Quantum Entanglement of locally perturbed thermal states

Joan Simón

University of Edinburgh and Maxwell Institute of Mathematical Sciences

Holographic Renormalization Group and Entanglement
Paris, January 26, 2015

Based on arXiv:1410.2287 and work in progress with P. Caputa, A. Stikonas, T. Takayanagi & K. Watanabe

Motivation

Time dependence of entanglement entropy in perturbed thermal states

- thermalisation
- scrambling time scale from first principles

In the context of holography :

- improve the holographic dictionary
- compare with Shenker & Stanford (shock-waves)

Today, we will explore dynamical aspects of quantum entanglement when a thermal density matrix is perturbed by a localised excitation

Outline

- CFT set-up
- Free scalar 2d CFT at $T = 0$ (warm-up)
- Large c 2d CFTs at finite T
- Comments on scrambling

Set-up

Consider an excited state in a 2d CFT

$$|\Psi_{\mathcal{O}}(t)\rangle = \sqrt{\mathcal{N}}\, e^{-iHt}\, e^{-\epsilon H} \mathcal{O}(0, -\ell)|0\rangle$$

- $\mathcal{O}$ is inserted at $t = 0$ and $x = -\ell$ and dynamically evolved afterwards
- ϵ is a small parameter regularising the UV behaviour of the local operator

Density matrix :

$$\begin{aligned}\rho(t) &= \mathcal{N}\, e^{-iHt}\, e^{-\epsilon H} \mathcal{O}(0, -\ell)|0\rangle\langle 0|\mathcal{O}^\dagger(0, -\ell)\, e^{iHt}\, e^{-\epsilon H} \\ &= \mathcal{N}\, \mathcal{O}(\omega_2, \bar{\omega}_2)|0\rangle\langle 0|\mathcal{O}^\dagger(\omega_1, \bar{\omega}_1)\end{aligned}$$

where $\omega_1 = -\ell + i(\epsilon - it)$, $\omega_2 = -\ell - i(\epsilon + it)$ $(\bar{\omega}_1 = -\ell - i(\epsilon - it))$

Set-up

Our calculations will be done in euclidean signature :

$$\omega = x + i\tau\,, \qquad \bar{\omega} = x - i\tau$$

We use the euclidean continuation : $\tau = it$

- The normalization factor $\mathcal{N}$ is fixed by $\text{Tr}(\rho(t)) = 1$
- The cut-off ϵ can be viewed as a separation in the insertion time appearing in $\rho(t)$

When computing entanglement entropy, we will consider the gluing of $\mathbb{R}^2$ copies at the endpoints of the region A defining the entangling region

- If A is the semi-infinite line $[0, \infty)$, the gluing is done so that the origin $\omega = \bar{\omega} = 0$ is extended from 2π to $2\pi n$ in the n-th copy
- The metric on the n-sheeted Σ_n is $(\omega = r\, e^{i\theta})$

$$ds^2 = dr^2 + r^2 d\theta^2 \quad 0 \leq \theta \leq 2\pi n$$

Set-up

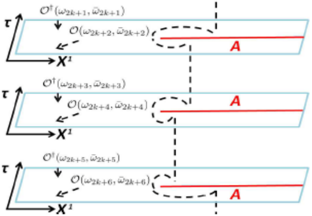
Following Cardy & Calabrese (replica trick)

$$\Delta S_A^{(n)} = \frac{1}{1-n} \log \left(\frac{\mathrm{Tr}\rho_A^n}{\mathrm{Tr}\left(\rho_A^{(0)}\right)^n} \right)$$

$$= \frac{1}{1-n} \log \left[\frac{\langle \mathcal{O}(\omega_1, \bar{\omega}_1) \mathcal{O}^\dagger(\omega_2, \bar{\omega}_2) \ldots \mathcal{O}^\dagger(\omega_{2n}, \bar{\omega}_{2n}) \rangle_{\Sigma_n}}{\left(\langle \mathcal{O}(\omega_1, \bar{\omega}_1) \mathcal{O}^\dagger(\omega_2, \bar{\omega}_2) \rangle_{\Sigma_1} \right)^n} \right]$$

