Torsional Newton-Cartan geometry in Lifshitz holography and non-relativistic FTs

Holographic renormalization group and entanglement

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based on work with:

Jelle Hartong and Elias Kiritsis

1409.1519 [1] & 1409.1522 [2] & to appear (soon)[3] & to appear (later)

and

Morten Holm Christensen, Jelle Hartong, Blaise Rollier

1311.4794 (PRD) & 1311.6471 (JHEP)

Introduction

- holography beyond original AdS-setup

- apply to study of strongly coupled CM systems
 non-relativistic scaling -> Schroedinger, Lifshitz, hyperscaling violating geometries

- how general is the holographic paradigm ?
 (nature of quantum gravity, black hole physics)

- appearance of novel geometric structures on the boundary (this talk: TNC)

- exotic theories of gravity can be viewed as Schwinger source functionals
 of non-rel QFTs (``metric'' couples to stress tensor)
 symmetries of FT -> symmetries of the coupled grav. theory and
 constrain form of source functionals

This talk: direct implementation of this in class of examples characterized
by Lifshitz scaling symmetry and extended Schroedinger sym.
+ holographic realization within context of bulk Lifshitz spacetime

Lifshitz symmetries

Many systems in nature exhibit critical points with non-relativistic scale invariance

Includes in particular scale invariance with dynamical exponent z>1

$$t \to \lambda^z t\,, \qquad \vec{x} \to \lambda \vec{x}$$

Such systems typically have Lifshitz symmetries:

$$\begin{aligned} D_z &: & \vec{x} &\to \lambda\vec{x} & t &\to \lambda^z t\,, \\ H &: & t &\to t + a\,, \\ P_i &: & x^i &\to x^i + a^i\,, \\ J_{ij} &: & x^i &\to R^i{}_j x^j\,. \end{aligned}$$

Lifshitz algebra (non-zero commutators, not involving rotations)

$$[D_z, H] = -zH\,, \qquad [D_z, P_i] = -P_i\,.$$

Schroedinger symmetries

example of symmetry group that also displays non-relativistic scaling and contains Lifshitz is Schroedinger group

additional symmetries:

Galilean boosts $G_i \ (x^i \to x^i + v^i t)$

particle number symmetry M

Schroedinger algebra

$$
\begin{aligned}
[D_z, H] &= -zH\,, & [D_z, P_i] &= -P_i\,, & [D_z, M] &= (z-2)M \\
[D_z, G_i] &= (z-1)G_i\,, & [H, G_i] &= P_i\,, & [P_i, G_j] &= M\delta_{ij}
\end{aligned}
$$

for z=2: additional special conformal generator K

Lifshitz spacetimes

Aim: construct holographic techniques for (strongly coupled) systems with NR symmetries

Lifshitz holography $$ds^2 = -\frac{dt^2}{r^{2z}} + \frac{1}{r^2}\left(dr^2 + d\vec{x}^2\right)$$

[Kachru,Liu,Mulligan]
[Taylor]

Taylor/Danielson,Thorlacius/Ross,Saremi/Ross/
Baggio,de Boer,Holsheimer/Mann,McNees/
Griffin,Horava,Melby-Thompson/
Korovin,Skenderis,Taylor/
Cheng,Hartnoll,Keeler/Baggio/Holsheimer/
Christensen,Hartong,NO,Rollier
Chemissany, Papadimitriou
Hartong,Kiritsis,NO

Some intuitions/expectations

- holography for bulk spacetimes with non-relativistic scaling
  -> some type of non-relativistic geometry on the boundary
  (Newton-Cartan or a generalization thereof)

besides energy momentum tensor, non-rel. theory also has a mass current T^μ
-> natural that there is an extra source coupling to it M_μ

- Newton-potential should enter the story

- expect Lifshitz sym (at the least, will see that there can be more)

Mini-intro to Newton-Cartan geometry

GR is a diff invariant theory whose tangent space is Poincare invariant

Newtonian gravity is diff invariant theory whose tangent space is the Bargmann algebra (non-rel limit of Poincare)

Andringa,Bergshoeff,Panda,de Roo

$$[J_{ab}, P_c] = -2\delta_{c[a}P_{b]}\,, \qquad [J_{ab}, G_c] = -2\delta_{c[a}G_{b]}\,,$$
$$[G_a, H] = -P_a\,, \qquad [G_a, P_b] = -\delta_{ab}Z\,,$$

centrally extended
Galilean algebra

here: only interested in geometrical framework; not in EOMs
boundary geometry in holographic setup is non-dynamical

