A bivariate two-point function for planar bicolored maps

Emmanuel Guitter (IPhT, CEA Saclay)

June 9, 2015

common work with É. Fusy

$$6, 13, 20, \cdots$$

$$6, 13, 20, \cdots$$

next 27

$$6, 13, 20, \cdots$$

next 27 (2022)

$$6, 13, 20, \cdots$$

next 27 (2022)

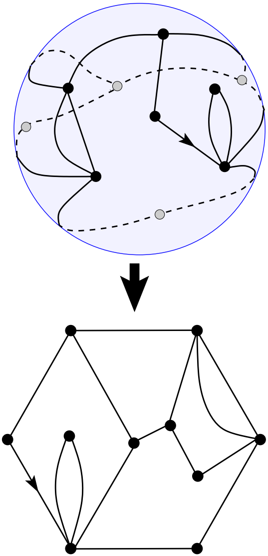
may be 26 ! (2021) (see OEIS®)

A109235	Floor(n*(e^2-1)/(e^2-2*e-1)).

6, 13, 20, 26, 33, 40, 46, 53, 60, 67, 73,

Planar bicolored maps

- Rooted planar map
  $\rightarrow$ canonical drawing in the plane



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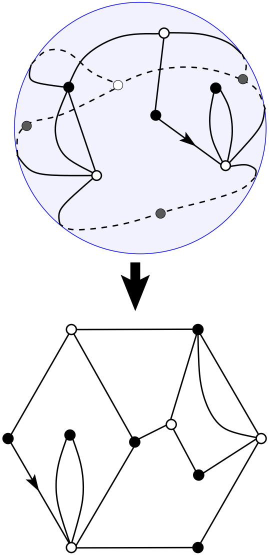
  $\rightarrow$ the map is bipartite (all faces of even degree)

  black-rooted (resp. white-rooted)
  $\rightarrow$ if the root vertex is black (resp. white)



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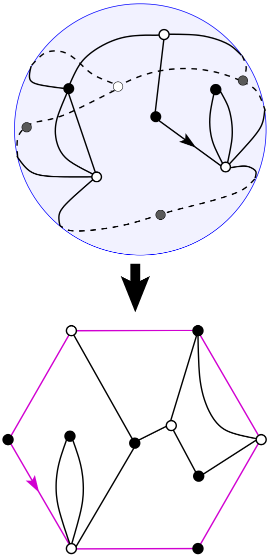
- with a boundary of length $2n$
  $\rightarrow$ the external face is of degree $2n$
  here $2n = 6$



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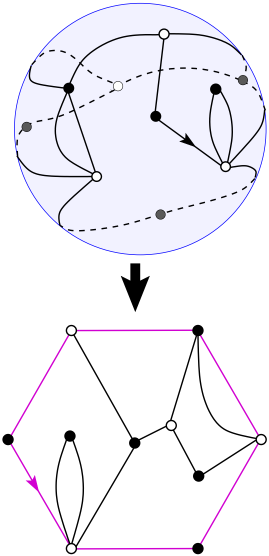
$\mathcal{M}_n^\bullet$ the set of black-rooted bicolored maps with a boundary of length $2n$

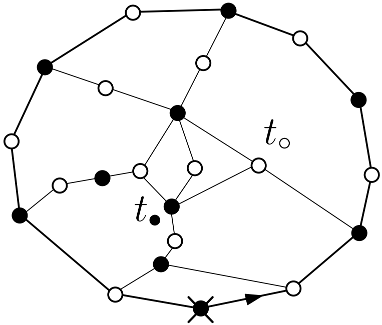


"Bivariate" generating function

weight $t_\bullet$ per black vertex
$t_\circ$ per white vertex
$+$ a standard control on the degree of the faces:
weight g_k per face of degree $2k$

$$F_n^\bullet(t_\bullet, t_\circ; g_1, g_2, \ldots) = \frac{1}{t_\bullet} \sum_{M \in \mathcal{M}_n^\bullet} w(M)$$

$$w(M) = t_\bullet^{\#black\,vert.} t_\circ^{\#white\,vert.} \times \prod_{\substack{inner \\ faces\ F}} g_{\frac{1}{2}degree(F)}$$



NB: By convention, no weight for the external face & no weight for the root vertex

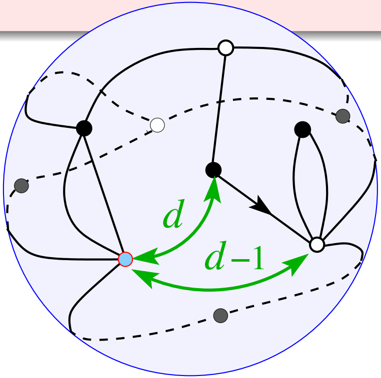
The g.f. for black-rooted bicolored maps is $G^\bullet = t_\bullet \sum_{n\geq 1} g_n F_n^\bullet$

The two-point function

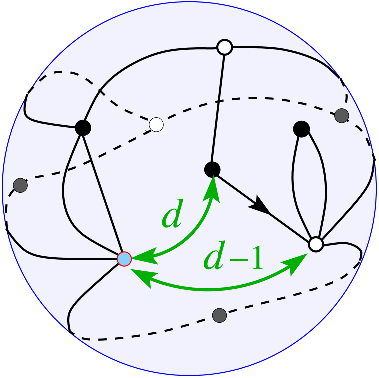
Pointed black-rooted map $\equiv$ black rooted map with an extra marked vertex of arbitrary (black or white) color

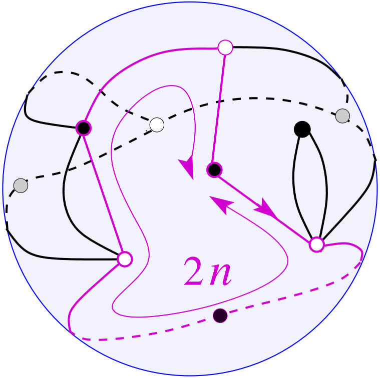
The distance-dependent two-point function

Def: $G^\bullet(d)$ is the g.f. of pointed black-rooted maps whose black (resp. white) extremities of the root edge are at distance d (resp. $d-1$) from the pointed vertex

$G^\bullet(d) =$ 

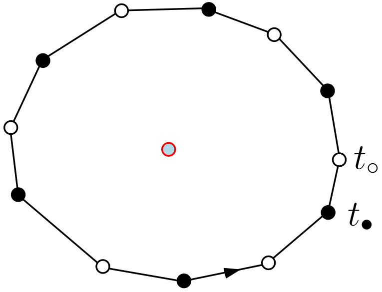
There is a direct connection between $G^\bullet(d)$ and $F_n^\bullet$





Pointed rooted maps

- Pointed black-rooted map with a boundary of length $2n$

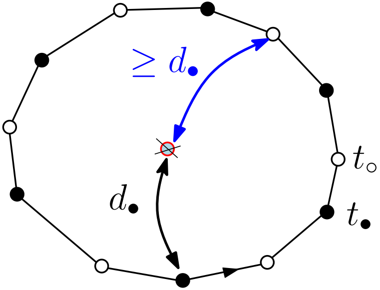


Pointed rooted maps

- Pointed black-rooted map with a boundary of length $2n$
- $\mathcal{M}_n^\bullet(d)$ set of these maps such that the distance $d_\bullet$ from the root vertex to the pointed vertex satisfies

$$d_\bullet \leq d$$

and all boundary vertices are at distance $\geq d_\bullet$ from the pointed vertex



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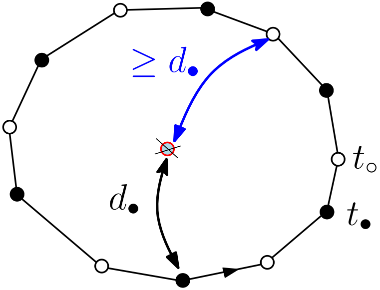
- Call $F_n^\bullet(d) = \sum_{M \in \mathcal{M}_n^\bullet(d)} \frac{1}{t_\circ(M)} w(M)$

with now the convention that the pointed vertex receives no weight (and no longer the root vertex)



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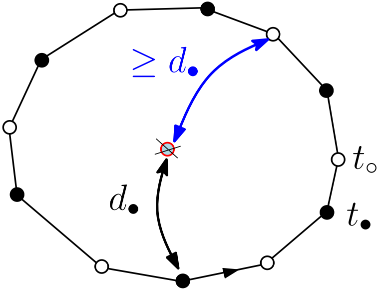
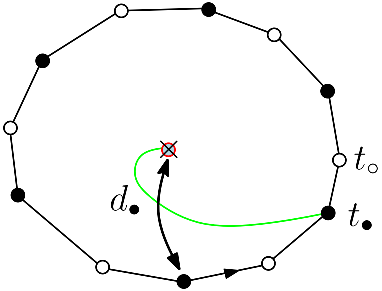
- $d = 0 \Leftrightarrow$ pointed vertex $=$ root vertex

$$\mathcal{M}_n^\bullet = \mathcal{M}_n^\bullet(0) \text{ and } F_n^\bullet = F_n^\bullet(0)$$



