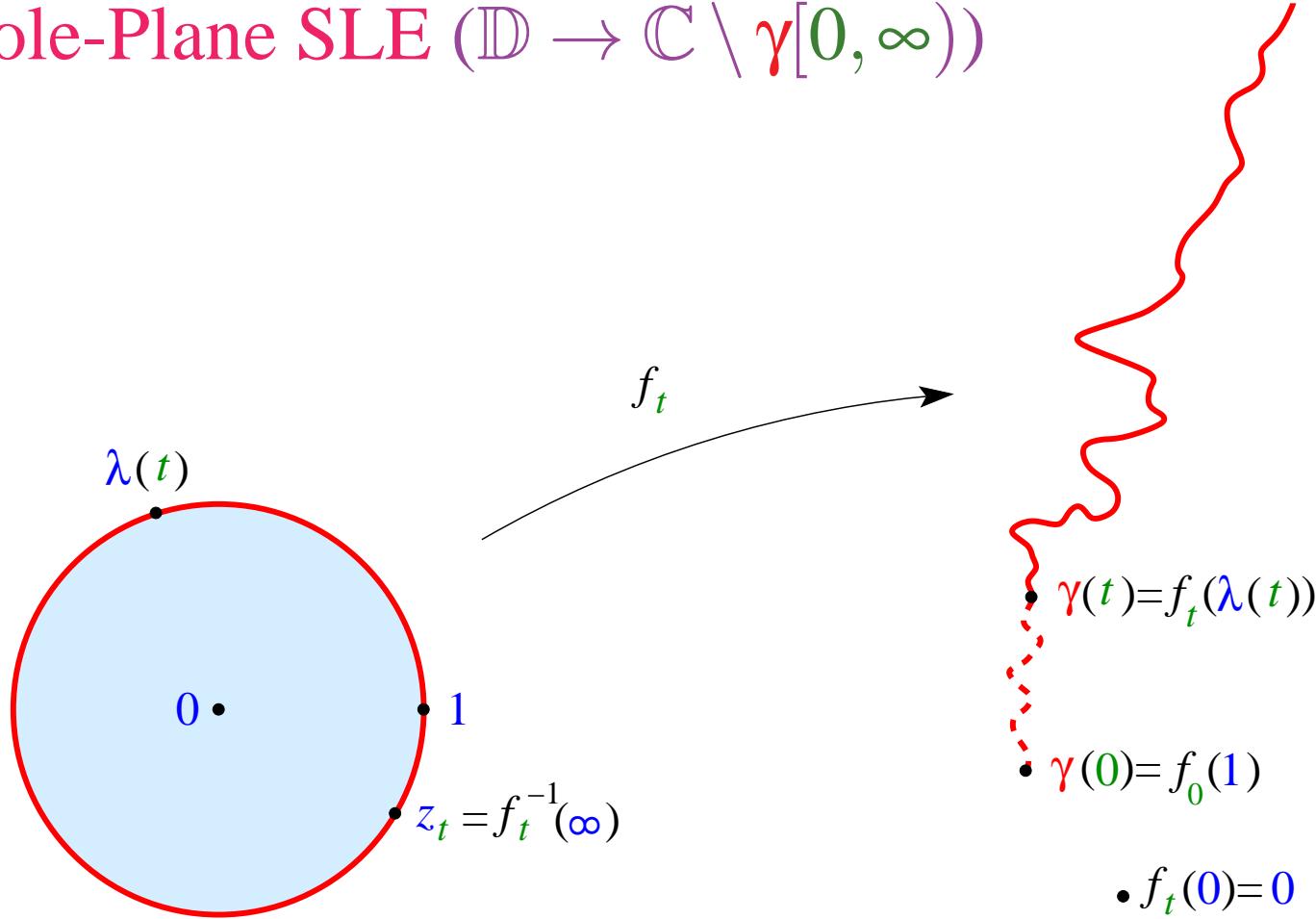
$$\mathbb{E}(e^{i\xi L_t}) = e^{-t\eta(\xi)},$$

where η the Lévy symbol. The function

$$\eta(\xi) = \kappa|\xi|^\alpha/2, \quad \alpha \in (0, 2]$$

is the Lévy symbol of the α -stable process. The normalization here is chosen so that it is SLE $_\kappa$ for $\alpha = 2$.

Whole-Plane SLE ($\mathbb{D} \rightarrow \mathbb{C} \setminus \gamma[0, \infty)$)



Loewner map $z \mapsto f_t(z)$ from the unit disk \mathbb{D} to the slit domain

$\Omega_t = C \setminus \gamma([t, \infty))$. One has $f_t(0) = 0, \forall t \geq 0$. At $t = 0$, $\lambda(0) = 1$, so that the image of $z = 1$ is at the tip $\gamma(0) = f_0(1)$ of the curve.

Series expansions

Let f_t be the whole-plane evolution generated by the Lévy process (L_t) with Lévy symbol η . We write

$$e^{-t} f_t(z) = z + \sum_{n=2}^{\infty} a_n(t) z^n; \quad e^{-t/2} f_t^{[2]}(z) = z + \sum_{n \geq 1} b_{2n+1}(t) z^{2n+1}.$$

Then the *conjugate* whole-plane LLE $e^{-iL_t} f_t(e^{iL_t} z)$ has the same law as $f_0(z)$, i.e., $e^{i(n-1)L_t} a_n(t) \stackrel{(\text{law})}{=} a_n(0)$. Similarly, the conjugate of the *oddified* whole-plane LLE $f_t^{[2]}(z) := z \sqrt{f_t(z^2)/z^2}$, $e^{-(i/2)L_t} f_t^{[2]}(e^{(i/2)L_t} z)$, has the same law as $f_0^{[2]}(z)$, i.e., $e^{inL_t} b_{2n+1}(t) \stackrel{(\text{law})}{=} b_{2n+1}(0)$.

Loewner's method

Recall that

$$f_t(z) = e^t \left(z + \sum_{n \geq 2} a_n(t) z^n \right).$$

Expanding both sides of Loewner's equation as power series and identifying coefficients yields the set of *recursion equations for $n \geq 2$*

$$\dot{a}_n(t) - (n-1)a_n(t) = 2 \sum_{k=1}^{n-1} k a_k(t) \bar{\lambda}^{n-k}(t),$$

with $a_1 = 1$; the dot means a t -derivative, and $\bar{\lambda}(t) = 1/\lambda(t)$, with $\lambda(t) = e^{i\sqrt{\kappa}B_t} [e^{i\xi L_t}]$.

Expected coefficients

Theorem 2. Setting $a_n := a_n(0)$ and $b_{2n+1} := b_{2n+1}(0)$, we have

$$\mathbb{E}(a_n) = \prod_{k=0}^{n-2} \frac{\eta_k - k - 2}{\eta_{k+1} + k + 1}, \quad n \geq 2,$$

$$\mathbb{E}(b_{2n+1}) = \prod_{k=0}^{n-1} \frac{\eta_k - k - 1}{\eta_{k+1} + k + 1}, \quad n \geq 1.$$

Corollary 2. If $\eta_1 = 3$, $\mathbb{E}(f'_0(z)) = 1 - z$ (*SLE*₆);

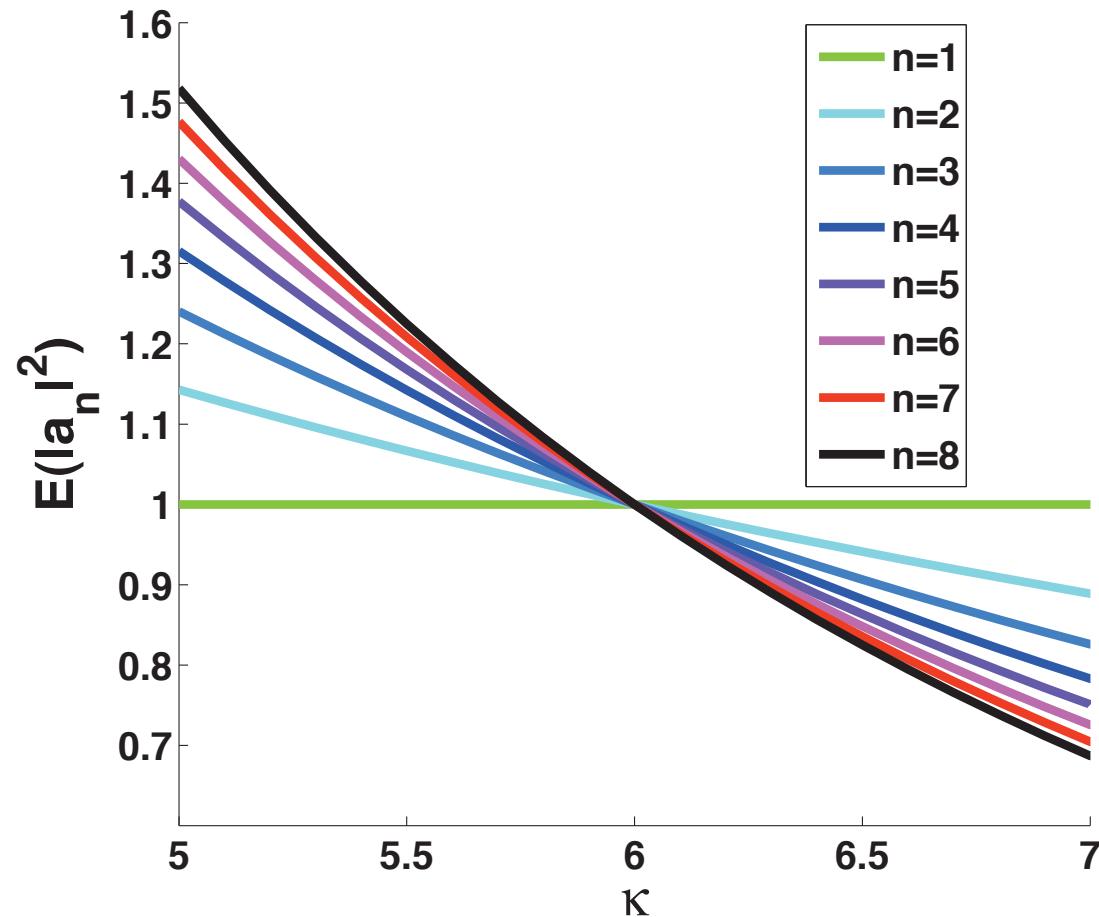
if $\eta_1 = 1$ and $\eta_2 = 4$, $\mathbb{E}(f'_0(z)) = (1 - z)^2$ (*SLE*₂);

if $\eta_1 = 2$, $\mathbb{E}(f_0^{[2]\prime}(z)) = 1 - z^2$ (*SLE*₄).