From Poincare to GR

GR is a diff invariant theory whose tangent space is Poincare invariant

- make Poincare local (i.e. gauge the translations and rotations)

$$
\begin{aligned}
A_\mu &= P_a e_\mu^a + \frac{1}{2} M_{ab} \omega_\mu{}^{ab} \\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu \, , A_\nu] = P_a R_{\mu\nu}{}^a(P) + \frac{1}{2} M_{ab} R_{\mu\nu}{}^{ab}(M) \\
\delta A_\mu &= \partial_\mu \Lambda + [A_\mu \, , \Lambda] \, , \qquad \Lambda = \xi^\mu A_\mu + \frac{1}{2} M_{ab} \lambda^{ab}
\end{aligned}
$$

GR (Lorentzian geometry) follows from curvature constraint

$$
R_{\mu\nu}{}^a(P) = 0 \left\{ \begin{array}{l}
\omega_\mu{}^{ab} = \text{spin connection: expr. in terms of } e_\mu^a \\
\delta A_\mu = \mathcal{L}_\xi A_\mu + \frac{1}{2} M_{ab} \partial_\mu \lambda^{ab} + \frac{1}{2} [A_\mu \, , M_{ab}] \lambda^{ab} \\
R_{\mu\nu}{}^{ab}(M) = \text{Riemann curvature 2-form} \\
\nabla_\mu \ \text{ defined via vielbein postulate}
\end{array} \right.
$$

From Bargmann to NC

Andringa,Bergshoeff,Panda,de Roo

Newtonian gravity is a diff invariant theory whose tangent space is Bargmann (make Bargmann local)

symmetry	generators	gauge field	parameters	curvatures
time translations	H	τ_μ	$\zeta(x^\nu)$	$R_{\mu\nu}(H)$
space translations	P^a	$e_\mu{}^a$	$\zeta^a(x^\nu)$	$R_{\mu\nu}{}^a(P)$
boosts	G^a	$\omega_\mu{}^a$	$\lambda^a(x^\nu)$	$R_{\mu\nu}{}^a(G)$
spatial rotations	J^{ab}	$\omega_\mu{}^{ab}$	$\lambda^{ab}(x^\nu)$	$R_{\mu\nu}{}^{ab}(J)$
central charge transf.	Z	m_μ	$\sigma(x^\nu)$	$R_{\mu\nu}(Z)$

curvature constraints $\quad R_{\mu\nu}(H) = R_{\mu\nu}{}^a(P) = R_{\mu\nu}(M) = 0.$

leaves as independent fields: $\quad \tau_\mu,\ e^a_\mu,\ m_\mu$

transforming as

$$
\begin{aligned}
\delta\tau_\mu &= \mathcal{L}_\xi \tau_\mu \\
\delta e^a_\mu &= \mathcal{L}_\xi e^a_\mu + \lambda^a \tau_\mu + \lambda^a{}_b e^b_\mu \\
\delta m_\mu &= \mathcal{L}_\xi m_\mu + \partial_\mu \sigma + \lambda_a e^a_\mu
\end{aligned}
$$

Overview of recent background

- bdry geometry for Lifshitz spacetimes is torsional Newton-Cartan geometry
  (novel extension of NC)

- first observed for specific z=2 example (in 4D): [Christensen,Hartong,NO,Rollier]
  * Scherk-Schwarz dim. reduction (null on bdry) from 5D AlAdS solution

- generalized to large class of arbitrary z (in EPD model) [Hartong,Kiritsis,NO]1

  sources: use vielbein formalism + appropriate lin. combo
  coupling of geometry to bdry -> vevs (stress tensor, mass current)
  & WIs in TNC covariant form

- TNC geometry arises by gauging the Schroedinger algebra [Bergshoeff,Hartong,Rosseel]

- coupling of TNC to non-rel FTs also considered directly [Jensen/Jensen,Karch]

- recent activity using NC/TNC in CM
  (strongly-correlated electron system, FQH)

[Son][Gromov,Abanov][Geracie,Son,Wu,Wu]
[Brauner,Endlich,Monin,Penco][Geracie,Son]
[Wu,Wu],[Geracie,Golkar,Roberts]

Symmetries: from Lifshitz to Schroedinger

- holography for Lifshitz spacetimes: Schroedinger symmetry acting on sources
-> strongly suggest that bdry theory can have Schroedinger invariance

[Hartong,Kiritsis,NO]1,2,3

Main overall points of this talk:

- appearance of global symmetries in non-relativistic field theories exhibits a new mechanism:
  * interplay between conserved currents and space-time isometries is different compared to relativistic case

- supported by considering the Lifshitz vacuum:
  = holographic dual of flat NC spacetime

-> Lifshitz holography dual to field theories on TNC spacetime

Plan & preview

1. holography for Lifshitz spacetimes and TNC geometry

time-like vielbein τ_μ, space-like vielbeins e^a_μ and a vector field M_μ,

2. scale invariant field theories on TNC backgrounds
   * the vector field can make a global U(1) into local sym.