Notice no twisted operators were inserted

- $\omega_{2k+1} = e^{2\pi i k} \omega_1$
- $\omega_{2k+2} = e^{2\pi i k} \omega_2$



Free scalar 2d CFT & finite region A

Compute the Renyi entropy variation for $n = 2$
Strategy : Map Σ_2 into Σ_1 using the conformal transformation (uniformization)

$$\frac{\omega}{\omega - L} = z^2$$

The 4-pt function determining the Renyi entropy will equal

$$\begin{aligned}
&\langle \mathcal{O}^\dagger(\omega_1,\bar\omega_1)\mathcal{O}(\omega_2,\bar\omega_2)\mathcal{O}^\dagger(\omega_3,\bar\omega_3)\mathcal{O}(\omega_4,\bar\omega_4)\rangle_{\Sigma_2} \\
&= \prod_{i=1}^{4}\left|\frac{d\omega_i}{dz_i}\right|^{-2\Delta_{\mathcal{O}}} \langle \mathcal{O}^\dagger(z_1,\bar z_1)\mathcal{O}(z_2,\bar z_2)\mathcal{O}^\dagger(z_3,\bar z_3)\mathcal{O}(z_4,\bar z_4)\rangle_{\Sigma_1} \\
&= \prod_{i=1}^{4}\left|\frac{d\omega_i}{dz_i}\right|^{-2\Delta_{\mathcal{O}}} |z_{13}z_{24}|^{-4\Delta_{\mathcal{O}}}\, G(z,\bar z)
\end{aligned}$$

where the cross-ratio $z = \frac{z_{12}z_{34}}{z_{13}z_{24}}$, with $z_{ij} = z_i - z_j$

Free scalar 2d CFT

Altogether

$$\frac{\langle \mathcal{O}^\dagger(\omega_1,\bar{\omega}_1)\mathcal{O}(\omega_2,\bar{\omega}_2)\mathcal{O}^\dagger(\omega_3,\bar{\omega}_3)\mathcal{O}(\omega_4,\bar{\omega}_4)\rangle_{\Sigma_2}}{\left(\langle \mathcal{O}^\dagger(\omega_1,\bar{\omega}_1)\mathcal{O}(\omega_2,\bar{\omega}_2)\rangle_{\Sigma_1}\right)^2} = |z|^{4\Delta_{\mathcal{O}}}|1-z|^{4\Delta_{\mathcal{O}}}\, G(z,\bar{z})$$

We will consider two different excitations with $\Delta_{\mathcal{O}} = \frac{1}{8}$

- When $\mathcal{O}_1 = e^{i\phi/2}$, then

$$G_1(z,\bar{z}) = \frac{1}{\sqrt{|z||1-z|}}$$

- When $\mathcal{O}_2 = \frac{1}{2}\left(e^{i\phi/2} + e^{-i\phi/2}\right)$,

$$G_2(z,\bar{z}) = \frac{|z|+1+|1-z|}{2}\, G_1(z,\bar{z})$$

Specific details

Our points $(z_i, \bar{z}_i)$ equal

$$z_1 = -z_3 = \sqrt{\frac{\ell - t - i\epsilon}{\ell + L - t - i\epsilon}}\,,$$
$$z_2 = -z_4 = \sqrt{\frac{\ell - t + i\epsilon}{\ell + L - t + i\epsilon}}\,.$$

In the limit of small ϵ we obtain

- $(z, \bar{z}) \to (0, 0)$ when $0 < t < \ell$ or $t > L + \ell$

$$z \simeq \frac{L^2\epsilon^2}{4(\ell - t)^2(L + \ell - t)^2}\,, \quad \bar{z} \simeq \frac{L^2\epsilon^2}{4(\ell + t)^2(L + \ell + t)^2}\,.$$