[See also **Kemppainen '10** for expectations of SLE coefficient moments.]

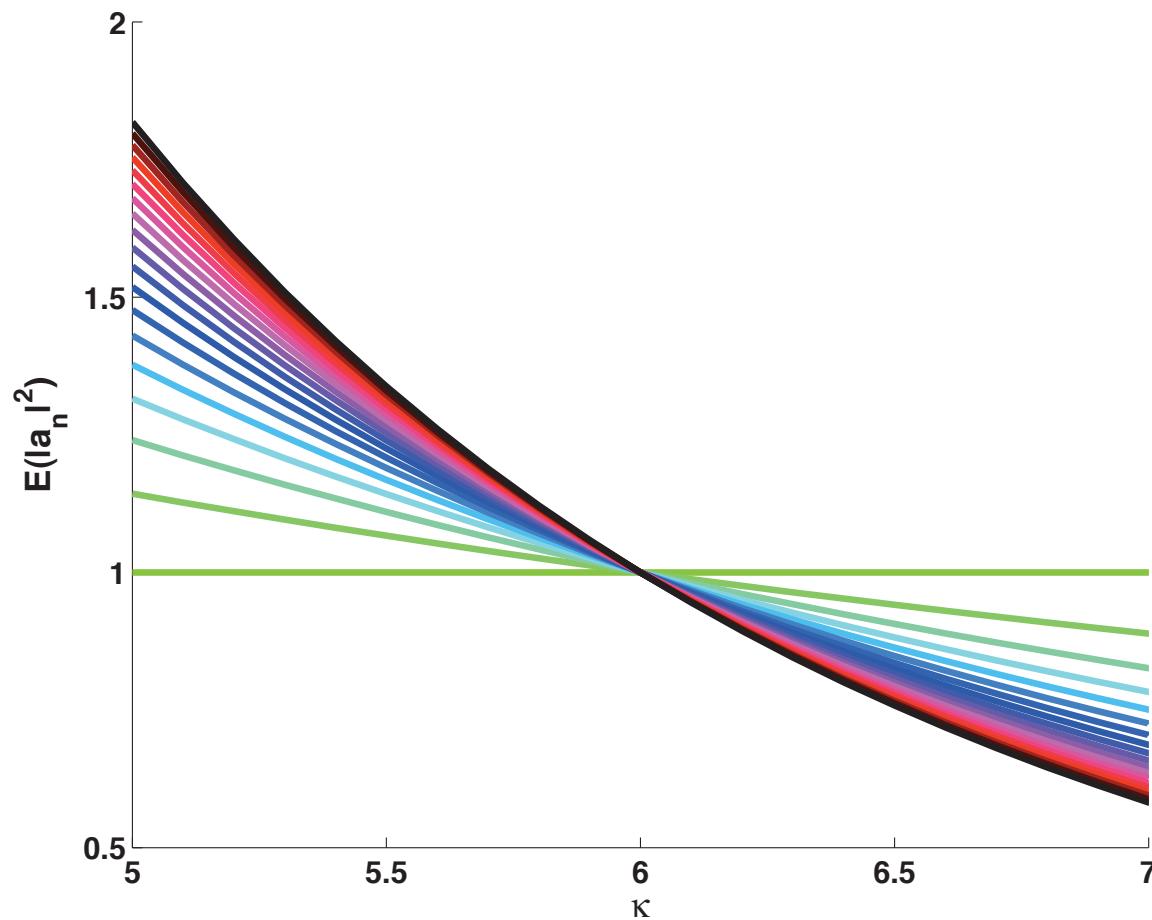
The Surprise: Expected Square Coefficients $\mathbb{E}(|a_n|^2)$