3. flat NC spacetime
   * comes with function M (in $M_\mu = \partial_\mu M$)
   * local symmetries can generate non-trivial orbit of equivalent M

4. scale-invariant field theories on flat NC
   * novel mechanism: M can be eaten up by physical fields generating extra global symmetries (e.g. Galilean boost) beyond Lif (-> Sch.)

5 . Lifsthiz vacuum
   * exhibits source M transforming under local Sch (Lif realized by Killing)
   * scalar probes on Lif bgr that are Sch invariant by similar mechanism as in FT
   * conserved (or improved) current: global U(1)

EPD model and AlLif spacetimes

- bulk theory

$$S = \int d^4x\sqrt{-g}\left(R - \frac{1}{4}Z(\Phi)F^2 - \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}W(\Phi)B^2 - V(\Phi)\right)$$

- admits Lifshitz solutions with z>1

For AlLif BCs useful to write:

$$ds^2 = \frac{dr^2}{R(\Phi)r^2} - E^0E^0 + \delta_{ab}E^aE^b\,, \qquad B_M = A_M - \partial_M\Xi$$

then AlLif BCs

[Ross],[Christensen,Hartong,NO,Rollier]
[Hartong,Kiritsis,NO]1

$$E^0_\mu \simeq r^{-z}\tau_\mu\,, \qquad E^a_\mu \simeq r^{-1}e^a_\mu\,, \qquad \Phi \simeq r^\Delta\phi\,,$$

$$B_\mu - \alpha(\Phi)E^0_\mu \simeq -r^{z-2}M_\mu\,.$$

Stueckelberg decompostion: $M_\mu = \tilde{m}_\mu - \partial_\mu\chi\,.$

Transformation of sources

use local bulk symmetries:
local Lorentz, gauge transformations and diffs preserving metric gauge

these symmetries induce an action on sources: $\tau_\mu \quad e^a_\mu \quad M_\mu$

$=$ action of Bargmann algebra plus local dilatations $=$ Schroedinger

$$\begin{aligned}
\delta\tau_\mu &= \mathcal{L}_\xi \tau_\mu + z\Lambda_D \tau_\mu \,, \\
\delta e^a_\mu &= \mathcal{L}_\xi e^a_\mu + \lambda^a \tau_\mu + \lambda^a{}_b e^b_\mu + \Lambda_D e^a_\mu \,, \\
\delta M_\mu &= \mathcal{L}_\xi M_\mu + e^a_\mu \lambda_a + (2-z)\Lambda_D M_\mu \,,
\end{aligned}$$

there is thus a Schroedinger Lie algebra valued connection given by

$$A_\mu = H\tau_\mu + P_a e^a_\mu + M m_\mu + \frac{1}{2} J_{ab}\omega_\mu{}^{ab} + G_a \omega_\mu{}^a + D b_\mu$$

with appropriate curvature constrains that reproduces trafos of the sources

Torsional Newton-Cartan (TNC) geometry

the bdry geometry is novel extension of NC geometry

- inverse veilbeins (v^μ, e^μ_a)

$$v^\mu \tau_\mu = -1\,, \qquad v^\mu e^a_\mu = 0\,, \qquad e^\mu_a \tau_\mu = 0\,, \qquad e^\mu_a e^b_\mu = \delta^b_a$$

can build Galilean boost-invariants

$$h_{\mu\nu} = e^a_\mu e^b_\nu \delta_{ab}$$

$$\begin{aligned} \hat{v}^\mu &= v^\mu - h^{\mu\nu} M_\nu\,, \\ \bar{h}_{\mu\nu} &= h_{\mu\nu} - \tau_\mu M_\nu - \tau_\nu M_\mu\,, \\ \tilde{\Phi} &= -v^\mu M_\mu + \tfrac{1}{2} h^{\mu\nu} M_\mu M_\nu\,, \end{aligned}$$