Enumeration by slice decomposition

- Take $M \in \mathcal{M}_n^\bullet(d)$ and draw the leftmost geodesic ($\equiv$ shortest) path from a boundary vertex to the pointed vertex



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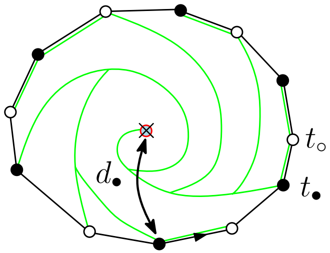
- Repeat the construction for all boundary vertices



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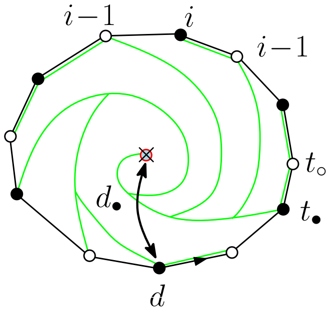
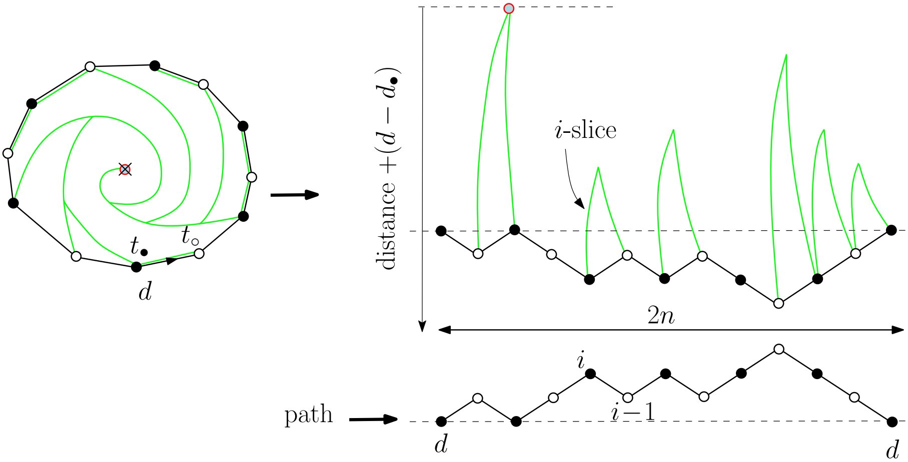
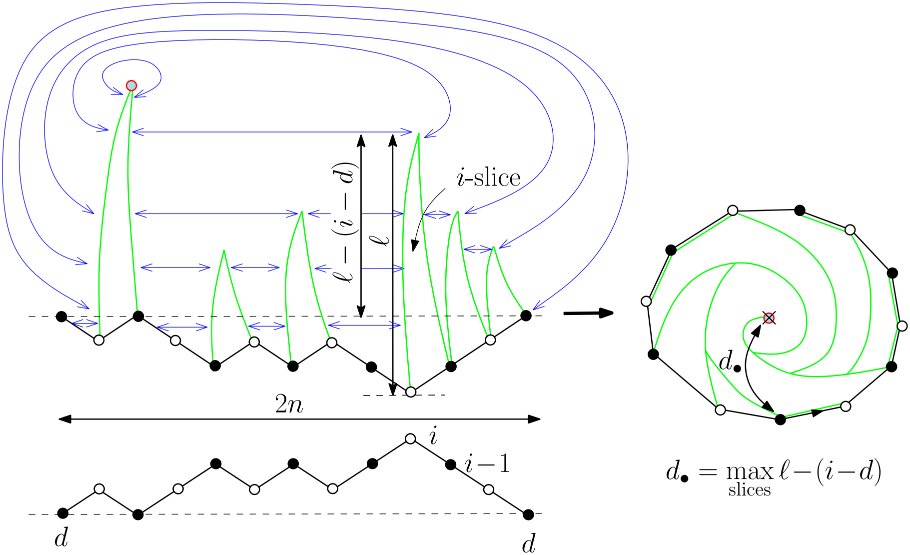


- Label each boundary vertex by $i =$ distance to ○ $+(d - d_\bullet)$.
  - for each sequence $i-1 \to i$, the geodesic follows the boundary
  - each sequence $i \to i-1$ gives rise to a new domain = "i-slice"

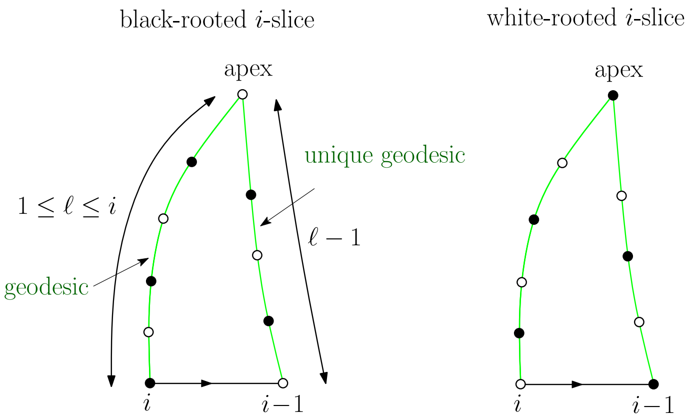


Path of length $2n$ made of ± 1 steps, with total height change 0, each "descending step" $i \to i-1$ equipped with an i-slice



$d_\bullet \leq d \leftrightarrow$ "height" ℓ of an i-slice such that $\ell \leq i$

Slices

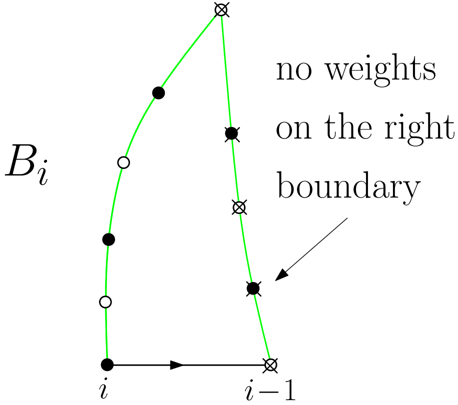


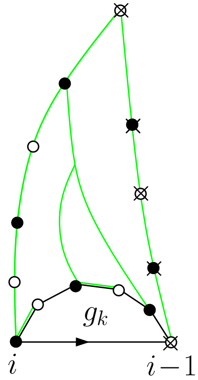
- left boundary = geodesic, of length ℓ, $\quad 1 \leq \ell \leq i$
- right boundary = unique geodesic, of length $\ell-1$

NB: i is only an upper bound on the length of the left boundary of the slice

Call $B_i \equiv B_i(t_\bullet, t_\circ, \{g_k\}_{k\geq 1})$ (resp. W_i) the g.f. for black-rooted (resp. white-rooted) i-slices

For a proper counting, put no weights on the right boundary





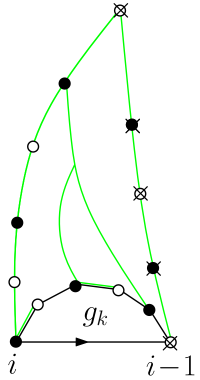
B_i and W_i are solution of the (non linear) system

$$B_i = t_\bullet + \sum_{k\geq 1} g_k Z^{\bullet\circ}_{i,i-1}(2k-1, \{B_j\}_{j\geq 1}, \{W_j\}_{j\geq 1})$$

$$W_i = t_\circ + \sum_{k\geq 1} g_k Z^{\circ\bullet}_{i,i-1}(2k-1, \{B_j\}_{j\geq 1}, \{W_j\}_{j\geq 1})$$

for $i \geq 1$ with $B_0 = W_0 = 0$.

where $Z^{\bullet\circ}_{i,i-1}(2k-1,\{B_j\}_{j\geq 1},\{W_j\}_{j\geq 1})$ denotes the g.f. for paths of length $2k-1$ from black height i to white height $i-1$ with weights B_j (resp. W_j) attached to each descending step $j \to j-1$ starting at a black (resp. a white) vertex



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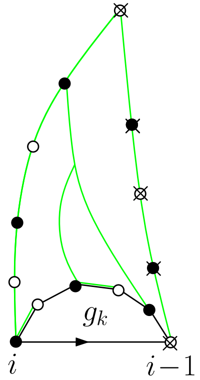
for $i \geq 1$ with $B_0 = W_0 = 0$.