Examples: For SLE_6

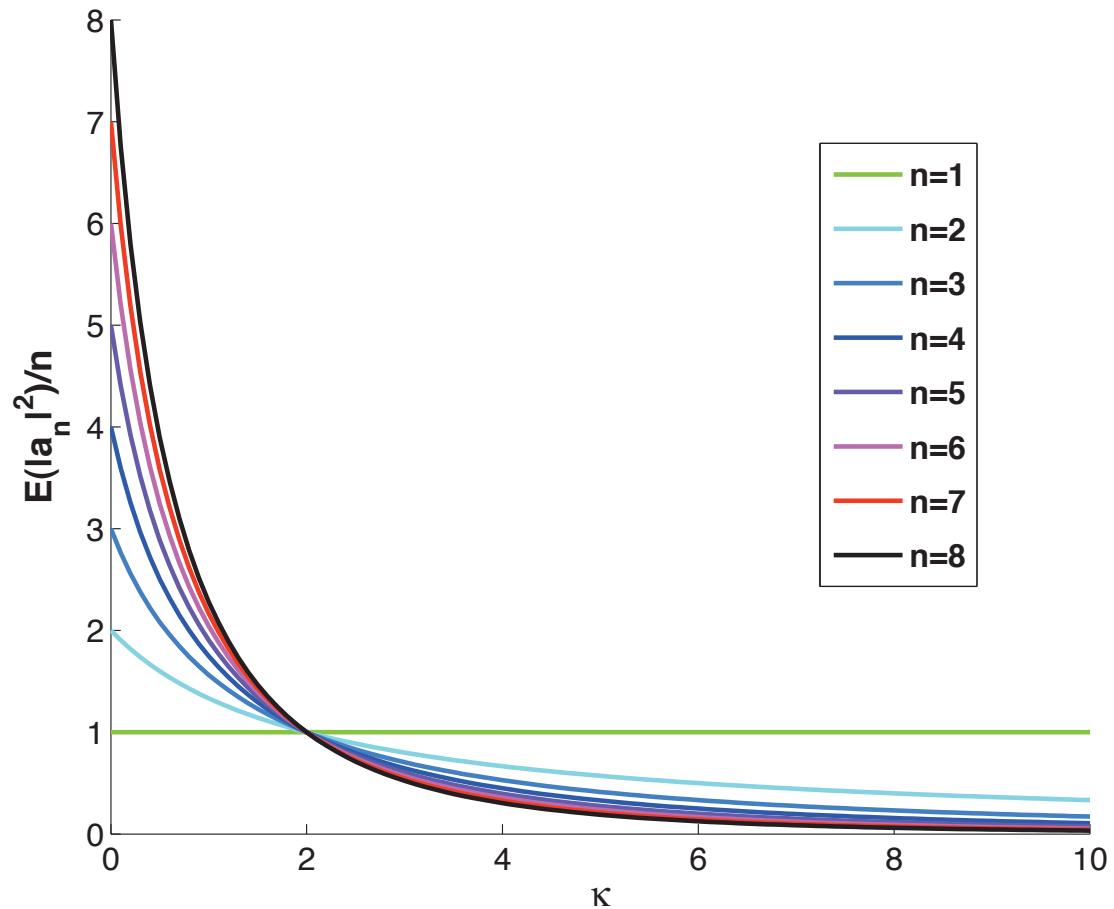


Expected Square Coefficients $\mathbb{E}(|a_n|^2)$

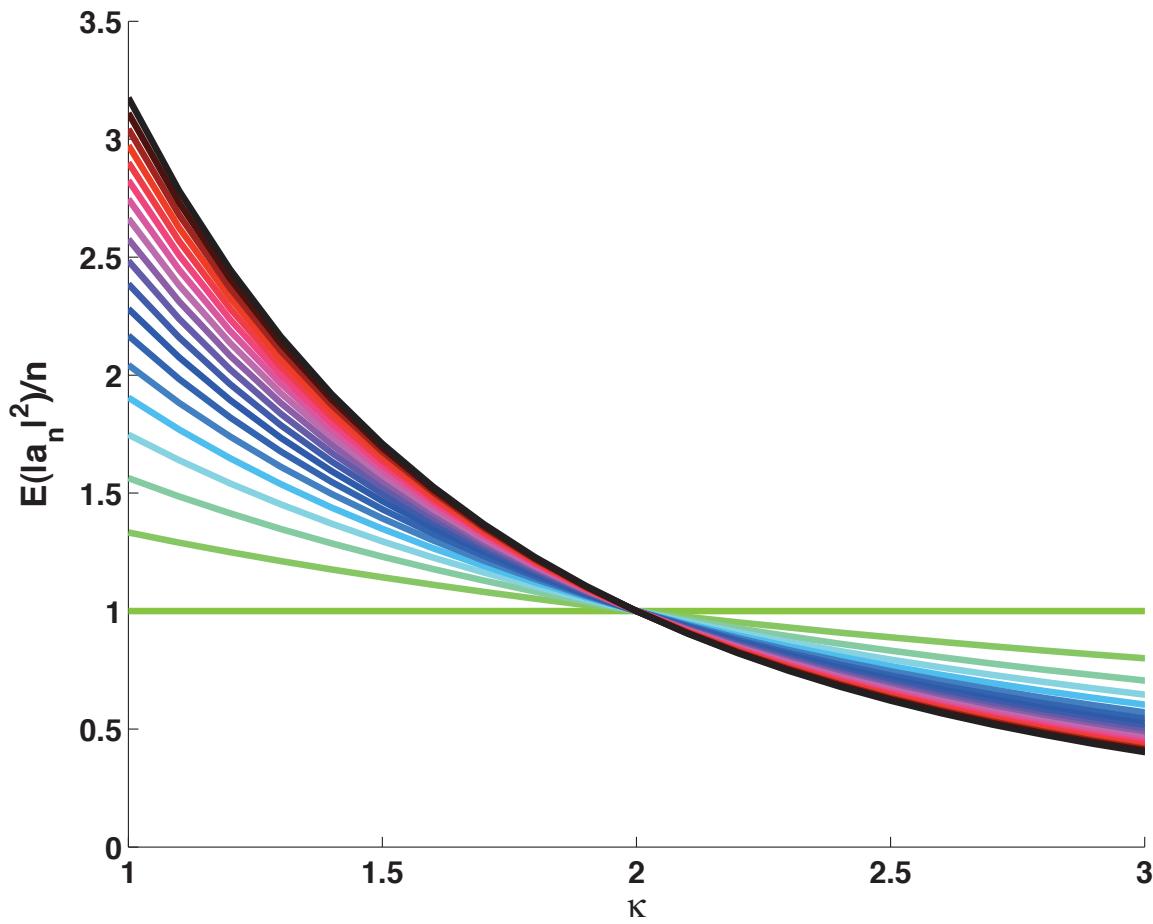
For SLE_6



Expected Square Coefficients $\mathbb{E}(|a_n|^2)/n$ For SLE_2



Expected Square Coefficients $\mathbb{E}(|a_n|^2)/n$ For SLE_2



Expected Square Coefficients

Examples: For SLE₆ and SLE₂

$$\mathbb{E}(|a_{\textcolor{red}{n}}|^2) = \textcolor{red}{1}, \kappa = 6, \forall n \geq 1; \quad \mathbb{E}(|a_{\textcolor{red}{n}}|^2) = \textcolor{red}{n}, \kappa = 2, \forall n \geq 1$$

$$\mathbb{E}(|a_4|^2) = \frac{8 \kappa^5 + 104 \kappa^4 + 4576 \kappa^3 + 18288 \kappa^2 + 22896 \kappa + 8640}{9 (\kappa + 10)(3\kappa + 2)(\kappa + 6)(\kappa + 1)(\kappa + 2)^2}.$$

[Recursion: $n \leq 4$; Computer assisted: $n \leq 8$ (formal), $n \leq 19$ (num.)]

Theorem 2.

- (i) if $\eta_1 = 3$, $\mathbb{E}(|a_{\textcolor{red}{n}}|^2) = \textcolor{red}{1}, \forall n \geq 1$ (SLE₆);
- (ii) if $\eta_1 = 1, \eta_2 = 4$, $\mathbb{E}(|a_{\textcolor{red}{n}}|^2) = \textcolor{red}{n}, \forall n \geq 1$ (SLE₂);
- (iii) if $\eta_1 = 2$, $\mathbb{E}(|b_{2n+1}|^2) = 1/(2n+1), \forall n \geq 1$ (SLE₄).

Derivative Moments

Theorem 3. *The whole-plane SLE $_{\kappa}$ map $f_0(z)$ has derivative moments*

$$\begin{aligned}\mathbb{E}[(f'_0(z))^{p/2}] &= (1-z)^\alpha, \\ \mathbb{E}[|f'_0(z)|^p] &= \frac{(1-z)^\alpha(1-\bar{z})^\alpha}{(1-z\bar{z})^\beta},\end{aligned}$$

for the special set of exponents $p = p(\kappa) := (6 + \kappa)(2 + \kappa)/8\kappa$, with $\alpha = (6 + \kappa)/2\kappa$ and $\beta = (6 + \kappa)^2/8\kappa$. [See also Loutsenko & Yermolayeva '12]

Corollary 4. *$p = 2$ case: for $\kappa = 6$:*

$$\mathbb{E}(f'_0(z)) = 1 - z, \quad \mathbb{E}(|f'_0(z)|^2) = \frac{(1-z)(1-\bar{z})}{(1-z\bar{z})^3};$$

for $\kappa = 2$:

$$\mathbb{E}(f'_0(z)) = (1-z)^2, \quad \mathbb{E}(|f'_0(z)|^2) = \frac{(1-z)^2(1-\bar{z})^2}{(1-z\bar{z})^4}.$$

Logarithmic Moments [D., Ho, Le, Zinsmeister '15]

Theorem 5. *The whole-plane SLE_κ map $f_0(z)$ has logarithmic moments*

$$\mathbb{E}\left(\left|z \frac{f'_0(z)}{f_0(z)}\right|^{\frac{(\kappa+2)}{\kappa}}\right) = \frac{(1-z)^{\frac{2}{\kappa}}(1-\bar{z})^{\frac{2}{\kappa}}}{(1-z\bar{z})^{\frac{2}{\kappa}}}.$$

Corollary 6. *p = 2 case, for κ = 2:*

$$\mathbb{E}\left(\left|z \frac{f'_0(z)}{f_0(z)}\right|^2\right) = \frac{(1-z)(1-\bar{z})}{1-z\bar{z}}.$$

For the whole-plane SLE_κ map, let $\log \frac{f_0(z)}{z} = \sum_{n \geq 1} \gamma_n z^n$; then, for κ = 2, $\mathbb{E}(|\gamma_n|^2) = 2/n^2$, $\forall n \geq 1$.

Generalized Moments [D., Ho, Le, Zinsmeister '15]

$$\mathbb{E} \left(\frac{|f'_0(z)|^p}{|f_0(z)|^q} \right).$$

Theorem 7. For $p = 0$, the whole-plane SLE $_{\kappa}$ map $f_0(z)$ has a special moment

$$\mathbb{E} \left(|z^{-1} f_0(z)|^{\frac{(\kappa+4)(\kappa+2)}{2\kappa}} \right) = \frac{(1-z)^{\frac{\kappa+4}{\kappa}} (1-\bar{z})^{\frac{\kappa+4}{\kappa}}}{(1-z\bar{z})^{\frac{(\kappa+4)^2}{2\kappa}}}$$

Generalized Moments

Theorem 8. [D., Ho, Le, Zinsmeister '15]

$$\begin{aligned}\mathbb{E} \left(z_1^{\frac{q}{2}} \frac{(f'(z_1))^{\frac{p}{2}}}{(f(z_1))^{\frac{q}{2}}} \overline{\left[z_2^{\frac{q}{2}} \frac{(f'(z_2))^{\frac{p}{2}}}{(f(z_2))^{\frac{q}{2}}} \right]} \right) &= \frac{(1-z_1)^\gamma (1-\bar{z}_2)^\gamma}{(1-z_1\bar{z}_2)^\beta}, \\ \mathbb{E} \left(|z|^q \frac{|f'(z)|^p}{|f(z)|^q} \right) &= \frac{(1-z)^\gamma (1-\bar{z})^\gamma}{(1-z\bar{z})^\beta}, \quad \beta = \frac{\kappa}{2}\gamma^2\end{aligned}$$

on the *parabola* in the (p, q) plane

$$\gamma = \frac{2p-q}{1+\frac{\kappa}{2}}, \quad \frac{\kappa}{2}\gamma^2 - \left(2 + \frac{\kappa}{2}\right)\gamma + p = 0.$$

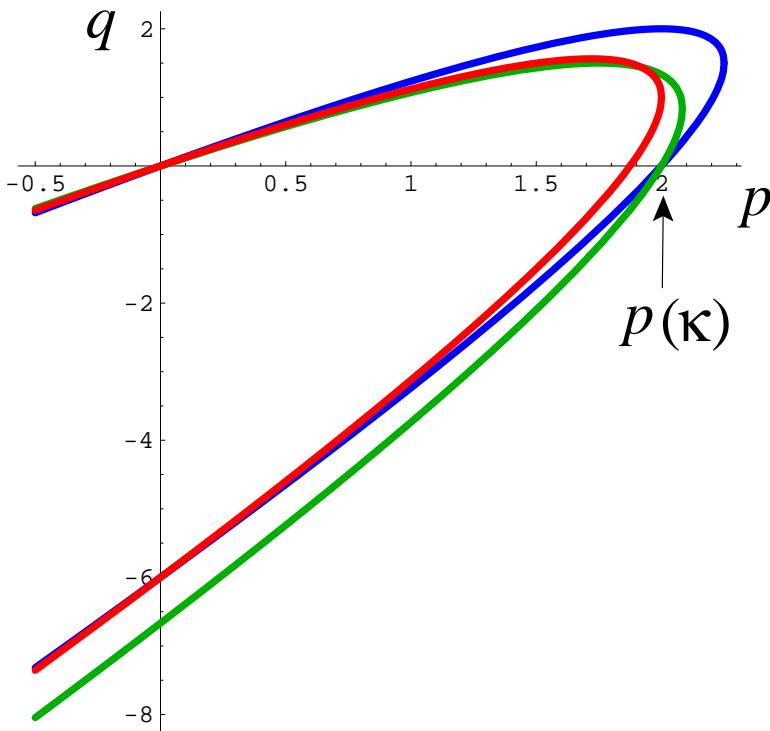


Figure 1: *Integral parabolic curves $(p, q)_\kappa$ of Theorem 7, for $\kappa = 2$ (blue), $\kappa = 4$ (red), and $\kappa = 6$ (green).* In addition to the origin, the $q = 0$ intersection point with the p -axis is at $p(\kappa) := (6 + \kappa)(2 + \kappa)/8\kappa$, with $p(2) = p(6) = 2$.

