A bijective proof for Hurwitz formula

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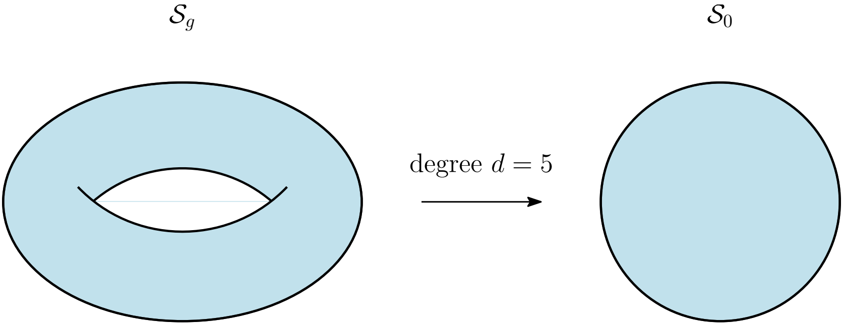
joint work with Enrica Duchi (LIAFA) and Gilles Schaeffer (LIX)

Conférence Itzykson, 11th June 2015

Hurwitz counting problem

I. in terms of ramified covers of $\mathcal{S}_0$



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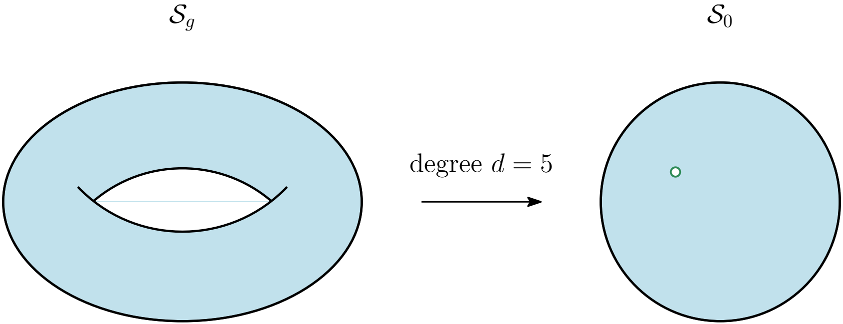

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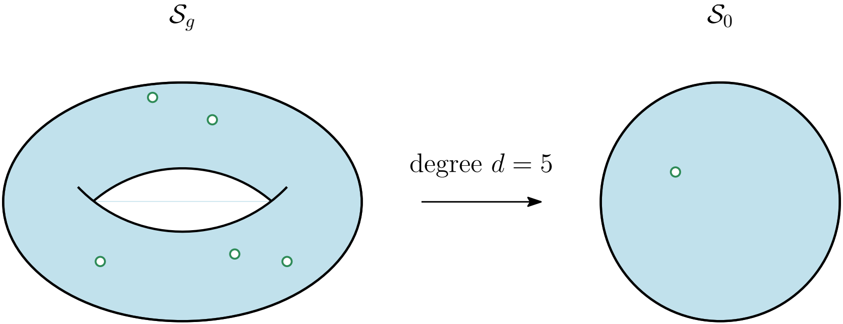

generic (unramified) point

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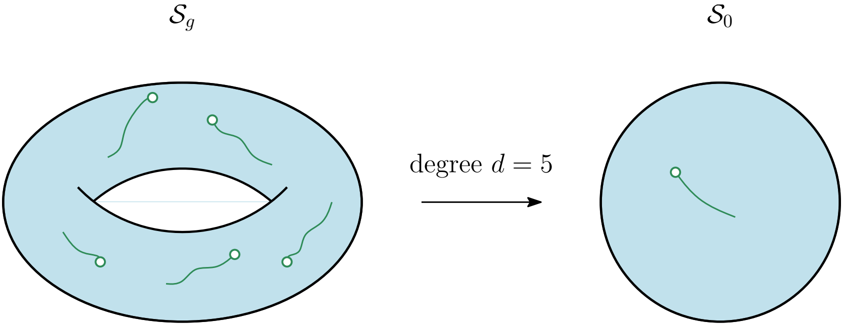