affine connection of TNC

$$\Gamma^\rho_{\mu\nu} = -\hat{v}^\rho \partial_\mu \tau_\nu + \frac{1}{2} h^{\rho\sigma} \left(\partial_\mu \bar{h}_{\nu\sigma} + \partial_\nu \bar{h}_{\mu\sigma} - \partial_\sigma \bar{h}_{\mu\nu} \right)$$

with torsion $\Gamma^\rho_{[\mu\nu]} = -\frac{1}{2} \hat{v}^\rho (\partial_\mu \tau_\nu - \partial_\nu \tau_\mu)$

$$\nabla_\mu \tau_\nu = 0\,, \qquad \nabla_\mu h^{\nu\rho} = 0\,,$$

Coupling FTs to TNC

[Hartong,Kiritsis,NO]

- action functional $S = S[\hat{v}^\mu, h^{\mu\nu}, \tilde{\Phi}]\,.$

$$\delta_{\rm bg} S = \int d^{d+1}x e \left[-\tau_\nu T^\nu{}_\mu \delta\hat{v}^\mu - \left(\hat{e}^a_\nu \hat{v}^\mu T^\nu{}_\mu\right) \hat{e}_{\sigma a} \tau_\rho \delta h^{\rho\sigma} \right.$$
$$\left. + \frac{1}{2}\left(\hat{e}^b_\nu e^{\mu a} T^\nu{}_\mu\right) \hat{e}_{\rho b}\hat{e}_{\sigma a} \delta h^{\rho\sigma} + \tau_\mu T^\mu \delta\tilde{\Phi} \right] \,,$$

EM tensor: $T^\mu{}_\nu = -\left(S^0_\nu + T^0 \partial_\nu \chi\right) v^\mu + \left(S^a_\nu + T^a \partial_\nu \chi\right) e^\mu_a\,.$

mass current $T^\mu = -T^0 v^\mu + T^a e^\mu_a$

tangent space projections provide
energy density, energy flux, momentum density, stress, mass density, mass current

- off-shell WIs

$$e^{-1}\partial_\mu\left(e T^\mu\right) = \langle O_\chi \rangle\,,$$
$$\hat{e}^a_\mu T^\mu - \tau_\nu e^{\mu a} T^\nu{}_\mu = 0\,.$$
$$\hat{e}^{[a}_\nu e^{b]\mu} T^\nu{}_\mu = 0\,.$$

Stueckelberg U(1)
local Galilean boosts
local rotations

Diffeomorphism and scale Ward identities

- diffeos -> on-shell WI

$$0 = e^{-1}\partial_\nu \left(e T^\nu{}_\mu\right) + T^\rho{}_\nu \left(\hat{v}^\nu \partial_\mu \tau_\rho - e^\nu_a \partial_\mu \hat{e}^a_\rho\right) + \tau_\nu T^\nu \partial_\mu \tilde{\Phi}\,.$$

* conserved currents $\partial_\nu \left(e K^\mu T^\nu{}_\mu\right) = 0\,.$

for K a TNC Killing vector:

$$\mathcal{L}_\xi \hat{v}^\mu = 0\,, \qquad \mathcal{L}_\xi h^{\mu\nu} = 0\,, \qquad \mathcal{L}_\xi \tilde{\Phi} = 0\,,$$

- if theory has scale invariance:
  can use TNC analogue of dilatation connection

$$-z\tau_\nu \hat{v}^\mu T^\nu{}_\mu + \hat{e}^a_\nu e^{\mu a} T^\nu{}_\mu + 2(z-1)\tau_\mu T^\mu \tilde{\Phi} = 0\,.$$

The Schroedinger model and deformations

- simplest toy model for coupling non-rel scale-inv theory to TNC (z=2)

$$S = \int d^{d+1}x e \left(-i\phi^\star \hat{v}^\mu \partial_\mu \phi + i\phi \hat{v}^\mu \partial_\mu \phi^\star - h^{\mu\nu} \partial_\mu \phi \partial_\nu \phi^\star - 2\tilde{\Phi}\phi\phi^\star - V_0 (\phi\phi^\star)^{\frac{d+2}{d}} \right)$$

* consider deformations preserving local scale inv $\qquad \phi = \frac{1}{\sqrt{2}} \varphi e^{i\theta}$