The B–S Equation

Beliaev and Smirnov (2005) obtained by martingale arguments the following equation for the *exterior whole-plane* case

$$(F(z) = F(re^{i\theta}), r \geq 1, \sigma = +1)$$

$$\begin{aligned} & p \left(\frac{r^4 + 4r^2(1 - r\cos\theta) - 1}{(r^2 - 2r\cos\theta + 1)^2} - \sigma \right) F + \frac{r(r^2 - 1)}{r^2 - 2r\cos\theta + 1} F_r \\ & - \frac{2r\sin\theta}{r^2 - 2r\cos\theta + 1} F_\theta + \Lambda F = 0. \end{aligned}$$

Proposition 1. *For the interior whole-plane Schramm (or Lévy)-Loewner evolution, the moments of the derivative modulus, $F(z) := \mathbb{E}(|f'_0(z)|^p)$, satisfy the same B–S equation, but with $\sigma = -1$, and $\Lambda = (\kappa/2)\partial^2/\partial\theta^2$ the generator of the driving Brownian process (or of the Lévy process).*

Holomorphic Coordinates

Switch to z, \bar{z} variables, instead of polar coordinates, and write $F(z)$ above as

$$F(z, \bar{z}) := \mathbb{E}(|f'_0(z)|^p) = \mathbb{E}[(f'_0(z))^{p/2}(\bar{f}'_0(\bar{z}))^{p/2}].$$

Using $\partial := \partial_z$, $\bar{\partial} := \partial_{\bar{z}}$, the equation then becomes

$$-\frac{\kappa}{2}(z\partial - \bar{z}\bar{\partial})^2 F + \frac{z+1}{z-1}z\partial F + \frac{\bar{z}+1}{\bar{z}-1}\bar{z}\bar{\partial} F - p \left[\frac{1}{(z-1)^2} + \frac{1}{(\bar{z}-1)^2} + (\sigma - 1) \right] F = 0.$$

Exterior/Interior whole-plane: $\sigma = \pm 1$.

- Consider the action of the differential operator $\mathcal{P}(D)$ on a function of the form $F(z, \bar{z}) = \varphi(z)\bar{\varphi}(\bar{z})P(z, \bar{z})$.
- Simplify it for the particular choice of a rotationally invariant $P(z, \bar{z}) := P(z\bar{z})$.
- Study the algebra generated by the action of $\mathcal{P}(D)$ on $\varphi(z) = \varphi_\alpha(z) := (1-z)^\alpha$, and $P(z\bar{z}) := (1-z\bar{z})^{-\beta}$, $\forall \alpha, \beta$.

Integral means spectrum

Definition 1. *The integral means spectrum of a conformal mapping f is the function defined on \mathbb{R} by*

$$\beta(p) := \overline{\lim}_{r \rightarrow 1} \frac{\log(\int_{\partial D} |f'(rz)|^p |dz|)}{\log(\frac{1}{1-r})}.$$

In the *stochastic* setting, one defines the *average* integral means spectrum

Definition 2.

$$\beta(p) := \overline{\lim}_{r \rightarrow 1} \frac{\log(\int_{\partial D} \mathbb{E} |f'(rz)|^p |dz|)}{\log(\frac{1}{1-r})}.$$

Define

$$\beta_0(p, \kappa) := -p + \frac{4+\kappa}{4\kappa} \left(4 + \kappa - \sqrt{(4+\kappa)^2 - 8\kappa p} \right),$$

$$\beta_{\text{lin}}(p, \kappa) := p - \frac{(4+\kappa)^2}{16\kappa}.$$

Average integral means spectrum $\bar{\beta}_0(p, \kappa)$ of the bulk of SLE_κ [D. '00; Binder & D. '03; Beliaev & Smirnov '05; Gwynne, Miller, Sun '14]

$$\begin{aligned}\bar{\beta}_0(p, \kappa) &= \beta_0(p, \kappa), \quad 0 \leq p \leq p_0^*(\kappa), \\ &= \beta_{\text{lin}}(p, \kappa), \quad p \geq p_0^*(\kappa), \\ p_0^*(\kappa) &:= \frac{3(4+\kappa)^2}{32\kappa}.\end{aligned}$$

Integral means spectra

The whole-plane SLE $_{\kappa}$, $f_0(z), z \in \mathbb{D}$, and its m -fold transforms,

$f_0^{[m]}(z) := z [f_0(z^m)/z^m]^{1/m}$, $m \geq 1$, have average integral means spectra

$\beta_m(p, \kappa)$ that exhibit a *phase transition* and are given, for $p \geq 0$, by

$$\beta_1(p, \kappa) = \max \left\{ \beta_0(p, \kappa), 3p - \frac{1}{2} - \frac{1}{2}\sqrt{1+2\kappa p} \right\},$$

$$\beta_2(p, \kappa) = \max \left\{ \beta_0(p, \kappa), 2p - \frac{1}{2} - \frac{1}{2}\sqrt{1+\kappa p} \right\},$$

$$\beta_m(p, \kappa) = \max \left\{ \bar{\beta}_0(p, \kappa), (1+2/m)p - \frac{1}{2} - \frac{1}{2}\sqrt{1+2\kappa p/m} \right\}.$$

The first spectrum β_1 has its transition point at

$$p^*(\kappa) := \frac{1}{16\kappa} \left((4+\kappa)^2 - 4 - 2\sqrt{4+2(4+\kappa)^2} \right) < p_0^*(\kappa).$$

[See also *Loutsenko & Yermolayeva '12, '13*]

Integral means spectrum: *Inner whole-plane SLE*

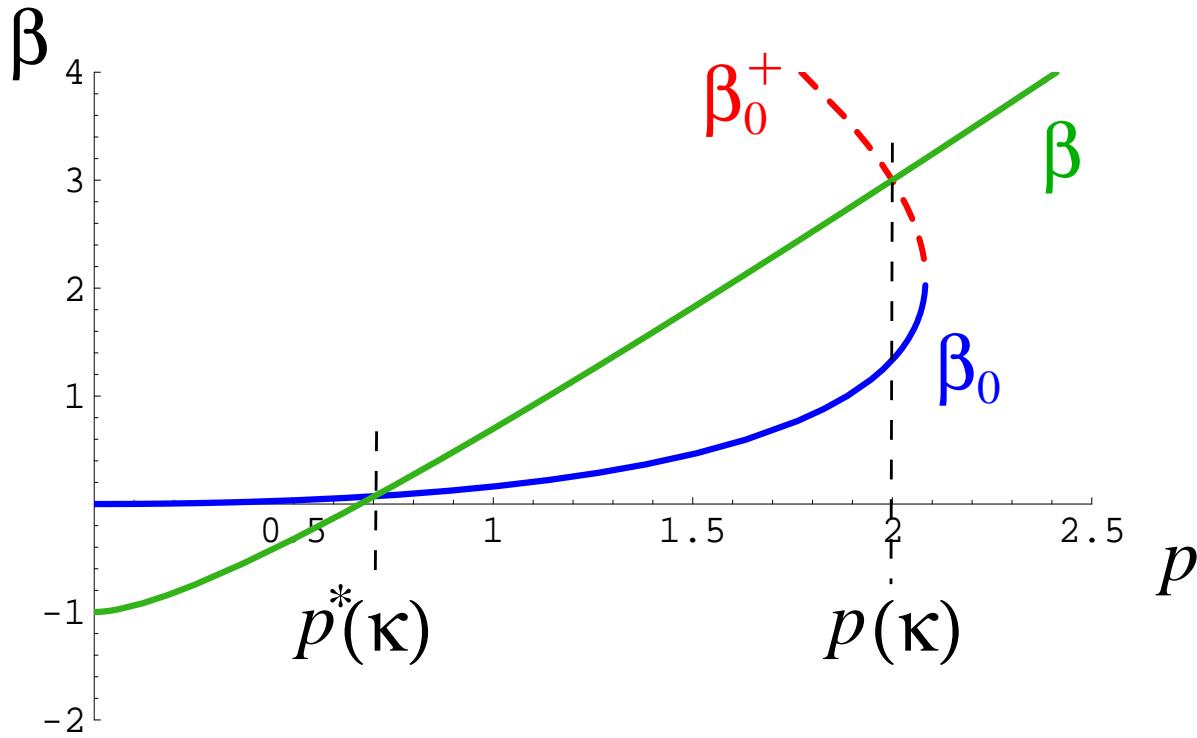


Figure 1: $\beta(p) = 3p - \frac{1}{2} - \frac{1}{2}\sqrt{1 + 2\kappa p}$

Theorem 9. [D., Nguyen, Nguyen, Zinsmeister '14] The average integral means spectrum $\beta(p, \kappa)$ of the unbounded whole-plane SLE $_{\kappa}$ has a phase transition at $p^*(\kappa)$ and a special point at $p(\kappa) := (6 + \kappa)(2 + \kappa)/8\kappa$ s.t.

$$\beta(p, \kappa) = \beta_0(p, \kappa), \quad 0 \leq p \leq p^*(\kappa);$$

$$\beta(p, \kappa) = 3p - \frac{1}{2} - \frac{1}{2}\sqrt{1 + 2\kappa p} \geq \beta_0(p, \kappa), \quad p^*(\kappa) \leq p \leq \min\{1 + \kappa/2, p(\kappa)\};$$

$$\beta(p, \kappa) \geq 3p - 1/2 - (1/2)\sqrt{1 + 2\kappa p}, \quad \min\{1 + \kappa/2, p(\kappa)\} \leq p \leq p(\kappa);$$

$$\beta(p, \kappa) \leq 3p - 1/2 - (1/2)\sqrt{1 + 2\kappa p}, \quad p(\kappa) < p.$$

- For $p > p^*(\kappa)$ the B–S solution ceases to be uniformly positive on $\partial\mathbb{D}$.
- Existence of a subsolution/supersolution for the parabolic operator $\mathcal{P}(D)[\Psi(z, \bar{z})\ell_{\delta}(z\bar{z})] \stackrel{\leq}{\geq} 0$ in some annulus of \mathbb{D} whose boundary includes $\partial\mathbb{D}$, corresponding respectively to $p \stackrel{\leq}{\geq} p(\kappa)$. Trial functions:
 $\Psi(z, \bar{z}) := (1 - z\bar{z})^{-\beta}|1 - z|^{2\gamma}, \quad \ell_{\delta}(z\bar{z}) := [-\log(1 - z\bar{z})]^{\delta}.$
- **γ -duality** in the space of hypergeometric B–S solutions on $\partial\mathbb{D}$.