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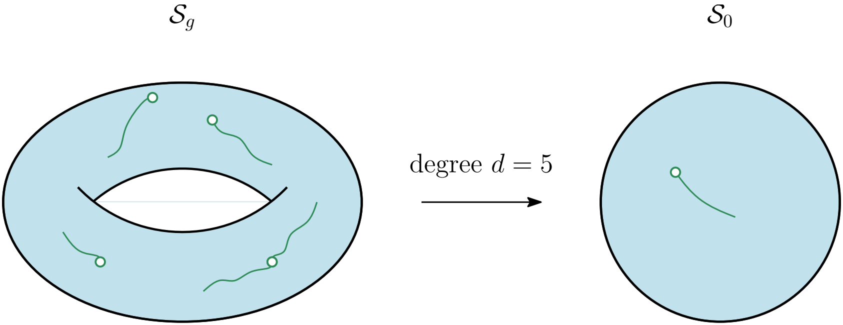

simple ramification point

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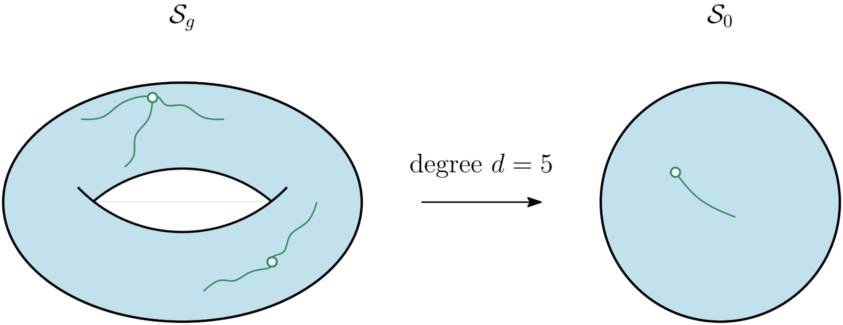

ramification point of type $(3, 2)$

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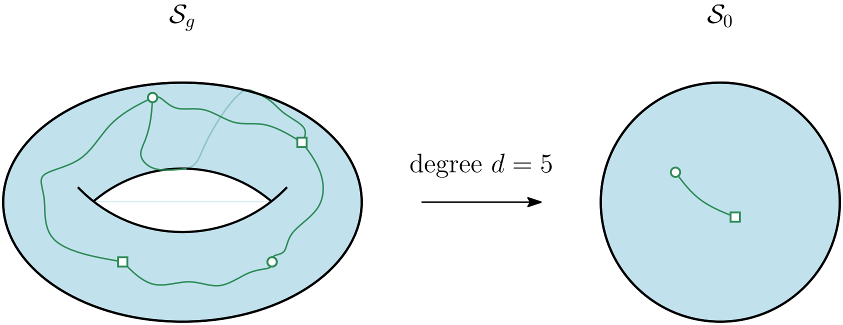

two ramification points of type $(3, 2)$

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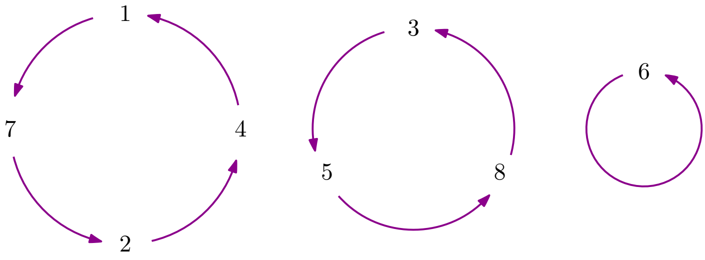
$\longrightarrow h_g(\mu, \nu)$

Hurwitz counting problem

II. in terms of products of permutations

cycle type of a permutation $\sigma \in \mathfrak{S}_d$: partition of d given by the lengths of the orbits

$\sigma = 7\,4\,5\,1\,8\,6\,2\,3 = (1\;7\;2\;4)\;(3\;5\;8)\;(6)$ has cycle type $(4, 3, 1)$:



$(1, 7, 2, 4)$ $(3, 5, 8)$ (6)

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let $\mu, \nu \vdash d$, $m = \ell(\mu)$, $n = \ell(\nu)$; count transitive $r+2$-tuples of permutations $(\rho, \sigma, \tau_1, \dots, \tau_r)$ s. t.