$\rightarrow$ two independent systems:
- one relating W_i with odd i and B_i with even i

- one relating W_i with even i and B_i with odd i

$\rightarrow$ how to solve them ?



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for $i \geq 1$ with $B_0 = W_0 = 0$.

We can shift all the path heights by i (i.e. consider paths from 0 to -1) provided we attach weights B_{j+i} and W_{j+i} to $j \to j-1$ steps

Sending $i \to \infty$, B_i and W_i tend to B and W respectively, which are slice g.f. with no bound on the boundary length, determined by the (closed) system

$$B = t_\bullet + \sum_{k \geq 1} g_k \mathbb{Z}^{\bullet\circ}_{0,-1}(2k-1; B, W)$$
$$W = t_\circ + \sum_{k \geq 1} g_k \mathbb{Z}^{\circ\bullet}_{0,-1}(2k-1; B, W) \ .$$

The path g.f. $\mathbb{Z}$ now involve homogeneous weights: B (resp. W) attached to any descending step starting with a black (resp. a white) vertex

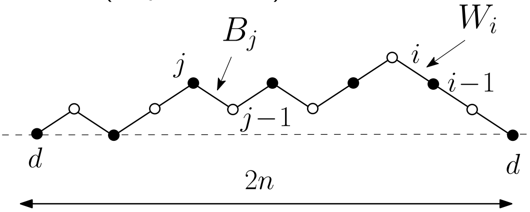
Back to $F_n^\bullet$

The slice decomposition allows us to relate B_i, W_i and $F_n^\bullet$:

- We have

$$F_n^\bullet(d) = Z_{d,d}^{\bullet\bullet+}(2n, \{B_i\}_{i\geq 1}, \{W_i\}_{i\geq 1})$$

  where $Z_{d,d}^{\bullet\bullet+}(2n, \{B_i\}_{i\geq 1}, \{W_i\}_{i\geq 1})$ denotes the g.f. for paths of length $2n$ from black height d to black height d, remaining above d, with weight B_i (resp. W_i) attached to any descending step $i \to i-1$ starting at a black (resp. a white) vertex



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$$F_n^\bullet = Z_{0,0}^{\bullet\bullet+}(2n, \{B_i\}_{i\geq 1}, \{W_i\}_{i\geq 1})$$

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and therefore

$$\sum_{n\geq 0} F_n^\bullet z^n = \cfrac{1}{1 - z\cfrac{W_1}{1 - z\cfrac{B_2}{1 - z\cfrac{W_3}{1 - z\cfrac{B_4}{1 - \cdots}}}}}$$

NB: involves only W_i with odd i and B_i with even i

Slice generating functions can be obtained from $F_n^\bullet$

Indeed, a standard result of the continued fraction theory (here of Stieltjes-type) says that

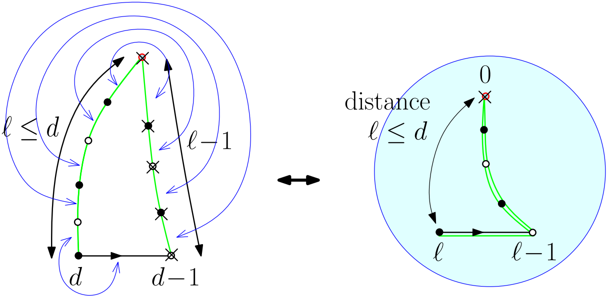
$$B_{2i} = \frac{h_i^{(0)}}{h_{i-1}^{(0)}} \Big/ \frac{h_{i-1}^{(1)}}{h_{i-2}^{(1)}} \qquad W_{2i-1} = \frac{h_{i-1}^{(1)}}{h_{i-2}^{(1)}} \Big/ \frac{h_{i-1}^{(0)}}{h_{i-2}^{(0)}}$$

in terms of the Hankel determinants

$$h_i^{(0)} = \det(F_{n+m}^\bullet)_{0 \le n,m \le i} \qquad h_i^{(1)} = \det(F_{n+m+1}^\bullet)_{0 \le n,m \le i}$$

To compute the other parity, simply exchange $t_\bullet$ and $t_\circ$

Back to the two-point function



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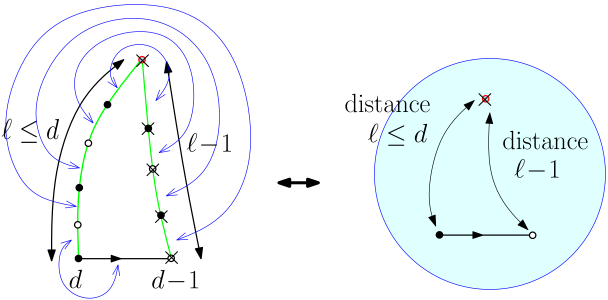
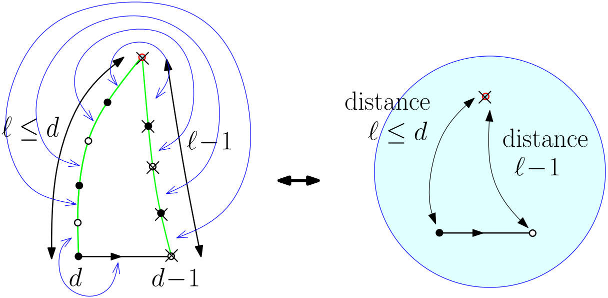


$$B_d = t_\bullet + \sum_{\ell \leq d} \frac{G^\bullet(\ell)}{(\delta_{\ell,\text{even}} t_\bullet + \delta_{\ell,\text{odd}} t_\circ)}$$

Back to the two-point function



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The twopoint function can be obtained from the slice g.f.

$$G^\bullet(d) = t_\circ (B^\bullet_d - B^\bullet_{d-1}), \qquad t_\circ = (\delta_{d,\mathrm{even}} t_\bullet + \delta_{d,\mathrm{odd}} t_\circ)$$

The recipe

1. Take a known formula for $F_n^\bullet$

2. Compute the Hankel determinants to get a formula for B_d (and W_d)

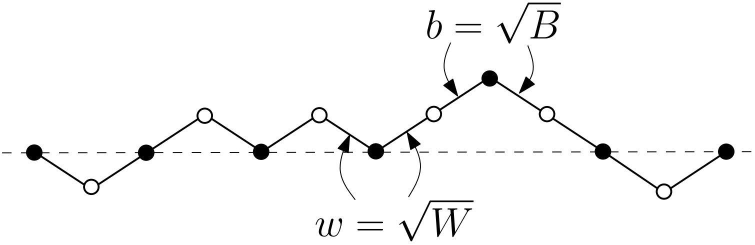
3. Deduce $G^\bullet(d)$

① An expression for $F_n^\bullet$

$F_n^\bullet$ can be expressed in terms of B and W via[1]

$$F_n^\bullet = \sum_{q\geq 0} \alpha_q \hat{\mathbb{Z}}_{0,0}^{\bullet\bullet+}(2n+2q) \qquad \alpha_q = \frac{B}{t_\bullet}\left(\delta_{q,0} - \sum_{k\geq q+1} g_k L_0(2k-2q-2)\right)$$

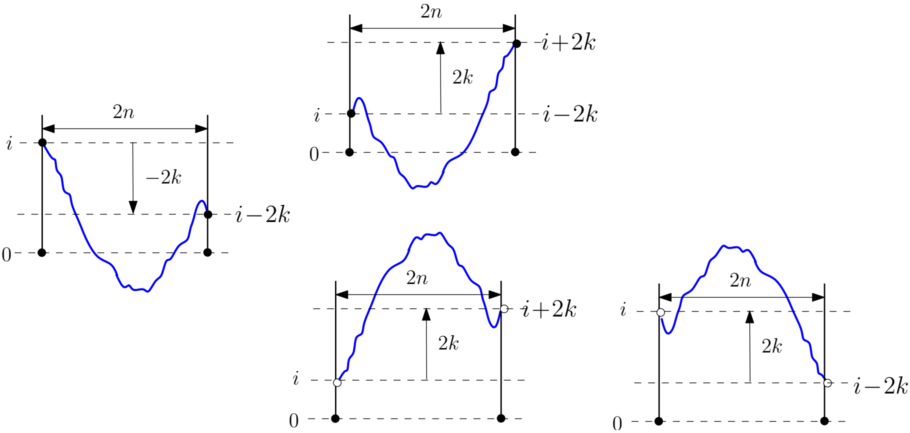
involving a linear combination of g.f. for paths of length $2n, 2n+2, 2n+4, \cdots$. Here, in $\hat{\mathbb{Z}}$, we decided to distribute the weights in a more symmetric way by setting $b \equiv \sqrt{B}$ and $w \equiv \sqrt{W}$