Integral means spectrum: Outer whole-plane SLE

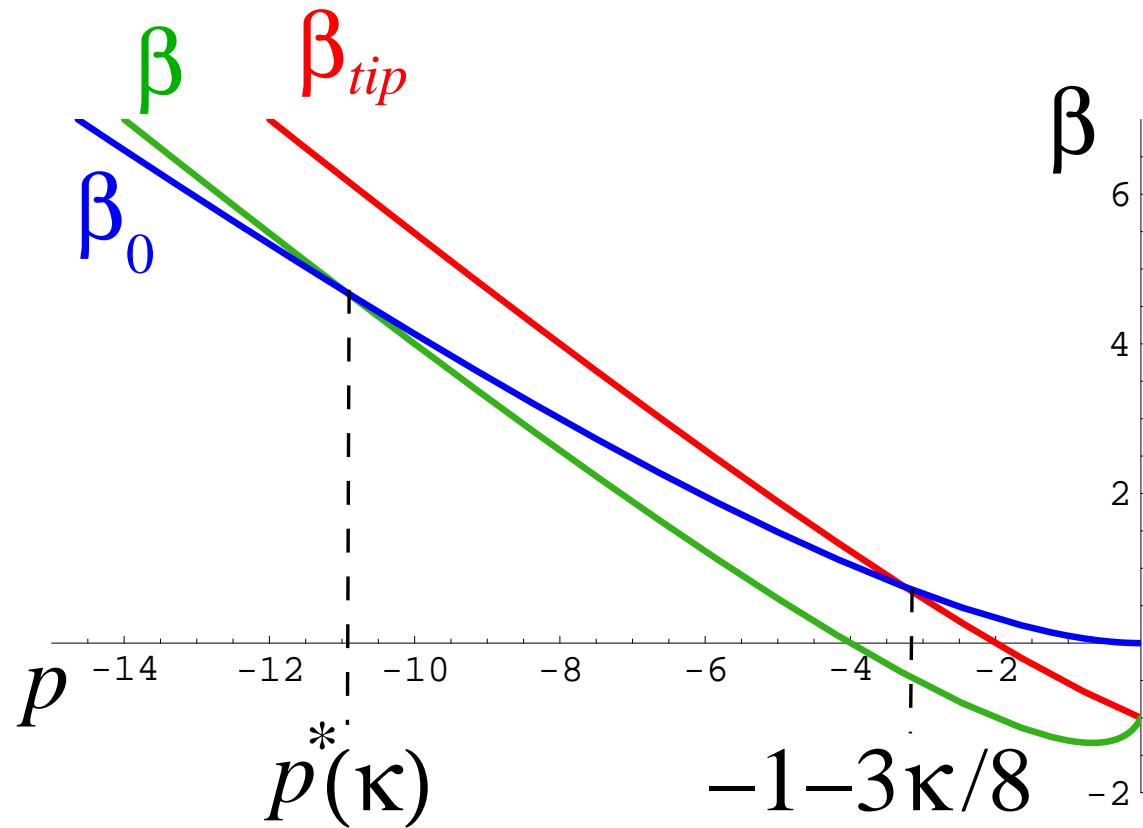


Figure 2: $\beta(p) = -p - \frac{1}{2} - \frac{1}{2}\sqrt{1 - 2\kappa p}$, $p^*(\kappa) = -(4 + \kappa)^2(8 + \kappa)/128$ (Beliaev, D., Zinsmeister '14)

Generalized Integral Means Spectrum

The *generalized integral means spectrum* $\beta(p, q)$, corresponding to the generalized moment, is the exponent such that

$$\int_{r\partial\mathbb{D}} \mathbb{E} \left(\frac{|f'(z)|^p}{|f(z)|^q} \right) |dz| \stackrel{(r \rightarrow 1^-)}{\asymp} (1 - r)^{-\beta(p, q)}.$$

The map:

$$\zeta \in \overline{\mathbb{D}} \mapsto f^{[-1]}(\zeta) := 1/f(z), z = 1/\zeta,$$

is the *exterior* whole-plane map from $\overline{\mathbb{D}} := \mathbb{C} \setminus \mathbb{D}$ to the slit plane considered by Beliaev and Smirnov.

$$\int_{r^{-1}\partial\mathbb{D}} \mathbb{E} \left(|f^{[-1]'}(\zeta)|^p \right) |d\zeta| = r^{2p-2} \int_{r\partial\mathbb{D}} \mathbb{E} \left(\frac{|f'(z)|^p}{|f(z)|^{2p}} \right) |dz|,$$

so the *standard* integral means spectrum at $(p, q = 0)$ for the *exterior* whole-plane map *coincides* with the *generalized* (p, q) integral means spectrum for $q = 2p$, for the *interior* whole-plane map.

m-fold transform

Let $f^{[m]}$ be the holomorphic branch of $[f(z^m)]^{1/m}$ whose derivative is equal to 1 at 0, f being a univalent function in \mathbb{D} such that $f(0) = 0$, $f'(0) = 1$. With inversion ($m = -1$), $m \in \mathbb{Z}^*$.

$$f^{[m]\prime}(z) = z^{m-1} f'(z^m) f(z^m)^{1/m-1},$$

$$\frac{|f^{[m]\prime}(z)|^p}{|f^{[m]}(z)|^q} = |z|^{p(m-1)} \frac{|f'(z^m)|^p}{|f(z^m)|^{p+\frac{q-p}{m}}},$$

so that

$$\beta_m(p, q) = \beta_1(p, q_m), \quad q_m := p + \frac{q-p}{m}.$$

For $q = 0$, $q_m = p(1 - 1/m)$.

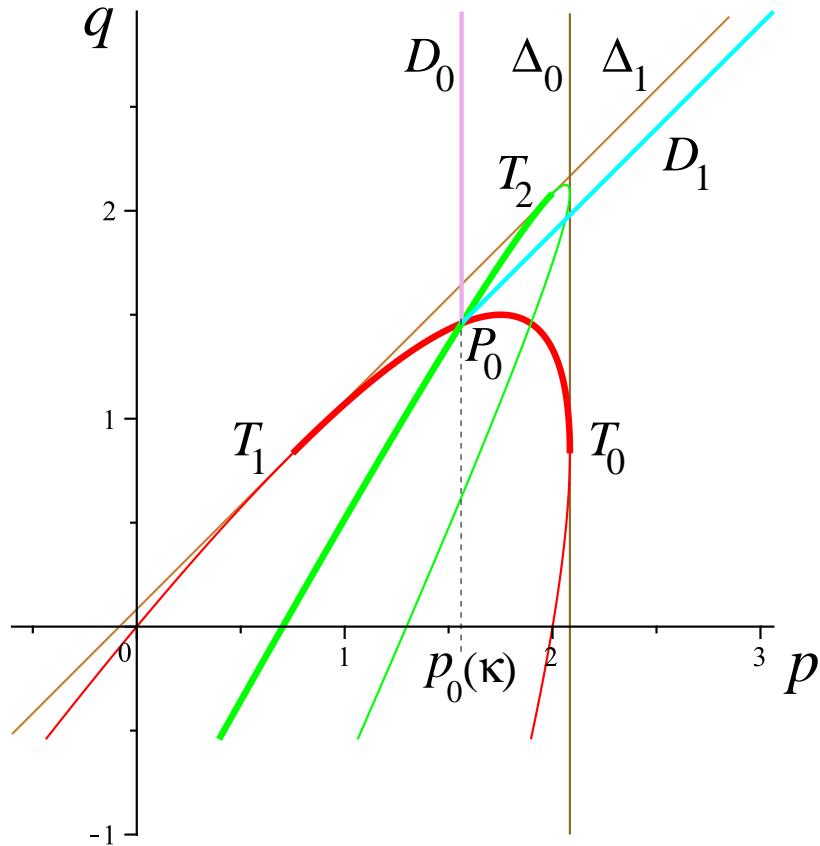
Average integral means spectrum for SLE_κ [D. '00; Hastings '02, Beliaev & Smirnov '05; Johansson & Lawler '09; Gwynne, Miller & Sun '14]:

$$\begin{aligned}\beta_{\text{tip}}(p, \kappa) &:= -p - 1 + \frac{1}{4}(4 + \kappa - \sqrt{(4 + \kappa)^2 - 8\kappa p}), \quad p \leq -1 - 3\kappa/8; \\ \beta_0(p, \kappa) &:= -p + \frac{4 + \kappa}{4\kappa} \left(4 + \kappa - \sqrt{(4 + \kappa)^2 - 8\kappa p} \right), \\ &\quad -1 - 3\kappa/8 \leq p \leq p_0^*(\kappa) := 3(4 + \kappa)^2 / 32\kappa; \\ \beta_{\text{lin}}(p, \kappa) &:= p - \frac{(4 + \kappa)^2}{16\kappa}, \quad p \geq p_0^*(\kappa).\end{aligned}$$