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Theorem (Hurwitz, 1891)

$$H_0(\mu) = d^{m-3} \cdot (m+d-2)! \cdot d! \cdot \prod_{i \geq 1} \frac{1}{m_i!} \left(\frac{i^i}{i!} \right)^{m_i}$$

(Some) proofs:

- [Hurwitz, 1891], reconstituted by [Strehl, 1996]
- [Goulden, Jackson, 1992]
- [Bousquet-Mélou, Schaeffer, 2000]
- [Lando, Zvonkine, 2000]
- [Borot, Eynard, 2010]

A very simple particular case: $H_0(d)$

Theorem (Dénes, 1959)

$$H_0(d) = (d-1)! \cdot d^{d-2}$$

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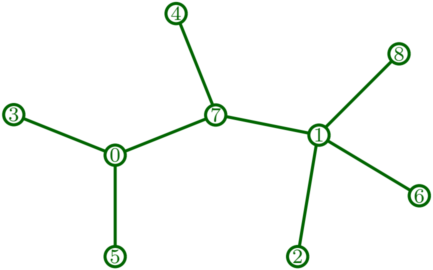

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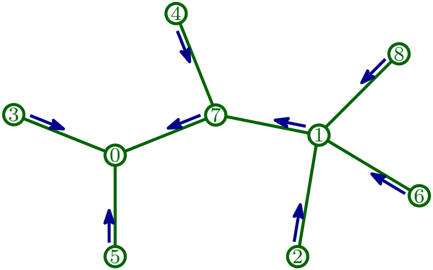

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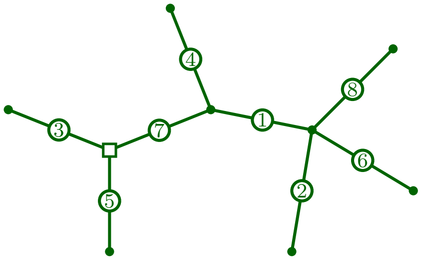

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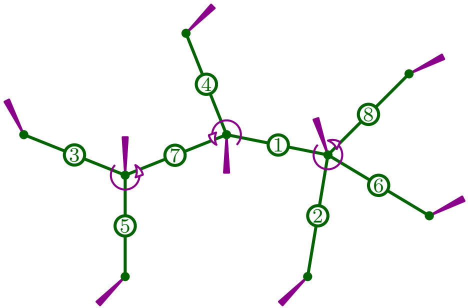

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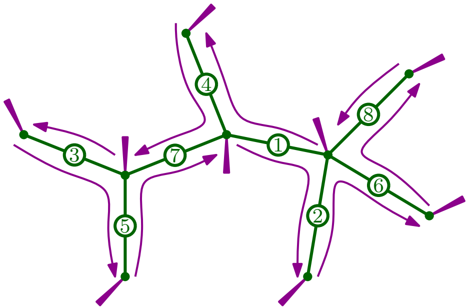

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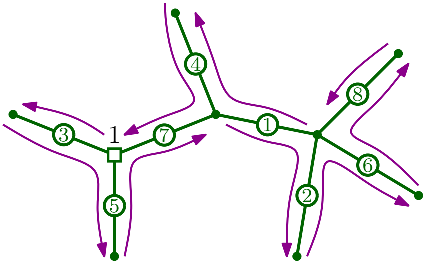

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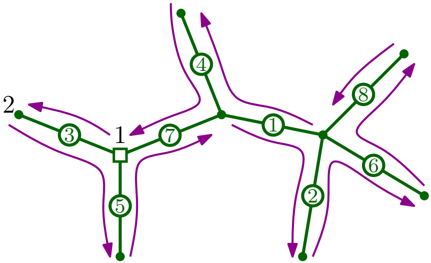

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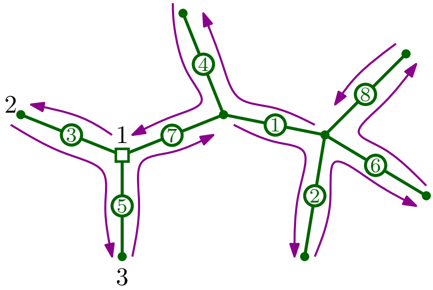

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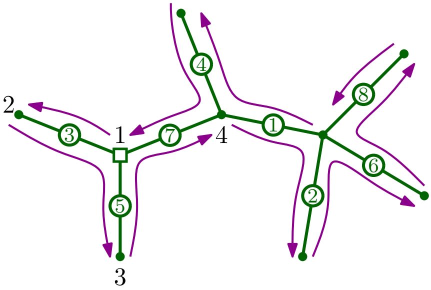