[1] can be proved slice decomposition - see the good authors

$\cdots$ and introduced

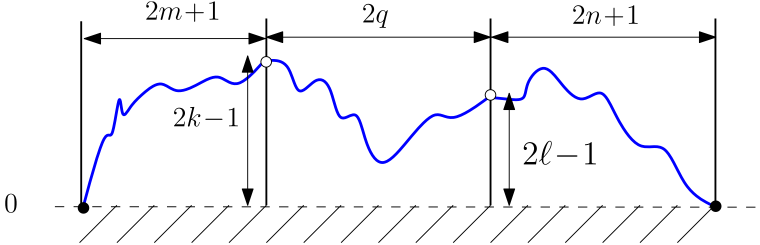
$$L_k(2n) \equiv \hat{\mathbb{Z}}^{\bullet\bullet}_{i,i-2k}(2n) \qquad (L_k(2n) = L_{-k}(2n), \quad L_k(2n) = \hat{\mathbb{Z}}^{\circ\circ}_{i,i-2k}(2n))$$



② Computing the Hankel determinant

Start with $h_i^{(1)}$:

$h_i^{(1)} = \det_{0\le n,m\le i}(F^\bullet_{n+m+1})$ where $F^\bullet_{n+m+1} = \sum_{q\ge 0} \alpha_q \hat{\mathbb{Z}}^{\bullet\bullet +}_{0,0}(2n+2m+2+2q)$



$$A^{\circ\circ +}_{2k-1,2\ell-1}(2q)$$

$$\hat{\mathbb{Z}}^{\bullet\bullet +}_{0,0}(2m+2n+2+2q) = \sum_{k=1}^{m+1}\sum_{\ell=1}^{n+1} \hat{\mathbb{Z}}^{\bullet\circ +}_{0,2k-1}(2m+1) A^{\circ\circ +}_{2k-1,2\ell-1}(2q) \hat{\mathbb{Z}}^{\circ\bullet +}_{2\ell-1,0}(2n+1)$$

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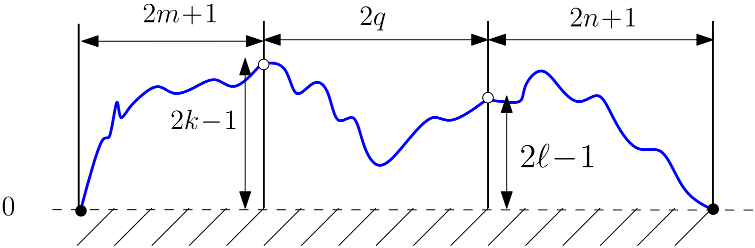
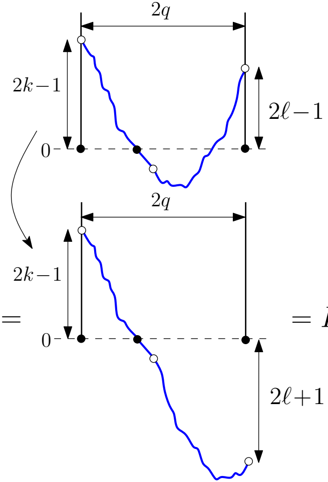


$A^{\circ\circ+}_{2k-1,2\ell-1}(2q)$

$$h_i^{(1)} = W^{i+1}(BW)^{\frac{i(i+1)}{2}} \det_{1\le k,\ell\le i+1}\Big(\sum_{q\ge 0} \alpha_q A^{\circ\circ+}_{2k-1,2\ell-1}(2q)\Big)$$

Reflection principle (to preserve the weights b and w, make a vertical reflection of the last part)

$$A^{\circ\circ+}_{2k-1,2\ell-1}(2q) = L_{k-\ell}(2q) - $$



$$= \quad = L_{k+\ell}(2q)$$

$$h_i^{(1)} = W^{i+1}(BW)^{\frac{i(i+1)}{2}} \det_{1\leq k,\ell\leq i+1}(C_{k-\ell} - C_{k+\ell}) \text{ where } C_k = \sum_{q\geq 0} \alpha_q L_k(2q)$$

From now on, assume faces with degree at most $2p+2$

$\Rightarrow \alpha_q = 0$ for $q > p$ $\quad \Rightarrow C_k = 0$ for $|k| > p$

Then it is a standard result that the wanted determinant can be expressed in terms of the roots x_a of the characteristic equation

$$0 = \sum_{k=-p}^{p} C_k x^k = C_0 + \sum_{k=1}^{p} C_k \left(x^k + \frac{1}{x^k} \right)$$

(which yields $2p$ solutions, $(x_a)_{1 \leq a \leq p}$ and $(1/x_a)_{1 \leq a \leq p}$), namely

$$D_i \equiv \det_{1 \leq k,\ell \leq i+1} (C_{k-\ell} - C_{k+\ell}) = (-1)^{p(i+1)} C_p^{i+1} \frac{\det\limits_{1 \leq a,a' \leq p} \left(x_a^{i+1+a'} - x_a^{-(i+1+a')} \right)}{\det\limits_{1 \leq a,a' \leq p} \left(x_a^{a'} - x_a^{-a'} \right)}$$

from which $h_i^{(1)}$ follows immediately

Heuristic explanation

Kernel of $(C_{k-\ell} - C_{k+\ell})_{1 \leq k,\ell \leq i+1}$

- $\sum_{\ell \in \mathbb{Z}} C_{k-\ell} x_a^{\ell} = \sum_{\ell \in \mathbb{Z}} C_{k-\ell} x_a^{-\ell} = 0, \quad a = 1, \cdots, p$

Heuristic explanation

Kernel of $(C_{k-\ell} - C_{k+\ell})_{1\le k,\ell\le i+1}$

- $\sum_{\ell\in\mathbb{Z}} C_{k-\ell} x_a^{\ell} = \sum_{\ell\in\mathbb{Z}} C_{k-\ell} x_a^{-\ell} = 0, \quad a = 1, \cdots, p$
- $\sum_{\ell\ge 1} (C_{k-\ell} - C_{k+\ell}) v_\ell = \sum_{\ell\ge 1} C_{k-\ell} v_\ell - \sum_{\ell\le -1} C_{k-\ell} v_{-\ell} = \sum_{\ell\in\mathbb{Z}} C_{k-\ell} v_\ell$

  provided $v_{-\ell} = -v_\ell$ for all ℓ. Choose:

  $$v_\ell^{(a)} = x_a^{\ell} - x_a^{-\ell} \qquad \ell \ge 1 \qquad \text{then } \sum_{\ell\ge 1} (C_{k-\ell} - C_{k+\ell}) v_\ell^{(a)} = 0$$

Heuristic explanation

Kernel of $(C_{k-\ell} - C_{k+\ell})_{1\leq k,\ell\leq i+1}$

- $\sum_{\ell\in\mathbb{Z}} C_{k-\ell}x_a^\ell = \sum_{\ell\in\mathbb{Z}} C_{k-\ell}x_a^{-\ell} = 0, \quad a = 1,\cdots,p$

- $\sum_{\ell\geq 1}(C_{k-\ell} - C_{k+\ell})v_\ell = \sum_{\ell\geq 1} C_{k-\ell}v_\ell - \sum_{\ell\leq -1} C_{k-\ell}v_{-\ell} = \sum_{\ell\in\mathbb{Z}} C_{k-\ell}v_\ell$

  provided $v_{-\ell} = -v_\ell$ for all ℓ. Choose:

$$v_\ell^{(a)} = x_a^\ell - x_a^{-\ell} \qquad \ell \geq 1 \qquad \text{then } \sum_{\ell\geq 1}(C_{k-\ell} - C_{k+\ell})v_\ell^{(a)} = 0$$

- To satisfy $\sum_{\ell=1}^{i+1}(C_{k-\ell} - C_{k+\ell})v_\ell = 0$ for $1 \leq k \leq i+1$, simply take a linear comb. of the $v_\ell^{(a)}$ such that $v_{i+2} = v_{i+3} = \cdots = v_{i+p+1} = 0$. A non-zero such combination exists if:

$$d_i \equiv \det_{1\leq a,a'\leq p} v_{i+a'+1}^{(a)} = 0$$

  In other words $d_i = 0 \Rightarrow D_i = 0$

- However d_i also vanishes whenever
  - $x_a = x_{a'}$ for some $a \neq a'$ (as it implies $v_\ell^{(a)} = v_\ell^{(a')}$)
  - $x_a = 1/x_{a'}$ for any a, a' (as it implies $v_\ell^{(a)} = -v_\ell^{(a')}$) and in

  particular (for $a = a'$) when $x_a = \pm 1$ (in which case $v_\ell^{(a)} = 0$).
  These cases correspond precisely to the zeros of

$$d_{-1} = \det_{1 \leq a, a' \leq p} v_{a'}^{(a)} = \frac{\prod_{a=1}^{p} (x_a^2 - 1) \prod_{1 \leq a < a' \leq p} (x_a - x_{a'})(1 - x_a x_{a'})}{\prod_{a=1}^{p} x_a^p}$$

  and we must suppress them by dividing d_i by d_{-1}.
  In other words $D_i \propto d_i / d_{-1}$

$$D_i \propto \frac{d_i}{d_{-1}}$$

- We obtain the proportionality constant by ensuring that the $(x_1 x_2 \cdots x_p)^{i+1}$ term coincides on both sides