- $(z, \bar{z}) \to (1, 0)$ when $\ell < t < L + \ell$

$$z \simeq 1 - \frac{L^2\epsilon^2}{4(\ell - t)^2(L + \ell - t)^2}\,, \quad \bar{z} \simeq \frac{L^2\epsilon^2}{4(\ell + t)^2(L + \ell + t)^2}\,.$$

Results & interpretation

$$\Delta S_A^{(2)}(\mathcal{O}_1) = 0 \quad \text{all times}$$

$$\Delta S_A^{(2)}(\mathcal{O}_2) = \begin{cases} 0 & 0 < t < \ell\,, \text{ or } t > \ell + L \\ \log 2 & \ell < t < \ell + L \end{cases}$$

- $\Delta S_A^{(2)}(\mathcal{O}_1) = 0$ because it can be viewed as a direct product state

$$e^{i\phi_L/2}|0\rangle_L \otimes e^{i\phi_R/2}|0\rangle_R$$

- Since $\mathcal{O}_2$ creates a maximally entangled state at $x = -\ell$ propagating in opposite directions

$$\frac{1}{\sqrt{2}}\left(e^{i\phi_L/2}|0\rangle_L \otimes e^{i\phi_R/2}|0\rangle_R + e^{-i\phi_L/2}|0\rangle_L \otimes e^{-i\phi_R/2}|0\rangle_R\right)$$

  - Causality makes both pairs to be in the complement of A for $0 < t < \ell$ and $t > \ell + L$
  - for $\ell < t < \ell + L$ one member of the pair lies in A.

Excitations at finite temperature

Same set-up as before, but now

1. we perturb a thermal state :

$$\rho(t) \equiv \mathcal{N}\mathcal{O}(x_2, \bar{x}_2)\, e^{-\beta H}\, \mathcal{O}^\dagger(x_1, \bar{x}_1)$$

   with

$$x_1 = t - l + i\epsilon \qquad \bar{x}_1 = -l - t - i\epsilon$$
$$x_2 = t - l - i\epsilon \qquad \bar{x}_2 = -l - t + i\epsilon\,.$$

2. A pair of operators will be inserted on a cylinder, separated $2i\epsilon$
3. When computing EE, we will be gluing these cylinders. Operator insertions in the n-th cylinder are

$$x_{2n-1} = x_1 + i(n-1)\beta, \qquad x_{2n} = x_2 + i(n-1)\beta$$

Comment on interpretation

Local excitation is not a state in the radial quantisation sense
Energy density : expectation value of the stress tensor component T_{tt} in the previous state

$$\langle T_{tt}(x,\bar{x})\rangle_{\mathcal{O}} \equiv \frac{\langle \mathcal{O}^\dagger(x_1,\bar{x}_1)\, T_{tt}(x,\bar{x})\mathcal{O}(x_2,\bar{x}_2)\rangle_{C_1}}{\langle \mathcal{O}^\dagger(x_1,\bar{x}_1)\mathcal{O}(x_2,\bar{x}_2)\rangle_{C_1}}.$$

For the particular insertion points :

$$\langle T_{tt}(x)\rangle_{\mathcal{O}} = \frac{\frac{4\pi^2\Delta_O}{\beta^2}\sin^2\left(\frac{2\pi\epsilon}{\beta}\right)}{\left(\cosh\left(\frac{2\pi(l-t+x)}{\beta}\right)-\cos\left(\frac{2\pi\epsilon}{\beta}\right)\right)^2} + \frac{\frac{4\pi^2\Delta_O}{\beta^2}\sin^2\left(\frac{2\pi\epsilon}{\beta}\right)}{\left(\cosh\left(\frac{2\pi(l+t+x)}{\beta}\right)-\cos\left(\frac{2\pi\epsilon}{\beta}\right)\right)^2} + \frac{\pi^2 c}{3\beta^2}\,.$$