- change the potential $\qquad V_0 \varphi^{\frac{2(d+2)}{d}} (1 + b\theta^2)$

- adding the term: $\qquad -a \int d^{d+1}x e \varphi^2 h^{\mu\nu} \tilde{\nabla}_\mu \partial_\nu \theta \quad - a \int d^{d+1}x e \varphi^2 e^\mu_a \mathcal{D}_\mu M^a$

- can show that a-deformed model has local symmetry

$$\delta M_\mu = \partial_\mu \alpha \,, \qquad \delta\theta = -\alpha \,, \qquad \text{giving on-shell WI} \qquad \partial_\mu \left(e T^\mu \right) = 0 \,,$$

* diffeos + local boosts (+ possibly local scale) induce trafos of type:

$$\tilde{N} \;:\; \qquad \delta v^\mu = 0 \,, \qquad \delta h^{\mu\nu} = 0 \,, \qquad \delta M_\mu = \partial_\mu \tilde{\sigma} \,,$$

-> posssibility of extra global symmetries (intimately connected to vector field)

The Lifshitz model

- other possibility: do not couple to Phi -> e.g. z=2 Lifshitz model

$$S=\int d^{d+1}xe\left[\frac{1}{2}\left(\hat{v}^{\mu}\partial_{\mu}\phi\right)^{2}-\frac{\lambda}{2}\left(h^{\mu\nu}\nabla_{\mu}\partial_{\nu}\phi\right)^{2}\right]$$

more generally $S = S[\hat{v}^{\mu}, h^{\mu\nu}].$ --> $S = S[\hat{v}^{\mu}, g^{\mu\nu}].$

$$g^{\mu\nu} = -\hat{v}^{\mu}\hat{v}^{\nu} + h^{\mu\nu}$$

situation considered in

[Hoyos,Kim,Oz]

special case: $S = S[g^{\mu\nu}].$

Flat NC spacetime

to study FTs on flat NC: first need to define notion of flat NC
- use global inertial coordinates

τ_μ	$=$	δ^t_μ	fixing diffs
$h^{\mu\nu}$	$=$	$\delta^{ij}\delta^\mu_i\delta^\nu_j$	fixing diffs and flat space
v^μ	$=$	$-\delta^\mu_t$	fixing boosts
$h_{\mu\nu}$	$=$	$\delta_{ij}\delta^i_\mu\delta^j_\nu$	forced by other choices
M_μ	$=$	$\partial_\mu M$	global inertial coordinates: $\Gamma^\rho_{\mu\nu}=0$
Φ_N	$=$	0	no Newton potential

choice of vector field is motivated by looking at geodesics

- flat space should include M=const.
  * will see that we can allow for more general choices:
    equiv. to M=const by local syms of the theory
      -> defines the notion of orbit of M

intermezzo: geodesics on NC spacetime

- worldline action of non-rel particle of mass m on NC background

$$S = \int d\lambda L = \frac{m}{2} \int d\lambda \frac{\bar{h}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}{\tau_\rho \dot{x}^\rho}$$

[Kuchar],
[Bergshoeff et al]

- gives the geodesic equation with NC connection $\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0\,,$

* reduces to Newton's law $\frac{d^2x^i}{dt^2} + \delta^{ij}\partial_j\Phi = 0\,,$

provided we take
$$M_t = \partial_t M + \Phi\,,$$
$$M_i = \partial_i M\,,$$

- EM and mass current from the action

$$T^\mu{}_\nu = -P_\nu \dot{x}^\mu \qquad\qquad P_\mu = p_\mu - mM_\mu$$

$$T^\mu = -m\dot{x}^\mu$$

residual coordinate trafos of flat NC

- trafos of the TNC geometry that leave flat NC invariant?

$$\Lambda_D = -\lambda - \delta_{z,2}\alpha t\,,$$
$$\xi^t = a + z\lambda t + \delta_{z,2}\alpha t^2\,,$$
$$\xi^i = v^i t + a^i + \lambda^i{}_j x^j + \lambda x^i + \delta_{z,2}\alpha t x^i$$
$$\lambda^i = -v^i - \delta_{z,2}\alpha x^i\,,$$

$$\delta M = \xi^t\partial_t M + \xi^i\partial_i M - (2-z)\lambda M - C - v^i x^i - \frac{1}{2}\delta_{z,2}\alpha x^i x^i\,.$$