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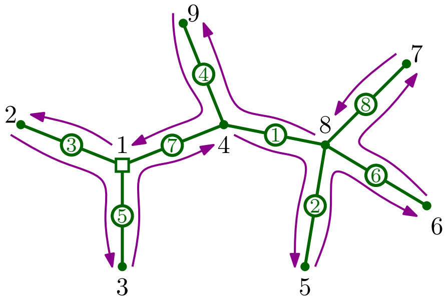

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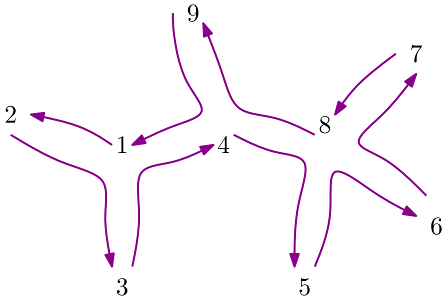

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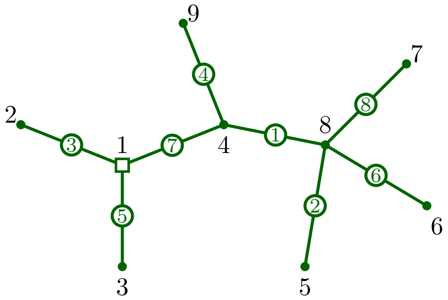


$$(1,2,3,4,5,6,7,8,9) = (7,8)(1,4)(6,8)(1,3)(4,9)(1,2)(5,8)(4,8)$$

First bijective proof of Hurwitz formula

Theorem (Duchi, P., Schaeffer, 2014)

The number of increasing maps with

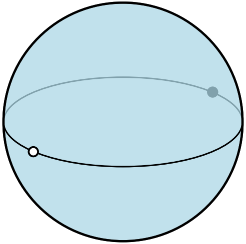
- d *labeled vertices*
- $m+d-2$ *labeled edges*
- m *faces, in which* μ *gives the distribution of descents*

is $H_0(\mu) = d^{m-3} \cdot (m+d-2)! \cdot d! \cdot \prod_{i\geq 1} \frac{1}{m_i!} \left(\frac{i^i}{i!}\right)^{m_i}$

(quite simple bijection with tree-like structures (cacti); uses a generic scheme [Albenque, P.] based on orientations without clockwise cycles)

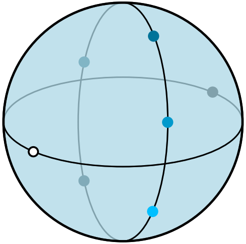
Drawbacks:

- more intricate for double Hurwitz numbers
- does not extend to higher genus



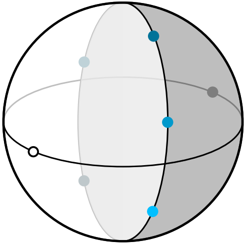
- two ramification points $\circ$ and $\bullet$

Hurwitz galaxies



- two ramification points ○ and ●
- r simple ramification points (blue shades)

Hurwitz galaxies



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- a closed curve through these r points, separating $\circ$ and $\bullet$

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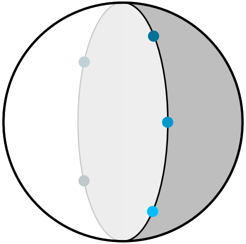

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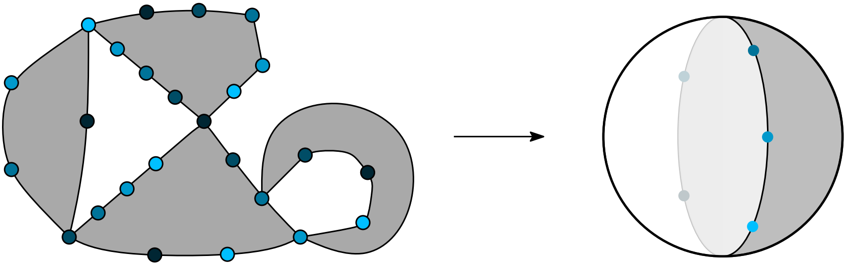

- degree $d = 6$, genus $g = 0$
- $d - 1 = 5$ vertices for each shade of blue (1 of degree 4)
- ramification over $\circ$: $\mu = (3, 2, 1)$
- ramification over $\bullet$: $\nu = (2, 2, 1, 1)$
- $\ell(\mu) = m = 3$ white faces , $\ell(\nu) = n = 4$ black faces