Whole-plane SLE_κ generalized integral means spectrum [D., Ho, Le, Zinsmeister '15]:

$$\beta_1(p, q, \kappa) := 3p - 2q - \frac{1}{2} - \frac{1}{2} \sqrt{1 + 2\kappa(p - q)}.$$

Matching bulk and generalized spectra

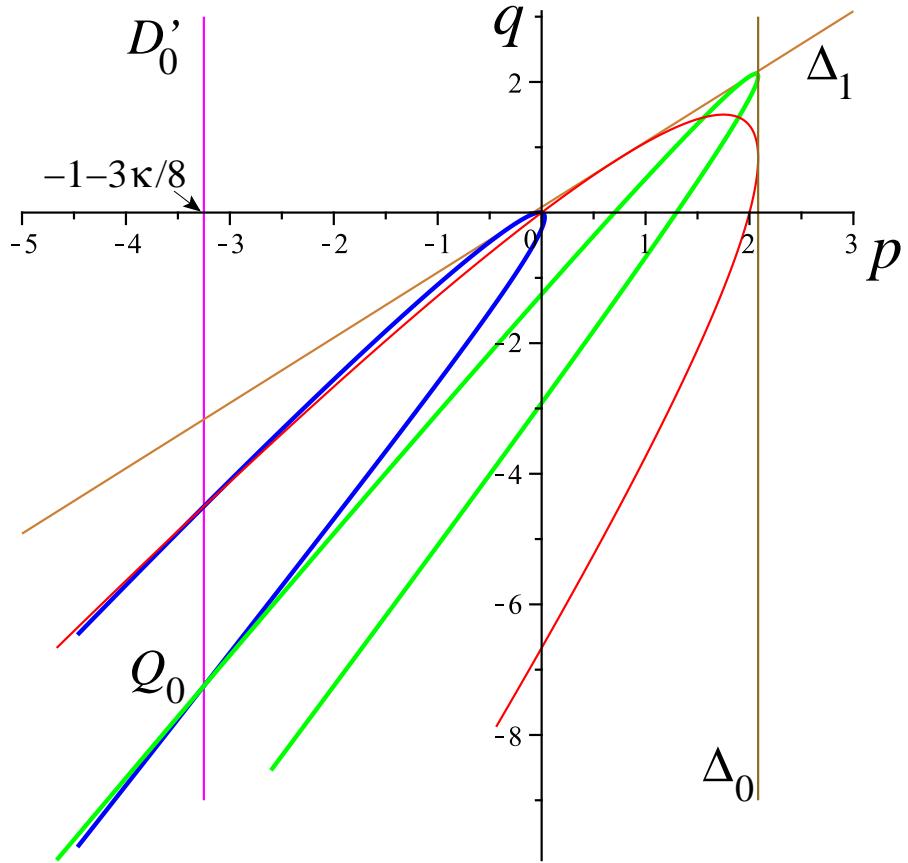


Red and Green (bold) parabolae: $\beta_0(p) = \beta_1(p, q)$.

Δ_0 and Δ_1 : boundaries for definition of β_0 and β_1 .

D_0 : β_0 to β_{lin} transition. D_1 : β_1 to β_{lin} transition.

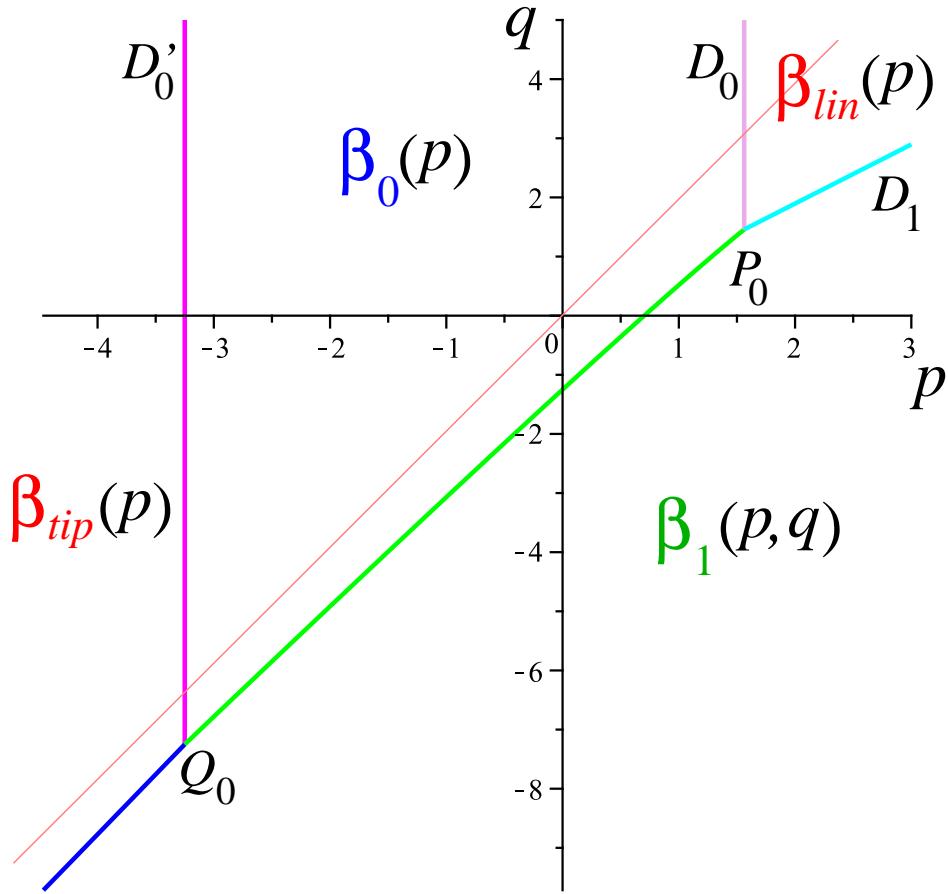
Matching tip and generalized spectra



Blue quartic: $\beta_{\text{tip}}(p) = \beta_1(p, q)$.

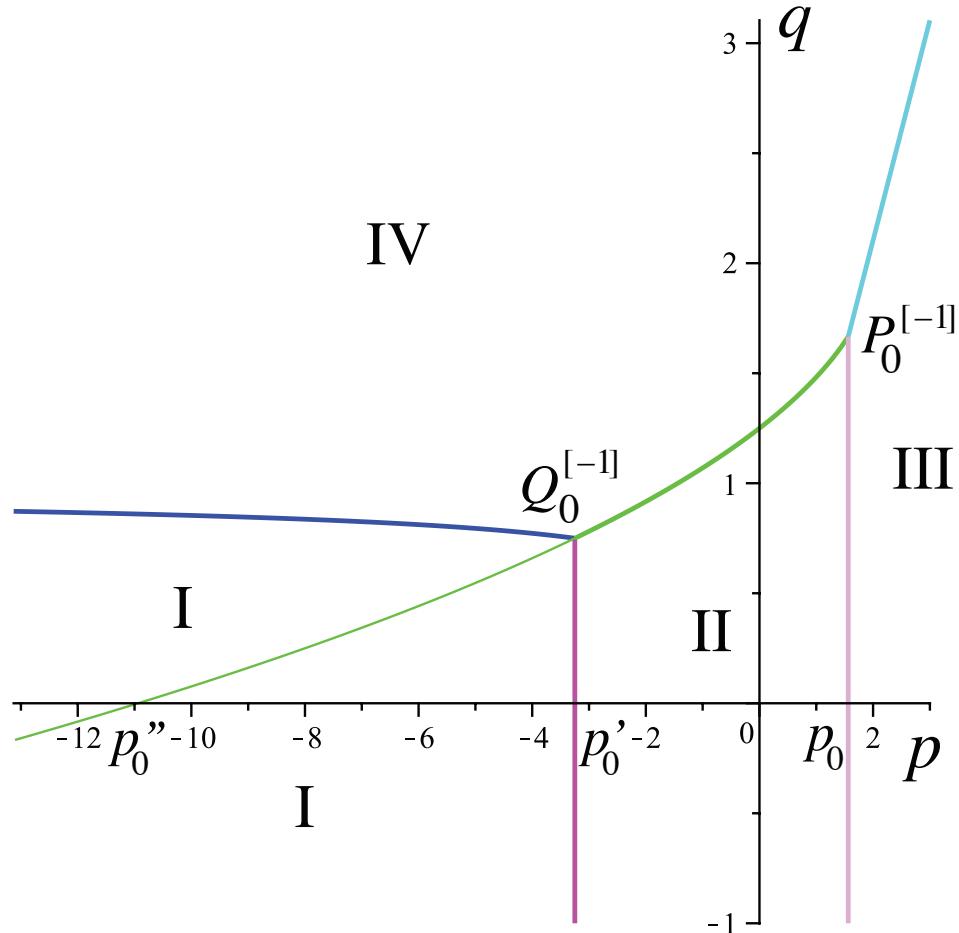
The tip spectrum holds to the left of the vertical line passing through Q_0 .