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- We obtain the proportionality constant by ensuring that the $(x_1 x_2 \cdots x_p)^{i+1}$ term coincides on both sides

$$D_i = (-1)^{p(i+1)} C_p^{i+1} \frac{\det\limits_{1\le a,a'\le p}\left(x_a^{i+1+a'} - x_a^{-(i+1+a')}\right)}{\det\limits_{1\le a,a'\le p}\left(x_a^{a'} - x_a^{-a'}\right)}$$

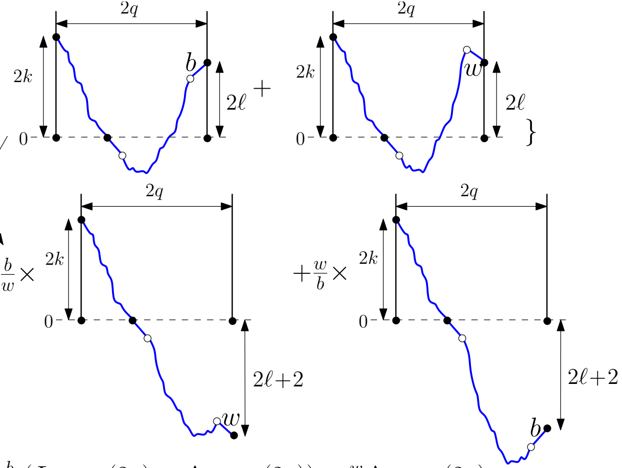
$$h_i^{(1)} = W^{i+1}(BW)^{\frac{i(i+1)}{2}}(-1)^{p(i+1)} C_p^{i+1} \frac{\det\limits_{1\le a,a'\le p}\left(x_a^{i+1+a'} - x_a^{-(i+1+a')}\right)}{\det\limits_{1\le a,a'\le p}\left(x_a^{a'} - x_a^{-a'}\right)})$$

Computing the Hankel determinant II

$h_i^{(0)}$ much more involved: $h_i^{(0)} = (BW)^{\frac{i(i+1)}{2}} \det\limits_{0\le k,\ell\le i}\left(\sum\limits_{q\ge 0} \alpha_q A^{\bullet\bullet+}_{2k,2\ell}(2q)\right)$

$A^{\bullet\bullet+}_{2k,2\ell}(2q) = L_{k-\ell}(2q) - \{ \quad + \quad \}$

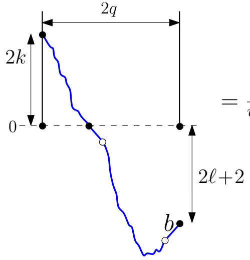
$= \frac{b}{w}\times \qquad + \frac{w}{b}\times$



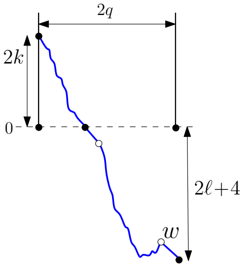
$$= \frac{b}{w}\left(L_{k+\ell+1}(2q) - \Delta_{k+\ell+1}(2q)\right) + \frac{w}{b}\Delta_{k+\ell+1}(2q)$$

$$= \frac{b}{w}L_{k+\ell+1}(2q) + \left(\frac{w}{b} - \frac{b}{w}\right)\Delta_{k+\ell+1}(2q)$$

$$\Delta_{k+\ell+1}(2q) =$$



$$= \frac{b}{w} \times$$



$$= \frac{b}{w}\left(L_{k+\ell+2}(2q) - \Delta_{k+\ell+2}(2q)\right)$$

$$A^{\bullet\bullet +}_{2k,2\ell}(2q) = L_{k-\ell}(2q) - c\, L_{k+\ell+1}(2q) + (c^2-1)\sum_{m\geq 2} L_{k+\ell+m}(2q)(-c)^{m-2}$$

where $c \equiv \frac{b}{w} = \sqrt{\frac{B}{W}}$

$$h_i^{(0)} \propto \bar{D}_i \equiv \det_{0\leq k,\ell\leq i}\left(C_{k-\ell} - c\, C_{k+\ell+1} + (c^2-1)\sum_{m\geq 2} C_{k+\ell+m}(-c)^{m-2}\right)$$

Heuristic argument

$$\sum_{\ell\geq 0}\Big(C_{k-\ell}-c\,C_{k+\ell+1}+(c^2-1)\sum_{m\geq 2}C_{k+\ell+m}(-c)^{m-2}\Big)w_\ell$$

$$=\sum_{\ell\geq 0}C_{k-\ell}w_\ell+\sum_{\ell\leq -1}C_{k-\ell}(-c\,w_{-\ell-1}+(c^2-1)\sum_{m=2}^{-\ell}(-c)^{m-2}w_{-\ell-m})$$

$$=\sum_{\ell\in\mathbb{Z}}C_{k-\ell}w_\ell \qquad \text{provided, for } \ell\leq -1$$

$$w_\ell=-c\,w_{-\ell-1}+(c^2-1)\sum_{m=2}^{-\ell}(-c)^{m-2}w_{-\ell-m}$$

which, by recursion is equivalent to

$$(w_\ell+w_{-\ell-2})+c(w_{\ell+1}+w_{-\ell-1})=0$$

Choose now:

$$w_\ell^{(a)}=\frac{c+x_a}{1+c\,x_a}x_a^\ell-x_a^{-\ell-1}\qquad \ell\geq 0$$

Then $\bar{d}_i\equiv\det_{1\leq a,a'\leq p}w_{i+a'}^{(a)}=0\Rightarrow\bar{D}_i=0$ and, eventually

$$\bar{D}_i\propto\frac{\bar{d}_i}{d_{-1}}$$

We end up with

$$h_i^{(0)} = (BW)^{\frac{i(i+1)}{2}} (-1)^{p(i+1)} C_p^{i+1} \prod_{a=1}^{p} (1+c\,x_a) \frac{\det\limits_{1\le a,a'\le p} \left(\gamma_a x_a^{i+a'} - x_a^{-(i+1+a')}\right)}{\det\limits_{1\le a,a'\le p} \left(x_a^{a'} - x_a^{-a'}\right)}$$

where $\gamma_a = \dfrac{c + x_a}{1 + c\,x_a}$

$\to$ can be proved rigorously

Final formula

Final formulas for slice g.f.

$$B_{2i} = B \frac{\det\limits_{1\le a,a'\le p}\left(x_a^{i+a'-1} - x_a^{-(i+a'-1)}\right) \det\limits_{1\le a,a'\le p}\left(\gamma_a x_a^{i+a'} - x_a^{-(i+a'+1)}\right)}{\det\limits_{1\le a,a'\le p}\left(\gamma_a x_a^{i+a'-1} - x_a^{-(i+a')}\right) \det\limits_{1\le a,a'\le p}\left(x_a^{i+a'} - x_a^{-(i+a')}\right)}$$

$$W_{2i+1} = W \frac{\det\limits_{1\le a,a'\le p}\left(\gamma_a x_a^{i+a'-1} - x_a^{-(i+a')}\right) \det\limits_{1\le a,a'\le p}\left(x_a^{i+a'+1} - x_a^{-(i+a'+1)}\right)}{\det\limits_{1\le a,a'\le p}\left(x_a^{i+a'} - x_a^{-(i+a')}\right) \det\limits_{1\le a,a'\le p}\left(\gamma_a x_a^{i+a'} - x_a^{-(i+a'+1)}\right)}$$

where $\gamma_a = \dfrac{c + x_a}{1 + c\,x_a}$

For the other parity, change $W \leftrightarrow B$, i.e. $c \leftrightarrow 1/c$, i.e. $\gamma_a \leftrightarrow 1/\gamma_a$