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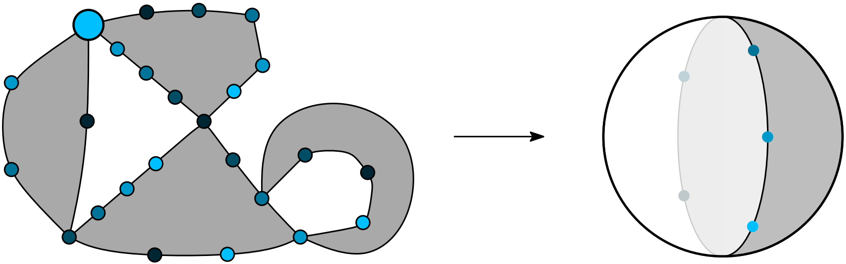

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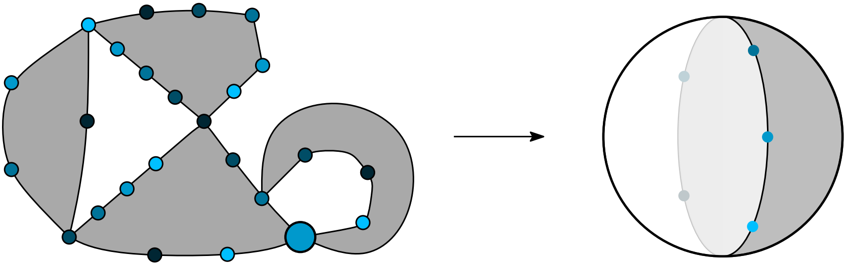

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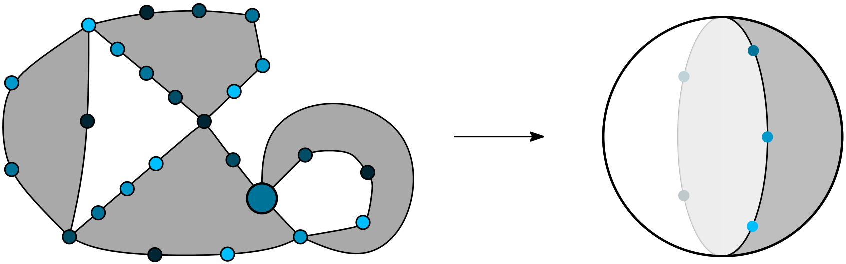

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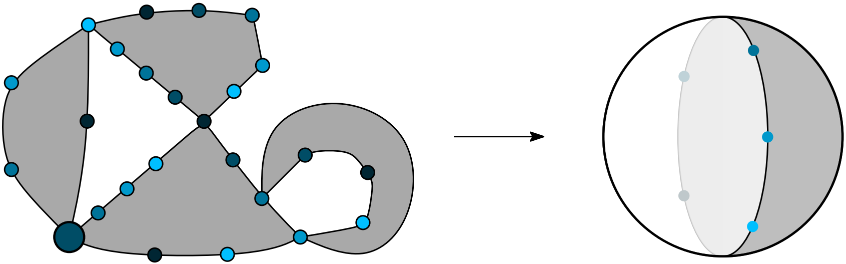

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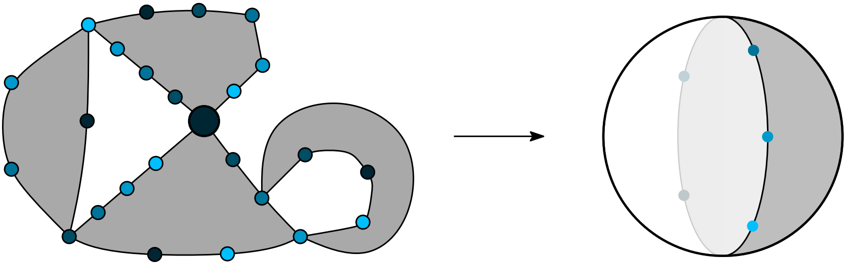

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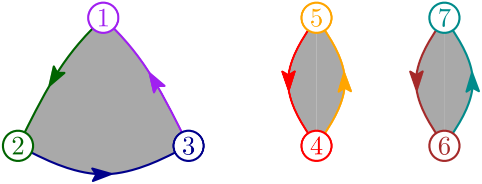
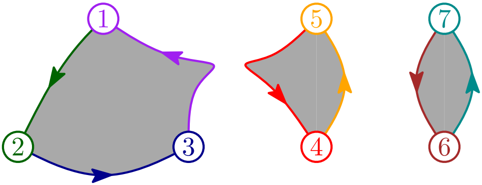
Hurwitz galaxies *vs* permutations

let $\rho = (1\ 2\ 3)\ (4\ 5)\ (6\ 7)$, $\tau_1 = (1\ 4)$, $\tau_2 = (1\ 6)$ and $\tau_3 = (2\ 7)$