* finite versions:

$$M'(x) = M(x) + C$$
$$t' = t + a \qquad M'(x') = M(x)$$
$$x'^i = x^i + a^i \qquad M'(x') = M(x)$$
$$x'^i = R^i{}_j x^j \qquad M'(x') = M(x)$$
$$t' = \lambda^z t \qquad x'^i = \lambda x^i \qquad M'(x') = \lambda^{2-z} M(x)$$
$$x'^i = x^i + v^i t \qquad t' = t \qquad M'(x') = M(x) - \tfrac{1}{2} v^i v^i t + v^i x^i$$

plus special conformal transformation for z=2

$$t' = \frac{t}{1-ct}\,, \qquad x'^i = \frac{x^i}{1-ct}\,, \qquad M'(x') = M(x) + \frac{c}{2}\frac{x^i x^i}{1-ct}\,.$$

Scale invariant FTs on flat NC

- role of M is non-trivial: consider the toy FT models

  - deformed Schroedinger model:

$$S=\int d^{d+1}x\left(-\varphi^2\left[\partial_t(\theta+M)+\frac{1}{2}\partial_i(\theta+M)\partial^i(\theta+M)+a\partial_i\partial^i(\theta+M)\right]\right.$$
$$\left.-\frac{1}{2}\partial_i\varphi\partial^i\varphi-V_0\varphi^{\frac{2(d+2)}{d}}\left(1+b\theta^2\right)\right),$$

  - Lifshitz model:

$$S=\int d^{d+1}x\left[\frac{1}{2}\left(\partial_t\phi+\partial^i M\partial_i\phi\right)^2-\frac{\lambda}{2}\left(\partial_i\partial^i\phi\right)^2\right].$$

can we remove M by local transformations (field redefinitions) ?
and get M=const. : depends on the model in question

b=0: $\tilde{\theta}=\theta+M$ $\Rightarrow$ Sch-invariant for a=0
Lif + Galilean boost for a not zero

(consequence of local U(1) symmetry)

b not zero
& Lifsthiz model $\Rightarrow$ only Lif invariance

More on the b=0 model

- in terms of the physical field $\tilde{\theta} = \theta + M$

$$S = \int d^{d+1}x \left(-\varphi^2 \left[\partial_t \tilde{\theta} + \frac{1}{2} \partial_i \tilde{\theta} \partial^i \tilde{\theta} + a \partial^2 \tilde{\theta} \right] - \frac{1}{2} \partial_i \varphi \partial^i \varphi - V_0 \varphi^{\frac{2(d+2)}{d}} \right)$$

Lifshitz invariance
+ Galilean boost

$$t = t' , \qquad x^i = x'^i - v^i t' ,$$
$$\tilde{\theta} = \tilde{\theta}' + \frac{1}{2} v^i v^i t' - v^i x'^i ,$$

Orbits of M

- the M functions related to M=const by residual trafos are characterized by

$$\tilde{\Phi} = \partial_t M + \frac{1}{2}\partial_i M \partial^i M = 0 .$$

$$0 = \partial_i \partial_j \partial^j M ,$$

$$0 = \partial_i \partial_j M - \frac{1}{d}\delta_{ij}\partial_k \partial^k M .$$

* maximal orbit underlies Sch symmetry (as in undeformed Sch model)

-> will be useful later when we look at Lif vacuum (in holography)

- families of M solutions:

$$M = C + \frac{(x^i - x_0^i)(x^i - x_0^i)}{2(t - t_0)}$$

$$M = C - \frac{1}{2}V^i V^i t + V^i x^i$$

TNC Killing vectors

[Kiritsis,Hartong,NO]2,3

- consider residual trafos with $\delta M = 0$

* correspond to conformal Killing vectors

$$
\begin{array}{rclrclrcl}
\mathcal{L}_K \tau_\mu & = & -z\Omega\tau_\mu\,, & \mathcal{L}_K \hat{v}^\mu & = & z\Omega\hat{v}^\mu\,, & \mathcal{L}_K \bar{h}_{\mu\nu} & = & -2\Omega\bar{h}_{\mu\nu} \\
\mathcal{L}_K h^{\mu\nu} & = & 2\Omega h^{\mu\nu}\,, & \mathcal{L}_K \Phi_N & = & 2(z-1)\Omega\Phi_N\,, & \mathcal{A}\Omega & = & 0
\end{array}
$$

$$
\begin{array}{ll}
M = \mathrm{cst} & H, D, P_i, J_{ij}\,, \\
M = \dfrac{x^2+y^2}{2t} & K, D, G_i, J_{ij}\,, \\
M = -\dfrac{1}{2}V^iV^it + V^ix^i & H, D, P_i, J_{ij}\,,
\end{array}
$$