Excitations at finite temperature

At finite temperature, Renyi entropies still satisfy

$$\Delta S_A^{(n)} = \frac{1}{1-n} \log \left[\frac{\langle \mathcal{O}^\dagger(\omega_1, \bar{\omega}_1) \mathcal{O}(\omega_2, \bar{\omega}_2) \ldots \mathcal{O}(\omega_{2n}, \bar{\omega}_{2n}) \rangle_{C_n}}{\left(\langle \mathcal{O}^\dagger(\omega_1, \bar{\omega}_1) \mathcal{O}(\omega_2, \bar{\omega}_2) \rangle_{C_1} \right)^n} \right]$$

but evaluated on an n-sheeted cylinder, C_n

Strategy :

- The plane and the cylinder are conformally related $z = e^{2\pi x/\beta}$
- Compose this map with the previous uniformization one

This will map the problem to the plane Σ_1 !!

Conformal maps at finite temperature

C_2 with coordinates $x = \sigma + i\tau$ and semi-infinite cuts at $\tau = 0$ can be mapped to Σ_1

$$z(x) = \sqrt{e^{\frac{2\pi x}{\beta}} - 1}$$

- For a finite interval cut $[0, L]$, the map could be

$$z(x)^2 = \frac{w(x) - 1}{w(x) - w(L)} \quad \text{with} \quad \omega(x) = e^{2\pi x/\beta}$$

Renyi entropy

The main correlator still satisfies

$$\frac{\langle \mathcal{O}(x_1,\bar{x}_1)\mathcal{O}^\dagger(x_2,\bar{x}_2)\mathcal{O}(x_3,\bar{x}_3)\mathcal{O}^\dagger(x_4,\bar{x}_4)\rangle_{C_2}}{\left(\langle \mathcal{O}(x_1,\bar{x}_1)\mathcal{O}^\dagger(x_2,\bar{x}_2)\rangle_{C_1}\right)^2} = |z_A(1-z_A)|^{4\Delta_O}\, G(z_A,\bar{z}_A)\,,$$

Thus, the Renyi entropy (n=2) equals

$$\Delta S_A^{(2)} = -\log\left(|z_A(1-z_A)|^{4\Delta_O}\, G(z_A,\bar{z}_A)\right)$$

where the cross-ratio satisfies

$$z_A \equiv \frac{z_{12}z_{34}}{z_{13}z_{24}} = \frac{1}{2}\left(1-\frac{z_1^2+z_2^2}{2z_1z_2}\right)$$

because $z_3 = z(x_1+i\beta) = -z_1$ and $z_4 = z(x_2+i\beta) = -z_2$

Large c 2d CFTs

Using Fateev & Ribault : $\Delta_{\mathcal{O}} \ll c$

$$G(z,\bar{z}) \simeq |z|^{-4\Delta_O} \Rightarrow \Delta S_A^{(2)} \simeq -4\Delta_O \log\left(|1-z_A|\right)$$

Analysis of the cross-ratios allows us to infer :

- $z_A,\ \bar{z}_A \to 0$ for $t < l$: $\Delta S_A^{(2)} \to 0$
- $z_A \to 1$, keeping $\bar{z}_A \to 0$ for $t > l$:

$$\Delta S_A^{(2)} \simeq 4\Delta_O \log\left(\frac{\beta}{\pi\epsilon}\right) + O(\epsilon^2) \quad t \gg \ell$$

Comparison with vacuum excitations

The same Renyi entropy growth for a localised excitation over the vacuum

$$\Delta S_A^{(2)} \simeq 4\Delta_O \log\left(\frac{2t}{\epsilon}\right) \quad t \gg \ell$$

- Finite temperature saturates the time logarithm growth at the scale

$$t_{\max} \simeq \frac{\beta}{2\pi}$$

Remark : this is the time required for a massive particle to reach the horizon of a BH

Comments on scrambling

- Consider a thermofield double set-up.
- Perturbed the system at t_ω by a primary localised operator $\mathcal{O}$
- Evolve unitarily