Hurwitz galaxies *vs* permutations

let $\rho = (1\ 2\ 3)\ (4\ 5)\ (6\ 7)$, $\tau_1 = (1\ 4)$, $\tau_2 = (1\ 6)$ and $\tau_3 = (2\ 7)$



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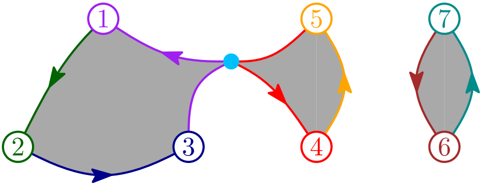

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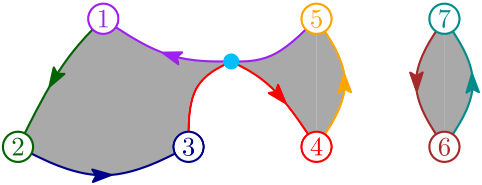

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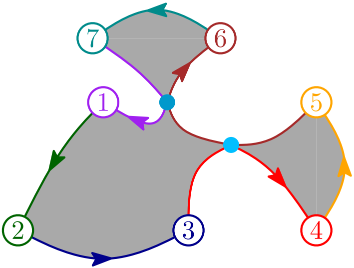

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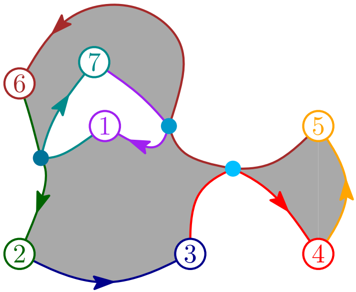

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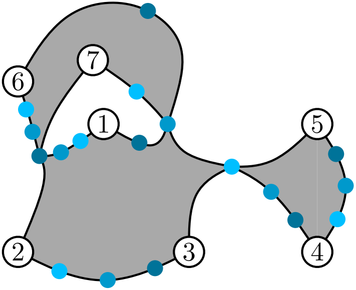

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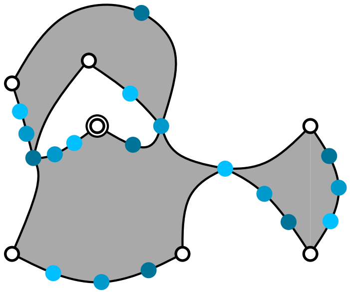

mark vertex 1, then remove the labels – we obtain r-Hurwitz galaxies $\rightarrow h_g^{\bullet}(\mu, \nu)$

Hurwitz galaxies *vs* permutations

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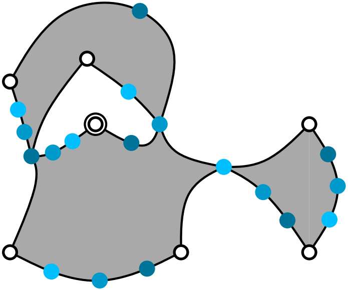
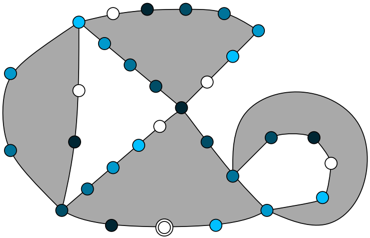


mark vertex 1, then remove the labels – we obtain r-Hurwitz galaxies $\longrightarrow h_g^\bullet(\mu, \nu)$

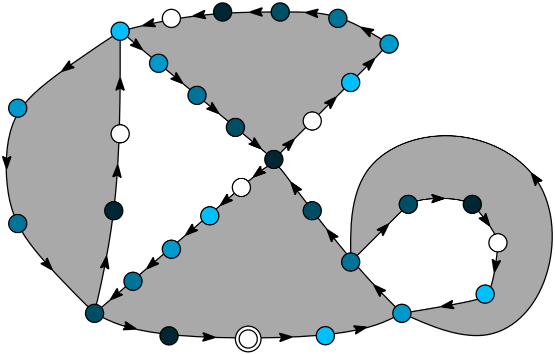
Lemma

$H_g(\mu, \nu) = (d-1)! \cdot h_g^\bullet(\mu, \nu)$ *and* $h_g^\bullet(\mu, \nu) = d \cdot h_g(\mu, \nu)$