- for each choice of M: CKVs form a Lifshitz subalgebra

$$
\begin{array}{ll}
H = \partial_t\,, & P_i = \partial_i\,, \\
G_i = t\partial_i\,, & J_{ij} = x_i\partial_j - x_j\partial_i\,, \\
D = zt\partial_t + x^i\partial_i\,, & K = t^z\partial_t + t^{z-1}x^i\partial_i\,,
\end{array}
$$

Local realization of Schr on M

CKVs can be used to generate maximal orbit: of Sch sym

$$
\begin{aligned}
H &= \partial_t\,, & P_i &= \partial_i\,, \\
G_i &= t\partial_i + x_i\tilde{N}\,, & J_{ij} &= x_i\partial_j - x_j\partial_i\,, \\
D &= zt\partial_t + x^i\partial_i\,,
\end{aligned}
$$

$\tilde{N}$ shifts of M

$$K = t^2\partial_t + tx^i\partial_i + \frac{1}{2}x^ix^i\tilde{N}\,.$$

Lifshitz vacuum (back to holography)

[Kiritsis,Hartong,NO]3

- sources in Lif holography transform under Sch

- can show that sources for Lif vacuum transform under Sch group (via bulk PBH trafos)

  * Killing symmetries = Lif subalgebra of Sch

  - in suitable bulk coords this is dual to:
  flat NC with CKVs spanning Lif and Sch realized locally on $M_\mu = \partial_\mu M$.

have seen: FTs on flat NC realize Sch with mechanism in which M is ``eaten'' up
(generators outside Lif are realized as projective transformations)

-> projective realizations of spacetime syms cannot be predicted by looking at Killing vectors

- can construct z=2 probe actions on Lifshitz bulk geometries that are invariant under Sch (in same manner as in FT setting)

One Lif metric for all M

[Kiritsis,Hartong,NO]3

- Lif metric in Poincare coords $ds^2 = \frac{dr^2}{r^2} - \frac{dt^2}{r^{2z}} + \frac{1}{r^2} dx^i dx^i \qquad B = \frac{dt}{r^z}.$

  (corresponds to M=const)

Lif metric for any M in flat NC-orbit

$$ds^2 = \left(\frac{dr}{r} - \frac{1}{d} \partial_i \partial^i M dt \right)^2 - \frac{dt^2}{r^{2z}} + \frac{1}{r^2} \left(dx^i - \partial^i M dt \right)^2 .$$

* not generally in radial gauge: but can do coord trafo to radial gauge that does not modify the sources

- trafo that close to bdry is bdry dependent rescaling and bdry diffeo
+ order (r^2) trivial bulk diffeo (bringing back to radial gauge)

Transforming to radial gauge (details)

$$M = x^i x^i/2t \qquad ds^2 = \left(\frac{dr}{r} - \frac{dt}{t}\right)^2 - \frac{dt^2}{r^{2z}} + \frac{1}{r^2}\left(dx^i - \frac{x^i}{t}dt\right)^2 \qquad B = \frac{dt'}{r'^2}$$

$$t' = -\frac{1}{T}, \qquad r' = -\frac{R}{T}, \qquad x'^i = -\frac{x^i}{T}.$$

$$ds^2 = -\frac{dT^2}{R^4} + \frac{dR^2}{R^2} + \frac{1}{R^2}dX^i dX^i\,, \qquad B = \frac{dT}{R^2}$$

$$T = -\frac{1}{t}\frac{1}{1 - \frac{1}{4}\frac{r^4}{t^2}},$$

$$R = -\frac{r}{t}\frac{1}{\left(1 - \frac{1}{4}\frac{r^4}{t^2}\right)^{1/2}},$$

$$X^i = -\frac{x^i}{t}.$$

$$ds^2 = \frac{dr^2}{r^2} - \frac{dt^2}{r^4} + \frac{1}{r^2}\delta_{ij}\left(1 - \frac{1}{4}\frac{r^4}{t^2}\right)\left(dx^i - \frac{x^i}{t}dt\right)\left(dx^j - \frac{x^j}{t}dt\right)$$

$$B = \frac{1 + \frac{1}{4}\frac{r^4}{t^2}}{1 - \frac{1}{4}\frac{r^4}{t^2}}\frac{dt}{r^2} - \frac{\frac{r^2}{t}}{1 - \frac{1}{4}\frac{r^4}{t^2}}\frac{dr}{r}.$$

Symmeries of Lif vacuum

- what is bulk realization of residual syms of flat NC ?