3 The expression for the two-point function $G^\bullet(d)$ follows immediately

Example

Quadrangulations

Faces with degree 4 only: $g_k = \delta_{k,2}$

$$B = t_{\bullet} + \sum_{k\geq 1} g_k \mathbb{Z}^{\bullet\circ}_{0,-1}(2k-1)$$

$$W = t_{\circ} + \sum_{k\geq 1} g_k \mathbb{Z}^{\circ\bullet}_{0,-1}(2k-1)\ .$$

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$$W = t_\circ + \sum_{k\geq 1} g_k \mathbb{Z}^{\circ\bullet}_{0,-1}(2k-1)\ .$$

$$B = t_\bullet + B(B+2W)\ , \quad W = t_\circ + W(W+2B)$$

Parametrization of $t_\bullet$ and $t_\circ$ by B and W via

$$t_\bullet = B(1-B-2W)\ , \quad t_\circ = W(1-W-2B)$$

Example

Quadrangulations

Faces with degree 4 only: $g_k = \delta_{k,2}$

$$C_k = \sum_{q\geq 0} \alpha_q L_k(2q) \text{ with } \alpha_q = \frac{B}{t_\bullet}\left(\delta_{q,0} - \sum_{k\geq q+1} g_k L_0(2k-2q-2)\right)$$

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$$\alpha_0 = \frac{B}{t_\bullet}(1-B-W),\ \alpha_1 = -\frac{B}{t_\bullet},\ C_0 = \alpha_0 + \alpha_1(B+W),\ C_1 = \alpha_1\sqrt{BW}$$

Parametrization of $t_\bullet$ and $t_\circ$ by B and W via

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Example

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$$0 = C_0 + \sum_{k=1}^{p} C_k \left(x^k + \frac{1}{x^k} \right)$$

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$$0 = 1 - 2(B+W) - c\,W \left(x + \frac{1}{x} \right) \quad c \equiv \sqrt{B/W}$$

$$\alpha_0 = \frac{B}{t_\bullet}(1-B-W),\ \alpha_1 = -\frac{B}{t_\bullet},\ C_0 = \alpha_0 + \alpha_1(B+W),\ C_1 = \alpha_1\sqrt{BW}$$

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$$t_\bullet = B(1-B-2W)\ , \quad t_\circ = W(1-W-2B)$$

Parametrization of B and W by x and c via

$$B = \frac{c^2 x}{c + 2x + 2c^2 x + c\,x^2}, \quad W = \frac{x}{c + 2x + 2c^2 x + c\,x^2}$$

Example

Quadrangulations

Faces with degree 4 only: $g_k = \delta_{k,2}$

$$B_{2i} = B\,\frac{(1-x^{2i})(1-\gamma x^{2i+3})}{(1-\gamma x^{2i+1})(1-x^{2i+2})},\;\; W_{2i+1} = W\,\frac{(1-\gamma x^{2i+1})(1-x^{2i+4})}{(1-x^{2i+2})(1-\gamma x^{2i+3})}$$

with $\gamma = \frac{c+x}{1+c\,x}$

Parametrization of $t_\bullet$ and $t_\circ$ by B and W via

$$t_\bullet = B(1-B-2W)\;, \quad t_\circ = W(1-W-2B)$$

Parametrization of B and W by x and c via

$$B = \frac{c^2 x}{c+2x+2c^2x+c\,x^2}, \quad W = \frac{x}{c+2x+2c^2x+c\,x^2}$$

Example

Quadrangulations

This leads to the expansions

$$B_1 = t_\bullet + t_\bullet(t_\bullet + t_\circ) + t_\bullet(2t_\bullet^2 + 5t_\bullet t_\circ + 2t_\circ^2) + t_\bullet(5t_\bullet^3 + 22t_\bullet^2 t_\circ + 22t_\bullet t_\circ^2 + 5t$$

$$B_2 = t_\bullet + t_\bullet(t_\bullet + 2t_\circ) + t_\bullet(2t_\bullet^2 + 9t_\bullet t_\circ + 6t_\circ^2) + t_\bullet(5t_\bullet^3 + 37t_\bullet^2 t_\circ + 57t_\bullet t_\circ^2 + 2$$

$$B_3 = t_\bullet + t_\bullet(t_\bullet + 2t_\circ) + t_\bullet(2t_\bullet^2 + 10t_\bullet t_\circ + 6t_\circ^2) + t_\bullet(5t_\bullet^3 + 44t_\bullet^2 t_\circ + 65t_\bullet t_\circ^2 +$$

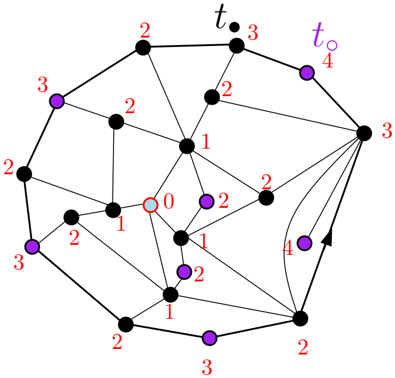
and

$$G^\bullet(1) = t_\bullet t_\circ(t_\bullet + t_\circ) + t_\bullet t_\circ(2t_\bullet^2 + 5t_\bullet t_\circ + 2t_\circ^2) + t_\bullet t_\circ(5t_\bullet^3 + 22t_\bullet^2 t_\circ + 22t_\bullet t_\circ^2$$

$$G^\bullet(2) = t_\bullet^2 t_\circ + 4t_\bullet^2 t_\circ(t_\bullet + t_\circ) + 5t_\bullet^2 t_\circ(3t_\bullet^2 + 7t_\bullet t_\circ + 3t_\circ^2) + t_\bullet^2 t_\circ(56t_\bullet^3 + 221t_\bullet^2 t$$

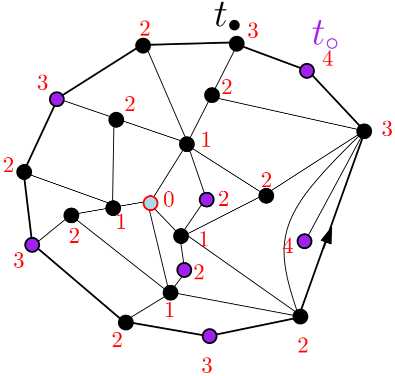
$$G^\bullet(3) = t_\bullet^2 t_\circ^2 + t_\bullet^2 t_\circ^2(7t_\bullet + 8t_\circ) + t_\bullet^2 t_\circ^2(37t_\bullet^2 + 95t_\bullet t_\circ + 47t_\circ^2) + t_\bullet^2 t_\circ^2(176t_\bullet^3 + 7$$

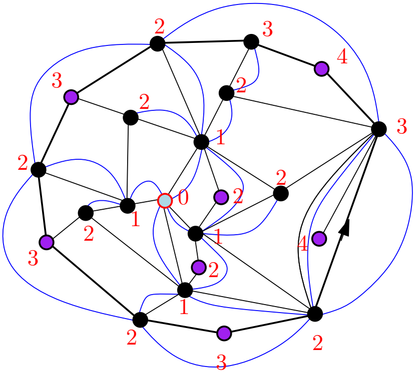
Another bivariate two-point function



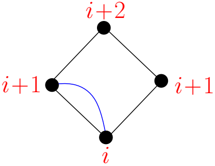
Consider a pointed rooted quadrangulation with a boundary of length $2n$ and assign a weight $t_\circ$ or $t_\bullet$ per vertex according to whether or not it is a local maximum for the distance to the pointed vertex.
Call $J_n(d)$ the corresponding g.f. with root/pointed vertex distance $\leq d$ and $J_n \equiv J_n(0)$.

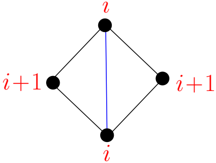
Another bivariate two-point function



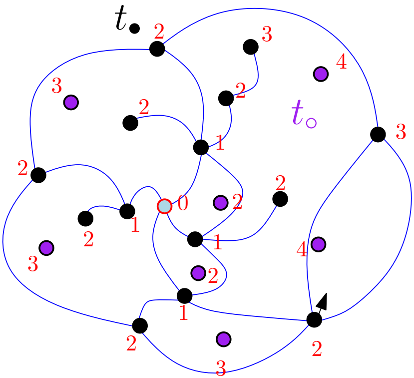


Apply the Ambjørn-Budd rule (inverse of Schaeffer's rule)



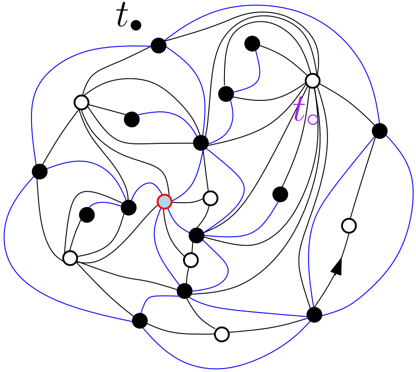


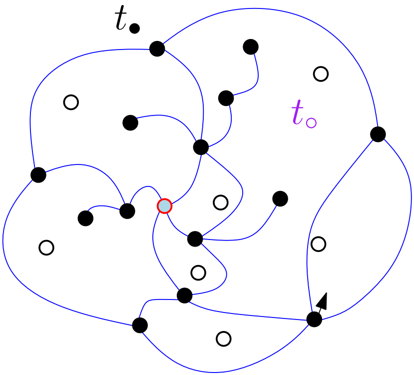
Another bivariate two-point function



Get a general rooted map with a boundary (of half the original boundary length)

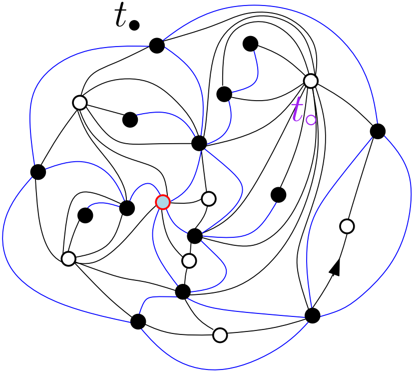
Another bivariate two-point function

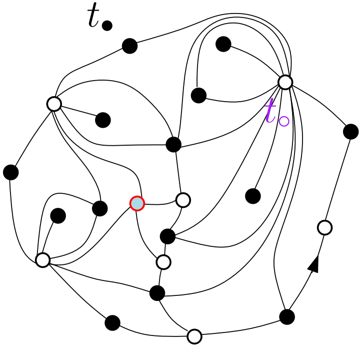




Use the standard equivalence between general maps and quadrangulations.