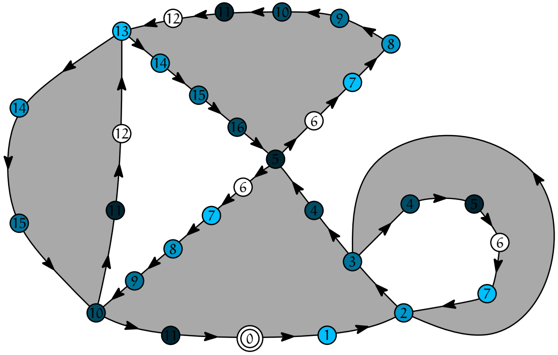
- rooted bicolored map of genus $g = m + n - 2 - r$
- $r + 1$ shades of vertices: d white vertices (including the root vertex), $d - 1$ of each other shade
- m white faces, n black faces, with face degree distribution given by μ and ν (up to a factor $r + 1$)

Orientation and distances in Hurwitz galaxies



- canonical orientation with the black face on the left
- distance according to this orientation
- $d(v) = c(v) \mod r + 1$

Orientation and distances in Hurwitz galaxies



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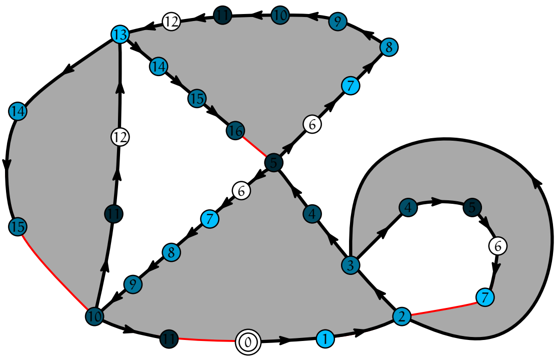
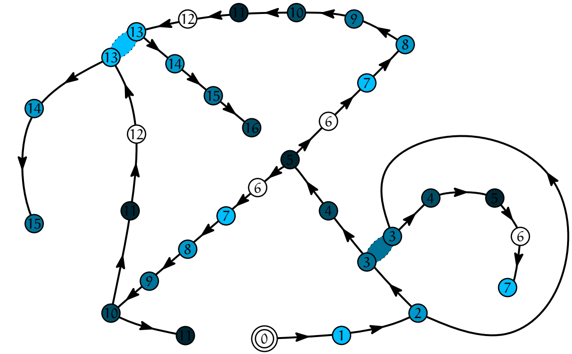
Geodesics of Hurwitz galaxies

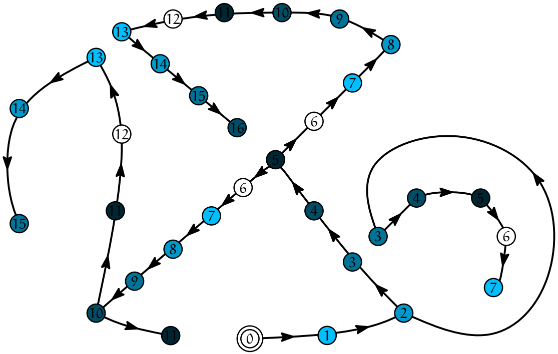


- at least one non geodesic edge around each face
- at least one incoming geodesic edge for each vertex