Measure the amount of entanglement at $t = 0$ using the mutual information

$$I(A : B; t_\omega) = S_A + S_B - S_{A\cup B}$$

We can ask what the time scale t_ω has to be so that the perturbation can not be distinguished from the original thermal state (scrambling time)

$$\Delta I(A : B; t_\omega) = \Delta S_A + \Delta S_B - \Delta S_{A\cup B} = 0$$

Thermofield double set-up

Consider two non-interacting 2d CFTs, say CFT_L and CFT_R, with isomorphic Hilbert spaces $\mathcal{H}_{L,R}$

Thermofield double state :

$$|\Psi_\beta\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\frac{\beta}{2}E_n} |n\rangle_L |n\rangle_R$$

- $|n\rangle_L$ is an eigenstate of the hamiltonian H_L acting on $\mathcal{H}_L$ with eigenvalue E_n (and similarly for $|n\rangle_R$).
- $|n\rangle_L$ is the CPT conjugate of the state $|n\rangle_R$
- Notation : $|n\rangle_L \otimes |n\rangle_R$ as $|n\rangle_L |n\rangle_R$.
- Thermal reduced density

$$\rho_R(\beta) = \text{tr}_{\mathcal{H}_L}\left(|\Psi_\beta\rangle\langle\Psi_\beta|\right) = \frac{1}{Z(\beta)} \sum_{n\in\mathcal{H}_R} e^{-\beta E_n} |n\rangle_R \langle n|_R \ ,$$

Thermofield double : observables

- Single sided correlators are thermal

$$\langle\Psi_\beta|\,\mathcal{O}_R(x_1,t_1)\ldots\mathcal{O}_R(x_n,t_n)\,|\Psi_\beta\rangle = \mathrm{tr}_{\mathcal{H}_R}\left(\rho_R(\beta)\mathcal{O}_R(x_1,t_1)\ldots\mathcal{O}_R(x_n,t_n)\right)\,.$$

- Two sided correlators : by analytic continuation

$$\langle\Psi_\beta|\,\mathcal{O}_L(x_1,-t)\ldots\mathcal{O}_R(x_n',t_n')\,|\Psi_\beta\rangle = \mathrm{tr}_{\mathcal{H}_R}\left(\rho_R(\beta)\mathcal{O}_R(x_1,t-i\beta/2)\ldots\mathcal{O}_R(x_n',t_n')\right)\,.$$

Will use this observation when computing Renyi entropies

CFT considerations

As discussed by Morrison & Roberts (see also Hartman & Maldacena) :

- *single* sided thermal correlation functions are computed on a *single* cylinder with periodicity $\tau \sim \tau + \beta$
- *two-sided* correlators involve a path integral over a cylinder with the same periodicity $\tau \sim \tau + \beta$, where *all* operators $\mathcal{O}_R$ are inserted at $\tau = i\beta/2$, whereas $\mathcal{O}_L$ are inserted at $\tau = 0$

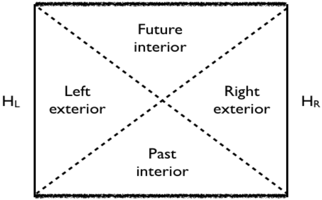
Set-up : Consider thermofield double state

- two finite intervals: $A = [L_1, L_2]$ in the left CFT_L and $B = [L_1, L_2]$ in the right CFT_R
- perturb the TFD by the insertion of a local primary operator $\mathcal{O}_L$ acting on CFT_L at $x = 0\,, t_L = -t_\omega$

Bulk interpretation

1. *Single* BH in thermal equilibrium : evolution by a boost ($H_R - H_L$)

$$H_{\text{tf}} = \mathbb{I}_L \otimes H_R - H_L \otimes \mathbb{I}_R \,.$$



   - Time propagates upwards in $\mathcal{H}_R$ and downwards in $\mathcal{H}_L$.
   - Thermofield double is (boost) invariant