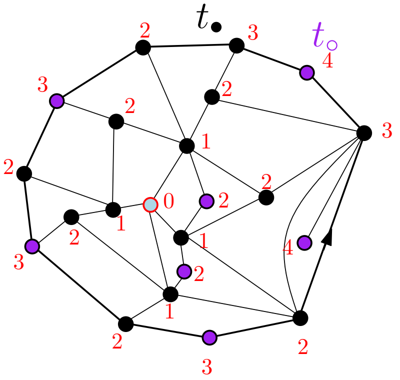
Another bivariate two-point function

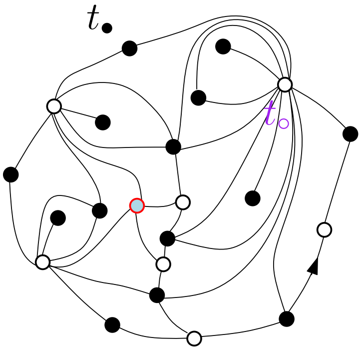




Get a bicolored quadrangulation with a boundary of the same length as the original quadrangulation

Another bivariate two-point function

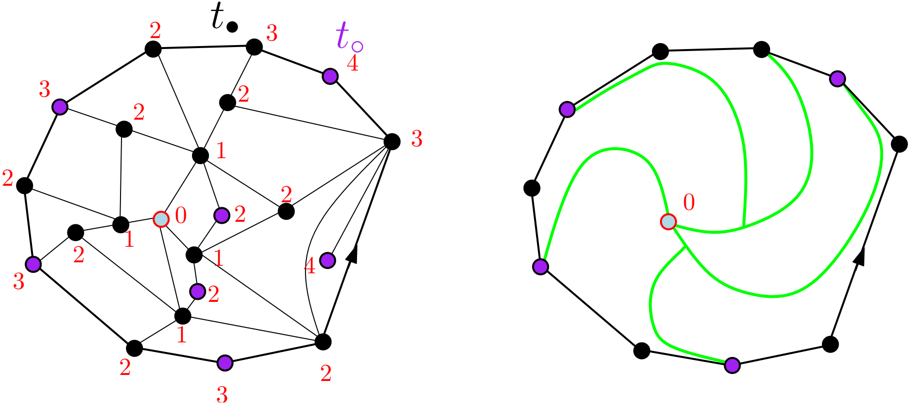




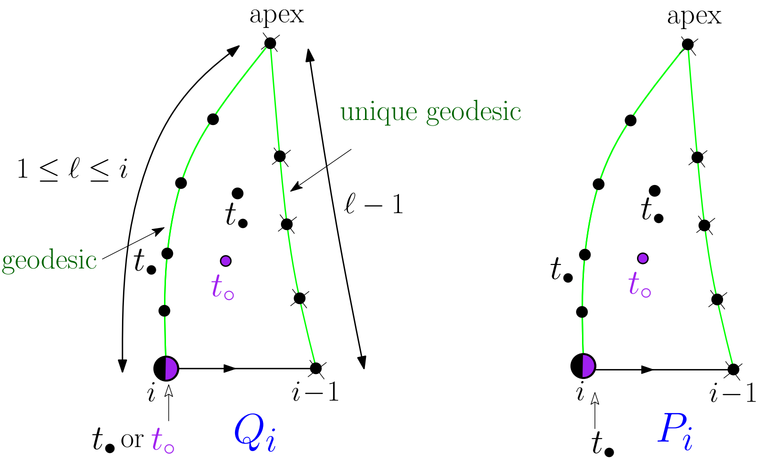
$d = 0$ is preserved, therefore

$$J_n = F_n^\bullet$$

If we now make a slice decomposition on the initial configurations

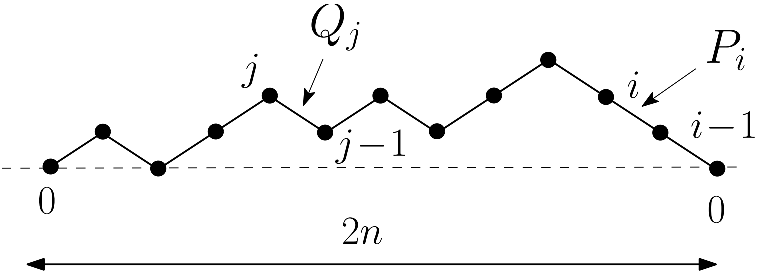


we are led to consider (two) new types of i-slices



Interest: knowing P_i and Q_i allows to immediately get the distance-dependent two-point function for planar maps with a weight $t_\bullet$ per vertex and $t_\circ$ per face !

and J_n is a sum over paths with the new weight distribution:



so that

$$\sum_{n\geq 0} F_n^{\bullet} z^n = \sum_{n\geq 0} J_n z^n = \frac{1}{1-(Q_1-P_1)z - \frac{P_1 z}{1-(Q_2-P_2)z-\frac{P_2 z}{1-\cdots}}}$$

The P_i and Q_i were computed by Ambjørn and Budd as the solution of the recursive system of equations

$$P_i = t_\bullet + P_i(P_{i-1} + Q_i + Q_{i+1}), \qquad Q_i = t_\circ + Q_i(P_{i-1} + P_i) + P_i Q_{i+1}$$

They get

$$Q_i = Q\frac{(1-y^i)(1-\alpha^2 y^{i+3})}{(1-\alpha y^{i+1})(1-\alpha y^{i+2})}, \qquad P_i = P\frac{(1-y^i)(1-\alpha y^{i+3})}{(1-y^{i+1})(1-\alpha y^{i+2})},$$

where

$$Q = t_\circ + Q(Q+2P), \qquad P = t_\bullet + P(P+2Q),$$

while y and α are obtained by inverting the relations

$$t_\bullet = \frac{y(1-\alpha y)^3(1-\alpha y^3)}{(1+y+\alpha y-6\alpha y^2+\alpha y^3+\alpha^2 y^3+\alpha^2 y^4)^2}$$
$$t_\circ = \frac{\alpha y(1-y)^3(1-\alpha^2 y^3)}{(1+y+\alpha y-6\alpha y^2+\alpha y^3+\alpha^2 y^3+\alpha^2 y^4)^2}.$$

Instead, we can decide to use the theory for our new type of continued fractions.
We have (see Di Francesco and Kedem 2010)

$$Q_i - P_i = \frac{H_{i-1}^{(0)}}{H_i^{(0)}} \Big/ \frac{H_i^{(1)}}{H_{i-1}^{(1)}} \qquad\qquad P_i = \frac{H_{i+1}^{(1)}}{H_i^{(1)}} \Big/ \frac{H_i^{(0)}}{H_{i-1}^{(0)}}$$

in terms of the "Hankel"-type determinants

$$H_i^{(0)} = \det(F^{\bullet}_{n+m-i-2})_{0\le n,m\le i} \qquad\qquad H_i^{(1)} = \det(F^{\bullet}_{n+m-i-1})_{0\le n,m\le i}$$

Problem: Requires $F_n^{\bullet}$ for negative n !!

For finite continued fractions, the $F_n^{\bullet}$ for n negative are related to the F_n for n positive (see Di Francesco Kedem) and this fixes the P_i and Q_i.

Instead, we can decide to use the theory for our new type of continued fractions.
We have (see Di Francesco and Kedem 2010)

$$Q_i - P_i = \frac{H_{i-1}^{(0)}}{H_i^{(0)}} \Big/ \frac{H_i^{(1)}}{H_{i-1}^{(1)}} \qquad P_i = \frac{H_{i+1}^{(1)}}{H_i^{(1)}} \Big/ \frac{H_i^{(0)}}{H_{i-1}^{(0)}}$$

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<u>Problem</u>: Requires $F_n^{\bullet}$ for negative n !!