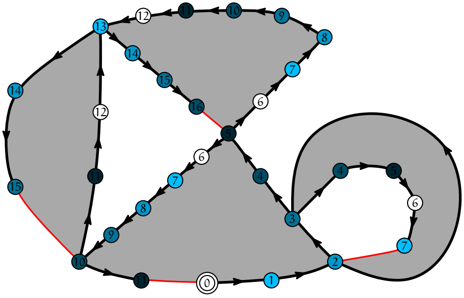
- keep only geodesic edges

$\Longrightarrow$ we get a (rooted, oriented) tree

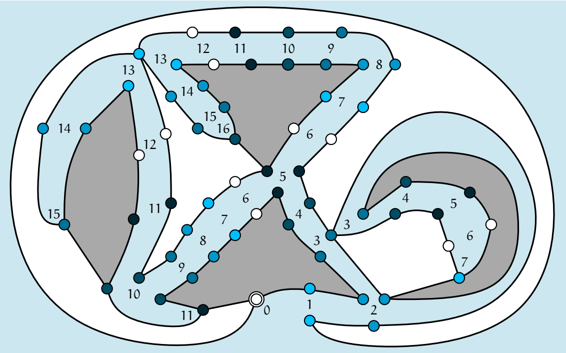


- keep only geodesic edges
- split vertices with 2 incoming geodesic edges

$\Longrightarrow$ we get a (rooted, oriented) tree



cut the surface along the tree



cut the surface along the tree $\Longrightarrow$ cactus of genus g with 1 boundary

- m_i white faces and n_i black faces of degree $i(r+1)$
- all vertices are incident to the boundary, with color condition
- exactly $d-1$ vertices of each color have at least one incoming white boundary edge

- corners of cacti can be canonically labeled
- this labeling has to be coherent on each vertex (automatic if $g = 0$)
- it has to be proper (the color of 0-labeled vertices is 0)

Lemma

Hurwitz galaxies of type (μ, ν) *are in bijection with proper coherent cacti of type* (μ, ν)

Lemma

each shift-equivalence class of coherent cacti contains $r + 1$ *elements, one of which is proper*

Theorem

Hurwitz galaxies of type (μ, ν) are in bijection with shift-equivalence classes of coherent edge-labeled Hurwitz mobiles of the same type.

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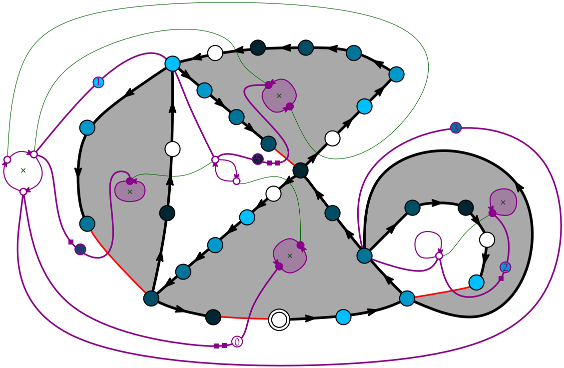

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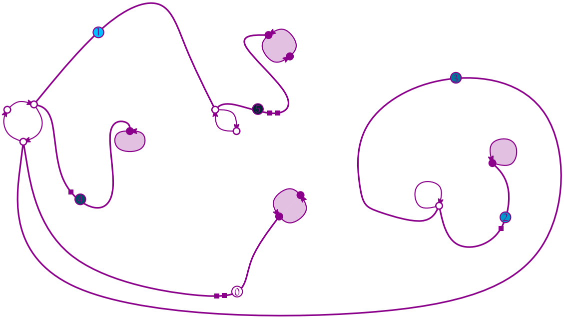

More precisely:

- d white nodes on m white polygons (m_i of size i)
- d black nodes on n black polygons (n_i of size i)
- $r + 1 = m + n - 1 + 2g$ weighted edges s.t.
    - 0-weight edges connect white nodes
    - positive weight edges connect a black and a white node
    - sum of weights of edges incident to an i-gon : i
- edge-labeled

Consequences if $g = 0$ and $\nu = 1^d$

- black polygons are 1-gons,
- they are incident to a single (positive) edge
- each white i-gon has i such pending edges
- white polygons and 0-weight edges form a Cayley cactus – m polygons attached by $m-1$ edges

Lemma

$$M_0(\mu, 1^d) = \binom{d+m-1}{m-1} \cdot \frac{1}{m} \binom{m}{m_1, \ldots, m_d} d^{m-2} \cdot \binom{d}{\mu} \cdot \prod_{i \geq 1} (i^i)^{m_i}$$

Corollary

Hurwitz formula

Consequences for genus 0 double Hurwitz numbers

skeleton of a mobile:

- contract each polygon
- remove 0-weight edges
- forget edge weights

the number of mobiles with a given skeleton is computable, leading to:

Theorem

$\bar{h}_0(x,y)$ *is an explicit sum of non negative terms indexed by skeletons.*

Consequences for genus 0 double Hurwitz numbers

Byproducts:

- Hurwitz formula again
- product formula in some special cases
- polynomiality in chambers (explicit sum of positive monomials)
- polynomiality of $h_0(\mu, \nu 1^{d-\nu})$