Generalized Multifractal Domains



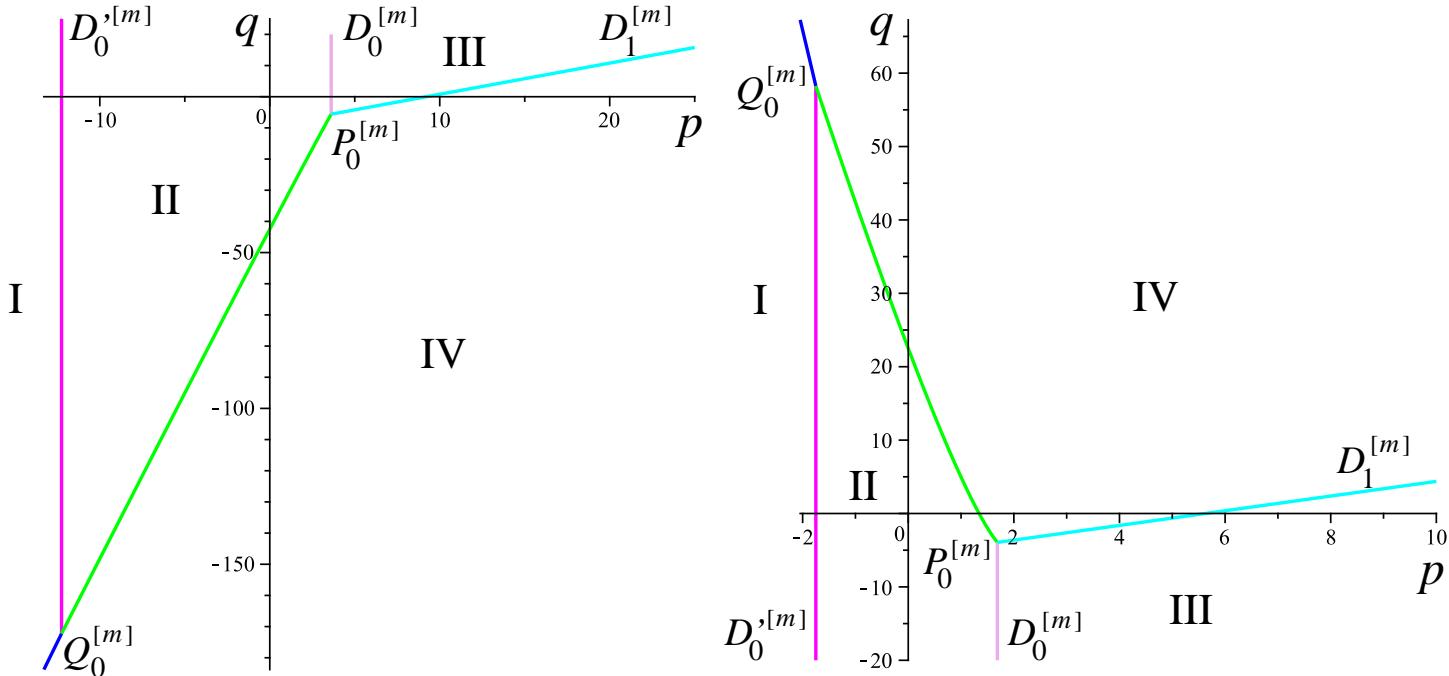
Multifractal domains for the inner whole-plane SLE. The B–S outer case corresponds to the straight line $q = 2p$ (coral).

m -Fold Spectrum, $m = -1$



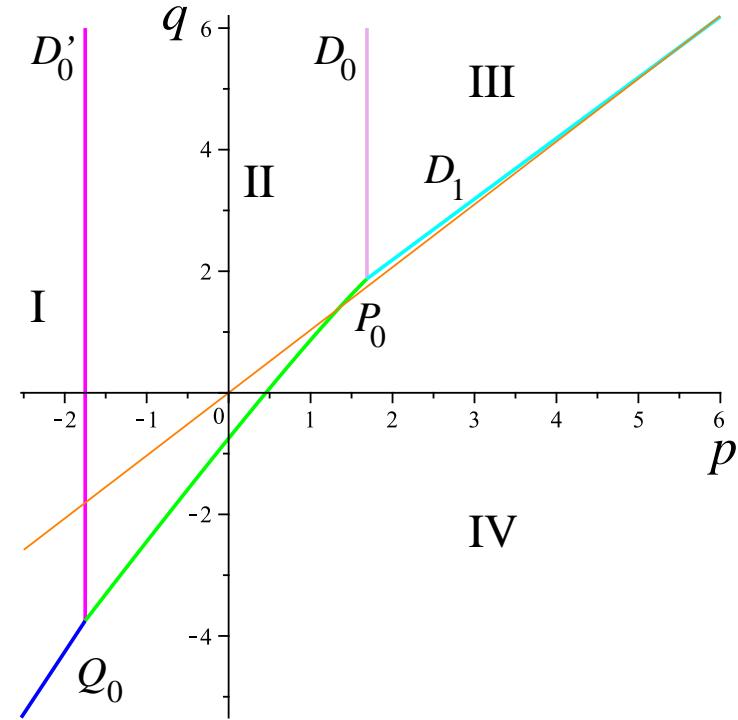
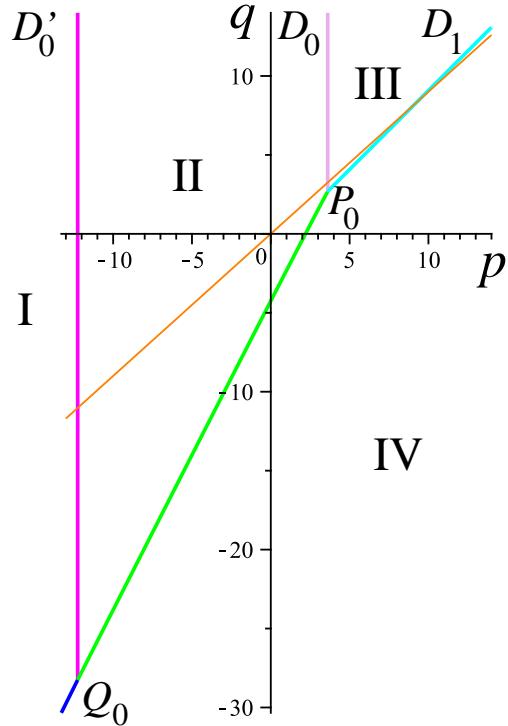
Phase diagram for the $m = -1$ case. (Here $\kappa = 6$.)

m -Fold Spectra, $m \in \mathbb{Z}$



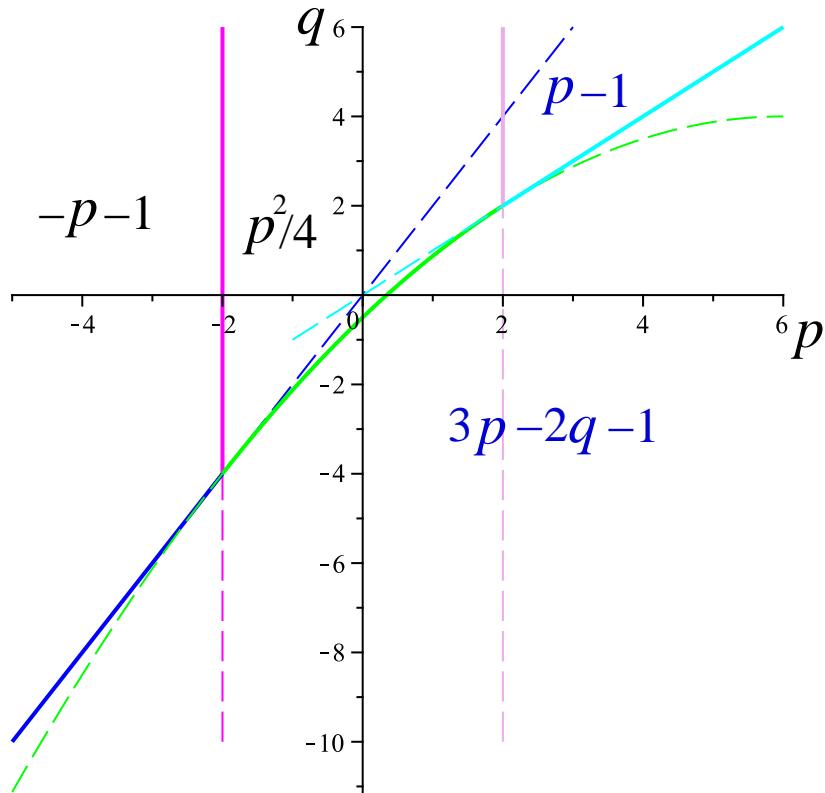
Phase diagram for m -fold whole-plane SLE $_{\kappa}$ and domains of validity of spectra β_{tip} (I), β_0 (II), β_{lin} (III), and β_m (IV). Left: For $m = +10$ and $\kappa = 30$, the $q = 0$ line successively crosses domains I, II, III and IV. Right: For $m = -30$ and $\kappa = 2$, it successively crosses I, II, IV and III.

m -Fold Spectra, $m \in \mathbb{Z}$



Phase diagram for m -fold whole-plane SLE_κ in the (p, q_m) -plane, and trajectories $q_m(p, q = 0)$ (coral color). Left: For $m = +10$ and $\kappa = 30$, the line successively crosses domains I, II, III and IV.
 Right: For $m = -30$ and $\kappa = 2$, it successively crosses I, II, IV and III.

Universal Generalized Spectrum



Universal generalized spectrum (including Brennan's and Kraetzer's conjectures).