For infinite continued fractions, the $F_n^{\bullet}$ for n negative are free !!
The knowledge of $F_n^{\bullet}$ for $n \ge 0$ is not sufficient to deduce P_i and Q_i.
Indeed, expanding the continued fraction as a power series to equate its coefficients with the $F_n^{\bullet}$, we immediately see that the system is underdeterminated.

Instead, we can decide to use the theory for our new type of continued fractions.
We have (see Di Francesco and Kedem 2010)

$$Q_i - P_i = \frac{H_{i-1}^{(0)}}{H_i^{(0)}} \Big/ \frac{H_i^{(1)}}{H_{i-1}^{(1)}} \qquad\qquad P_i = \frac{H_{i+1}^{(1)}}{H_i^{(1)}} \Big/ \frac{H_i^{(0)}}{H_{i-1}^{(0)}}$$

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Problem: Requires $F_n^\bullet$ for negative n !!

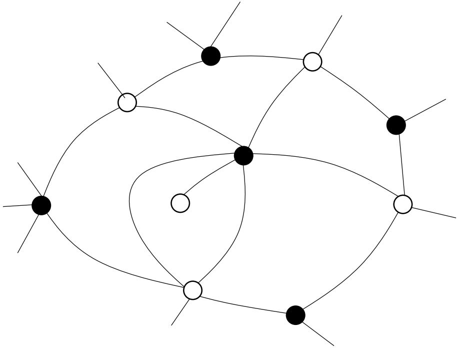
Still, we may decide to use the same relation as for the finite continued fraction case to define the $F_n^\bullet$ for n negative from the F_n for n positive (why this choice ??).
We then get a particular solution for P_i and Q_i and it precisely reproduces the Ambjørn-Budd formulas.

Thank You

An integrable system with three colors

System of equations for bicolored quadrangulations:

$B_i = t_\bullet + B_i(W_{i-1} + B_i + W_{i+1}),\ W_i = t_\circ + W_i(B_{i-1} + W_i + B_{i+1})$



An integrable system with three colors

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$B_i = t_\bullet + B_i(W_{i-1} + B_i + W_{i+1}), \ W_i = t_\circ + W_i(B_{i-1} + W_i + B_{i+1})$

Solution:

$$B_{2i} = B\frac{(1-x^{2i})(1-\gamma x^{2i+3})}{(1-\gamma x^{2i+1})(1-x^{2i+2})}, \ W_{2i+1} = W\frac{(1-\gamma x^{2i+1})(1-x^{2i+4})}{(1-x^{2i+2})(1-\gamma x^{2i+3})}$$

where B and W are parametrizations of $t_\bullet$ and $t_\circ$ via

$$t_\bullet = B - B(B+2W), \quad t_\circ = W - W(W+2B)$$

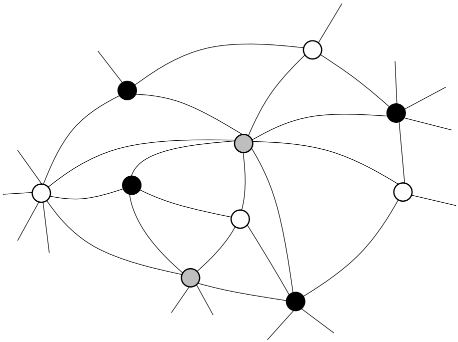
while $\gamma = \frac{c+x}{1+cx}$ with $c = \sqrt{B/W}$ and $0 = 1-2(B+W) - \sqrt{BW}\left(x+\frac{1}{x}\right)$,
i.e. c and x are themselves parametrizations of B and W via

$$B = \frac{c^2 x}{c+2x+2c^2x+c\,x^2}, \quad W = \frac{x}{c+2x+2c^2x+c\,x^2}$$

An integrable system with three colors

System of equations for tricolored triangulations:

$$T_i = t_\bullet + T_i(U_{i-1} + V_{i+1}),\ U_i = t_\circ + U_i(V_{i-1} + T_{i+1}),\ V_i = t_{\circ} + V_i(T_{i-1} + U_{i+1})$$



An integrable system with three colors

System of equations for tricolored triangulations:

$$T_i = t_\bullet + T_i(U_{i-1} + V_{i+1}),\ U_i = t_\circ + U_i(V_{i-1} + T_{i+1}),\ V_i = t_{\circ} + V_i(T_{i-1} + U_{i+1})$$

Solution:

$$T_{3i} = T\frac{(1-x^{3i})(1-\alpha x^{3i+4})}{(1-\alpha x^{3i+1})(1-x^{3i+3})}, \qquad U_{3i+2} = U\frac{(1-x^{3i+2}/\gamma)(1-x^{3i+6})}{(1-x^{3i+3})(1-x^{3i+5}/\gamma)}$$

$$V_{3i+1} = V\frac{(1-\alpha x^{3i+1})(1-x^{3i+5}/\gamma)}{(1-x^{3i+2}/\gamma)(1-\alpha x^{3i+4})}$$

where T, U and V are parametrizations of $t_\bullet, t_\circ$ and $t_{\circ}$ via

$$t_\bullet = T - T(U+V) \qquad t_\circ = U - U(V+T) \qquad t_{\circ} = V - V(T+U)$$

while α and γ are expressed in terms of three quantities c, d and x via

$$\alpha = \frac{d + c\,x + x^2}{1 + d\,x + c\,x^2} \qquad \gamma = \frac{1 + d\,x + c\,x^2}{c + x + d\,x^2}$$

while c, d and x are themselves parametrizations of T, U and V via

$$T = \frac{c\,d\,x}{(c+x)(1+d\,x)} \quad U = \frac{d\,x}{(c+x)(d+c\,x)} \quad V = \frac{c\,x}{(d+c\,x)(1+d\,x)}$$

A formula for $F_n^\bullet$

$d_\bullet \leq d \Leftrightarrow \{d_\bullet = 0$ or $1 \leq d_\bullet \leq d\}$ $\quad F_n^\bullet(d) = F_n^\bullet + F_n^\bullet(1 \to d)$

where $F_n^\bullet(1 \to d)$ is the g.f. for maps with $1 \leq d_\bullet \leq d$

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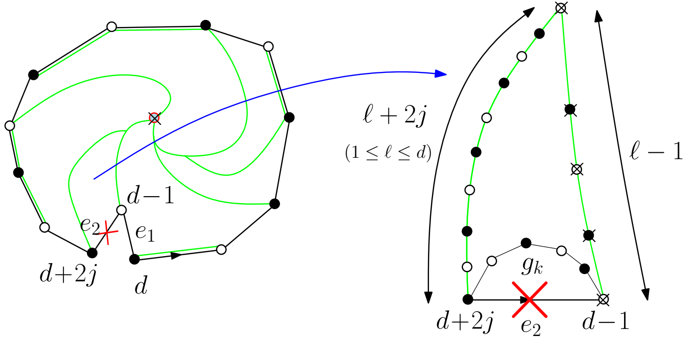

$$F_n^\bullet(1 \to d) = \frac{1}{t_\bullet} \sum_{j \ge 1} Z^{\bullet\bullet +}_{d,d+2j}(2n) \sum_{k \ge 1} g_k Z^{\bullet\circ}_{d+2j,d-1}(2k-1)$$

$$F_n^{\bullet} = Z_{d,d}^{\bullet\bullet+}(2n) - \frac{1}{t_\bullet}\sum_{j\geq 1} Z_{d,d+2j}^{\bullet\bullet+}(2n)\sum_{k\geq 1} g_k Z_{d+2j,d-1}^{\bullet\circ}(2k-1)$$

- The l.h.s. is independent of d (so the r.h.s. is a conserved quantity)
- We can shift path heights by d and send $d \to \infty$, which allows us to express $F_n^{\bullet}$ in terms of B and W via

$$F_n^{\bullet} = \mathbb{Z}_{0,0}^{\bullet\bullet+}(2n) - \frac{1}{t_\bullet}\sum_{j\geq 1} \mathbb{Z}_{0,2j}^{\bullet\bullet+}(2n)\sum_{k\geq 1} g_k \mathbb{Z}_{2j,-1}^{\bullet\circ}(2k-1)$$

and, after simple manipulations, we arrive at

$$F_n^{\bullet} = \sum_{q\geq 0} \alpha_q \hat{\mathbb{Z}}_{0,0}^{\bullet\bullet+}(2n+2q) \qquad \alpha_q = \frac{B}{t_\bullet}\left(\delta_{q,0} - \sum_{k\geq q+1} g_k L_0(2k-2q-2)\right)$$