-> bulk diffeos that preserve the form of the Lif(M)

- trafos for given M in M=const orbit: Lifshitz
- delta M trafos lie in Sch algebra

generators of PBH transformations that preserve the boundary conditions span the Schroedinger algebra

-> can give rise to global Schroedinger invariance

Particle number current

local transformations of source M -> WI for $\partial_\mu T^\mu$

$$\delta S^{\text{ren}}_{\text{on-shell}}[M] = -\int d^{d+1}x \partial_\mu T^\mu \delta M$$

can show $$\partial_\mu T^\mu = -\partial_t \lambda_1 - \partial_i(\lambda_1 \partial^i M) - \partial_i \partial_j \partial^j \lambda^i + \left(\partial_i \partial_j - \frac{1}{d}\delta_{ij}\partial_k \partial^k\right)\lambda^{ij}$$
$$= -\partial_t \lambda_1 - \partial_i(\lambda_1 \partial^i M) + \left(\partial_i \partial_j \Lambda^{ij} - \frac{1}{d}\partial_i \partial^i \Lambda^k{}_k\right) \equiv \partial_\mu J^\mu \,,$$

-> local Sch inv of on-shell action with flat NC bcs can lead to conserved current

$$\partial_\mu \left(T^\mu - J^\mu\right) = 0 \,.$$

so possible to have conserved particle number associated to local shifts in M (generated by Galilean and special conformal)

Schroedinger invariant probe actions

natural probe action for (z=2, d=2) Lifshitz spacetime

use covariant characterization of Lif

$$ds^2 = (-B_M B_N + \gamma_{MN})\, dx^M dx^N$$

$$B^2 = -1 \qquad \gamma_{MN} \text{ is orthogonal to } B^M$$

$$S = \int d^4x \sqrt{-g}\,\left(\gamma^{MN}\partial_M\phi^\star\partial_N\phi + iq\phi^\star B^M\partial_M\phi - iq\phi B^M\partial_M\phi^\star - (m^2 - q^2)\phi^\star\phi\right)$$

omit $-B^M\partial_M\phi^\star B^N\partial_N\phi$ in $S = \int d^4x\sqrt{-g}\,(D_M\phi^\star D^M\phi - m^2\phi^\star\phi)$

equation of motion is

$$r^2\left(\partial_i\partial^i\phi + 2iqD_t\phi\right) + r^2\partial_r^2\phi - 3r\partial_r\phi - (m^2 - q^2)\phi = 0\,.$$

$$D_t = \partial_t + \partial^i M\partial_i + \frac{1}{2}\partial^2 M r\partial_r$$

eat up M: $\phi = \exp[-iqM - \frac{i}{4}qr^2\partial^2 M]\tilde\phi$ + use all props of M

$$r^2\left(\partial_i\partial^i\tilde\phi + 2iq\partial_t\tilde\phi\right) + r^2\partial_r^2\tilde\phi - 3r\partial_r\tilde\phi - (m^2 - q^2)\tilde\phi = 0$$

-> Sch in UV, flow to Lif in IR ?

Summary

have shown:
Lif vacuum dual to flat NC has local action of Sch group acting on remaining source (M): subgroup is Lif, generated by Killing vectors

boundary theory can have conserved current related to particle number

both precisely in same manner as Sch syms arise in FTs on flat NC

- to show that boundary theory is Sch invariant under global Sch syms
-> need to know type of matter fields living on space & coupling to geometry

one can indeed construct scalar probes on bulk Lif that are inv. under Sch

Next steps

- new perspective on existing results: ncomparison to linearized perturbations
  relation between $\tilde{\Phi}$ and ψ.
- EMD model (emergence of TNC, and role of U(1) ?)
- adding other exponents: (logarithmic running of scalar) alpha/zeta-deformation

$$A_a = r^{-z-\zeta}\alpha_{(0)}\tau_{(0)a}$$

[Kiritsis,Goutereaux][Gath,Hartong,Monteiro,NO]
[Khveshchenko][Karch][Hartnoll,Karch]

- most general soln of Lif for bdry = NC + Newton potential
- 3D bulk (Virasoro-Schroedinger)
- applications to hydrodynamics:
  black branes with zero/non-zero particle number density ? Galilean perfect fluids

  Lifsthiz hdyro: [Hoyos,Kim,Oz]

- Schroedinger holography
- HL gravity and Einstein-aether theories
- adding charge

The end