2. Approximate description of the state at $t = 0$ of *two* AdS black holes ($H_R + H_L$)

$$H = \mathbb{I}_L \otimes H_R + H_L \otimes \mathbb{I}_R \equiv H_R + H_L \,.$$

Time propagates upwards in both boundaries

Calculation of S_A

$$S_A = -\lim_{n\to 1} \frac{1}{n-1} \log\left(\operatorname{Tr} \rho_A^n(t)\right)$$

where

$$\operatorname{Tr} \rho_A^n(t) = \frac{\langle \psi(x_1, \bar{x}_1) \sigma(x_2, \bar{x}_2) \tilde{\sigma}(x_3, \bar{x}_3) \psi^\dagger(x_4, \bar{x}_4) \rangle_{C_n}}{\left(\langle \psi(x, \bar{x}_1) \psi^\dagger(x_4, \bar{x}_4) \rangle_{C_1}\right)^n}$$

with the insertion points

$$\begin{aligned} x_1 &= -i\epsilon\,, \quad x_2 = L_1 - t_\omega - t, \quad x_3 = L_2 - t_\omega - t\,, \quad x_4 = +i\epsilon \\ \bar{x}_1 &= +i\epsilon\,, \quad \bar{x}_2 = L_1 + t_\omega + t, \quad \bar{x}_3 = L_2 + t_\omega + t\,, \quad \bar{x}_4 = -i\epsilon \end{aligned}$$

with conformal dimensions

$$H_\psi = n h_\psi\,, \quad H_\sigma = n h_\sigma = n \frac{c}{24}\left(n - \frac{1}{n}\right)$$

Conformal maps

1. From the cylinder to the plane

$$\omega(x) = e^{2\pi x/\beta}$$

2. Standard map : $\omega_1 \to 0$, $\omega_2 \to z$, $\omega_3 \to 1$ and $\omega_4 \to \infty$

$$z(\omega) = \frac{(\omega_1 - \omega)\omega_{34}}{\omega_{13}(\omega - \omega_4)}$$

where the cross-ratio satisfies

$$z = \frac{\omega_{12}\omega_{34}}{\omega_{13}\omega_{24}}$$

Result

$$S_A^{(n)} = \frac{c(n+1)}{6}\log\left(\frac{\beta}{\pi\epsilon_{UV}}\sinh\frac{\pi(L_2-L_1)}{\beta}\right) - \frac{1}{n-1}\log\left(|1-z|^{4H_\sigma}G(z,\bar{z})\right)$$

where

$$G(z,\bar{z}) = \langle\psi|\,\sigma(z,\bar{z})\tilde{\sigma}(1,1)\,|\psi\rangle$$

Using the large c results derived by Fitzpatrick, Kaplan & Walters in the limit $n \to 1$

$$\Delta S_A = \frac{c}{6}\log\left(\frac{z^{\frac{1}{2}(1-\alpha_\psi)}\bar{z}^{\frac{1}{2}(1-\bar{\alpha}_\psi)}(1-z_\psi^{\alpha})(1-\bar{z}^{\bar{\alpha}_\psi})}{\alpha_\psi\bar{\alpha}_\psi(1-z)(1-\bar{z})}\right)$$

where $\alpha_\psi = \sqrt{1-\frac{h_\psi}{c}}$.

Cross-ratios

The cross-ratios are

$$z = \frac{\sinh\left(\frac{\pi x_{12}}{\beta}\right)\sinh\left(\frac{\pi x_{34}}{\beta}\right)}{\sinh\left(\frac{\pi x_{13}}{\beta}\right)\sinh\left(\frac{\pi x_{24}}{\beta}\right)}$$

$$\simeq 1 - \frac{2\pi i\epsilon}{\beta}\frac{\sinh\frac{\pi(L_2-L_1)}{\beta}}{\sinh\frac{\pi(L_2-t-t_\omega)}{\beta}\sinh\frac{\pi(L_1-t-t_\omega)}{\beta}} + \mathcal{O}(\epsilon^2)$$