Packing Spectrum

The *packing spectrum* [Makarov] is defined as

$$\textcolor{red}{s}(p) := \beta(p) - p + 1.$$

For the unbounded whole-plane SLE $_{\kappa}$, we have for $p \geq p^*(\kappa)$

$$\begin{aligned}\textcolor{red}{s}(p, \kappa) &= \beta(p, \kappa) - p + 1 \\ &= 2p + \frac{1}{2} - \frac{1}{2}\sqrt{1 + 2\kappa p}.\end{aligned}$$

Consider its *inverse function*

$$\begin{aligned}p = p(\textcolor{red}{s}, \kappa) &:= \frac{\textcolor{red}{s}}{2} + \frac{\kappa}{8} u_{\kappa}^{-1}(\textcolor{red}{s}), \\ u_{\kappa}^{-1}(\textcolor{red}{s}) &:= \frac{1}{2\kappa} \left(\kappa - 4 + \sqrt{(4 - \kappa)^2 + 16\kappa \textcolor{red}{s}} \right)\end{aligned}$$

(KPZ formula)

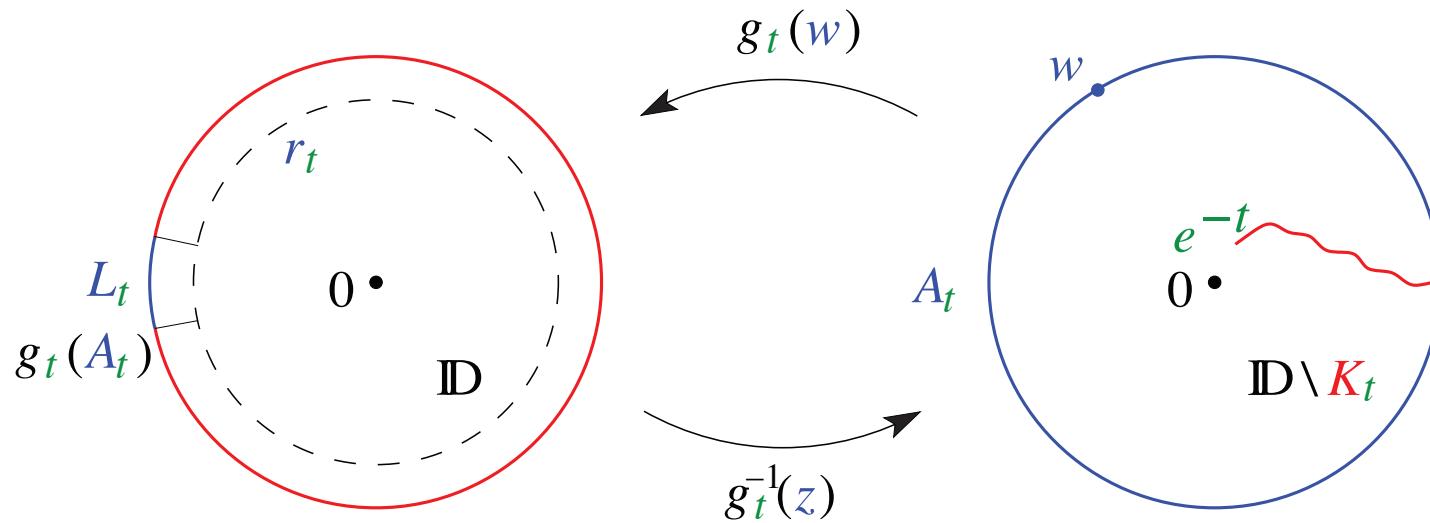
Relation to Tip & Derivative Exponents

(Non-standard) tip multifractal exponents obtained by quantum gravity [D. '00], corresponding geometrically to the extremity of an SLE_κ path avoiding a packet of s independent Brownian motions.

Differ from the ones associated to the standard SLE tip multifractal spectrum [Hastings '02, Beliaev & Smirnov '05, Johansson & Lawler '09].

Identical to the derivative exponents obtained for radial SLE_κ [Lawler, Schramm & Werner '01].

(Inverse) Radial SLE Map



$$f_0(z) \stackrel{\text{(law)}}{=} \lim_{t \rightarrow +\infty} [e^t g_t^{-1}(z)].$$

Derivative exponents

Lemma 1. (Lawler, Schramm, Werner '01) Let

$$A_t := \partial\mathbb{D} \setminus \overline{K}_t,$$

which is either an arc on $\partial\mathbb{D}$ or $A_t = \emptyset$. Let $s \geq 0$, and set

$$p = p(s, \kappa) := \frac{s}{2} + \frac{1}{16} \left(\kappa - 4 + \sqrt{(4 - \kappa)^2 + 16\kappa s} \right).$$

Let $\mathcal{H}(\theta, t)$ denote the event $\{w = \exp(i\theta) \in A_t\}$, and set

$$\begin{aligned} \mathcal{F}(\theta, t) &:= \mathbb{E} \left[|g'_t(\exp(i\theta))|^s 1_{\mathcal{H}(\theta, t)} \right], \\ q = q(s, \kappa) &:= u_\kappa^{-1}(s) = \frac{\kappa - 4 + \sqrt{(4 - \kappa)^2 + 16\kappa s}}{2\kappa}, \\ \mathcal{F}(\theta, t) &\asymp \exp(-pt) (\sin(\theta/2))^q, \quad \forall t \geq 1, \quad \forall \theta \in (0, 2\pi). \end{aligned}$$

Harmonic measure

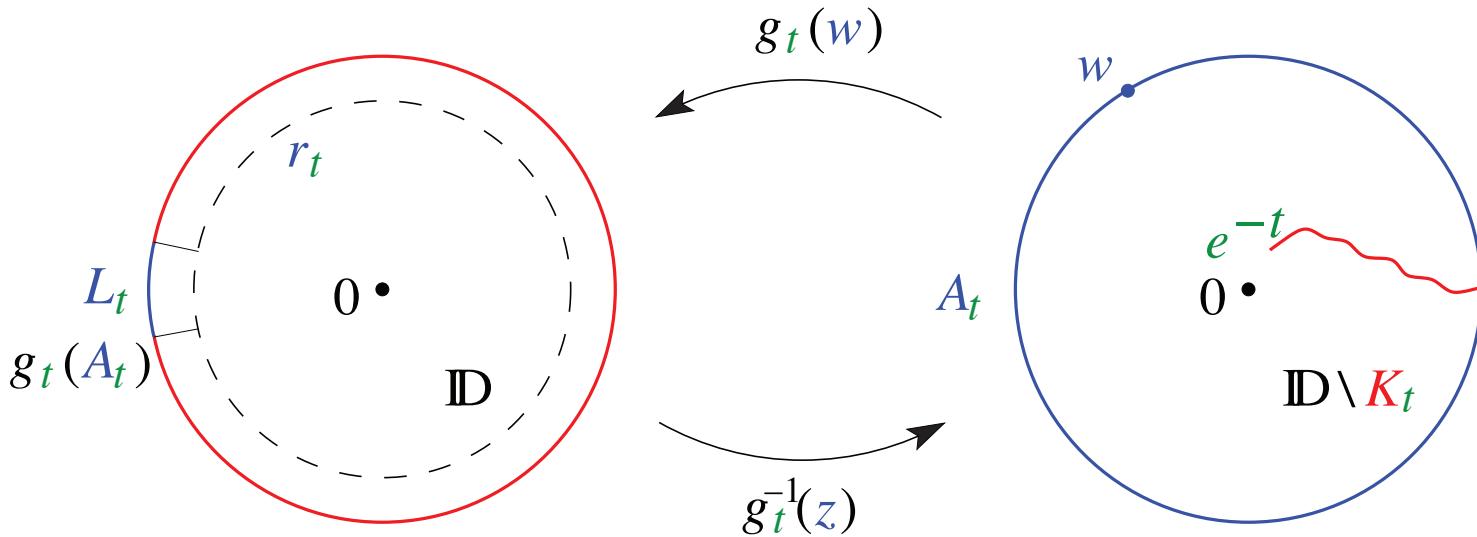


Figure 3: $f_0(z) \stackrel{\text{(law)}}{=} \lim_{t \rightarrow +\infty} e^t g_t^{-1}(z)$, where $z \mapsto g_t^{-1}(z)$ maps \mathbb{D} to the slit domain $\mathbb{D} \setminus K_t$ (K_t SLE hull). The length $L_t := |g_t(A_t)|$ of the image of the boundary set $A_t := \partial\mathbb{D} \setminus \overline{K_t}$ is the $(2\pi) \times$ the **harmonic measure** of A_t as seen from 0 in $\mathbb{D} \setminus K_t$, with $\mathbb{E}[L_t^s] \asymp e^{-p(s,\kappa)t}$ for $t \rightarrow +\infty$ [LSW '01].

Packing spectrum & derivative exponents

The average integral means spectrum involves evaluating, for the whole-plane SLE map $f_0(z)$, the integral

$$\mathbb{I}_p(r) := \int_{\partial D} \mathbb{E} [|f'_0(rz)|^p] |dz|,$$

on a circle of radius $r < 1$ concentric to $\partial\mathbb{D}$, and looking for the smallest $\beta(p)$ such that

$$(1-r)^{\beta(p)} \mathbb{I}_p(r) \stackrel{r \rightarrow 1}{<} +\infty.$$

For $p \geq p^(\kappa)$, the integrand behaves like a distribution and the circle integral concentrates in the vicinity of the pre-image point of infinity by the whole-plane map, $z_0 := f_0^{-1}(\infty) \in \partial\mathbb{D}$. In the large- t approximation to f_0 , this is the neighborhood of $g_t(A_t)$.*

Condensation

The circle integral there is the *restricted* integral in the image w -unit circle

$$I_p(t) := \int_{A_t} e^{pt} |g'_t(w)|^s |dw|; \quad s = s(p) = \beta(p) + 1 - p,$$

From LSW's Lemma above

$$\mathbb{E}[I_p(t)] \asymp \int_0^{2\pi} \sin^q(\theta/2) d\theta < +\infty.$$

By defining the *stochastic radius* $r_t := 1 - L_t \rightarrow 0$, this can be recast as

$$\mathbb{E} \left[(1 - r_t)^{\beta(p)} \int_{\partial\mathbb{D}} |\tilde{f}'_t(r_t z)|^p |dz| \right] \asymp 1, \quad t \rightarrow +\infty,$$

where $f_0(z) \stackrel{\text{(law)}}{=} \lim_{t \rightarrow +\infty} [\tilde{f}_t(z) := e^t g_t^{-1}(z)]$. This is (*formally*) reminiscent of the definition of the average integral means spectrum, hinting at why *the derivative exponent $p = p(s, \kappa)$ is the inverse function of the unbounded whole-plane packing spectrum $s(p, \kappa)$* . □