$$\bar{z} = \frac{\sinh\left(\frac{\pi \bar{x}_{12}}{\beta}\right)\sinh\left(\frac{\pi \bar{x}_{34}}{\beta}\right)}{\sinh\left(\frac{\pi \bar{x}_{13}}{\beta}\right)\sinh\left(\frac{\pi \bar{x}_{24}}{\beta}\right)}$$

$$\simeq 1 + \frac{2\pi i\epsilon}{\beta}\frac{\sinh\frac{\pi(L_2-L_1)}{\beta}}{\sinh\frac{\pi(L_2+t+t_\omega)}{\beta}\sinh\frac{\pi(L_1+t+t_\omega)}{\beta}} + \mathcal{O}(\epsilon^2)$$

Final result

Analysing the imaginary parts, we reach the conclusions :

- $(z,\bar{z}) \to (1,1)$ for $t+t_\omega < L_1$ and $t+t_\omega > L_2$
- $(z,\bar{z}) \to (e^{2\pi i},1)$ for $L_1 < t+t_\omega < L_2$

The importance of this monodromy has been emphasized by several groups including Asplund, Bernamonti, Galli & Hartman and Roberts & Stanford

The only non-trivial variation in the entanglement entropy is for $L_1 < t+t_\omega < L_2$

$$\Delta S_A = \frac{c}{6}\log\left[\frac{\beta}{\pi\epsilon}\frac{\sin\pi\alpha_\psi}{\alpha_\psi}\frac{\sinh\frac{\pi(L-t-t_\omega)}{\beta}\sinh\frac{\pi(t+t_\omega)}{\beta}}{\sinh\left(\frac{\pi L}{\beta}\right)}\right]$$

where $L = L_2 - L_1$

Calculation of S_B

Very similar, but with different insertion points :

$$\mathrm{Tr}\,\rho_A^n(t) = \frac{\langle \psi(x_1,\bar{x}_1)\sigma(x_5,\bar{x}_5)\tilde{\sigma}(x_6,\bar{x}_6)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_n}}{\left(\langle \psi(x,\bar{x}_1)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_1}\right)^n}$$

with the insertion points

$$x_1 = -i\epsilon\,, \quad x_5 = L_2 + i\frac{\beta}{2} \pm t, \qquad x_6 = L_1 + i\tfrac{\beta}{2} \pm t\,, \quad x_4 = +i\epsilon$$
$$\bar{x}_1 = +i\epsilon\,, \quad \bar{x}_5 = L_2 - i\frac{\beta}{2} \mp t, \qquad \bar{x}_6 = L_1 - i\tfrac{\beta}{2} \mp t\,, \quad \bar{x}_4 = -i\epsilon$$

We always obtain the expected thermal answer at all times

$$S_B = \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\sinh\frac{\pi L}{\beta}\right)$$

Calculation of $S_{A\cup B}$

Very similar, but with different insertion points :

$$\mathrm{Tr}\,\rho^n_{A\cup B}(t) = \frac{\langle\psi(x_1,\bar{x}_1)\sigma(x_2,\bar{x}_2)\tilde{\sigma}(x_2\bar{x}_3)\sigma(x_5,\bar{x}_5)\tilde{\sigma}(x_6,\bar{x}_6)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_n}}{\left(\langle\psi(x,\bar{x}_1)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_1}\right)^n}$$

with the insertion points

$$\begin{aligned} x_1 &= -i\epsilon\,, \quad x_2 = L_1 - t_\omega - t, \quad x_3 = L_2 - t_\omega - t\,, \quad x_4 = +i\epsilon \\ \bar{x}_1 &= +i\epsilon\,, \quad \bar{x}_2 = L_1 + t_\omega + t, \quad \bar{x}_3 = L_2 + t_\omega + t\,, \quad \bar{x}_4 = +i\epsilon \\ x_5 &= L_2 + i\frac{\beta}{2} \pm t, \qquad x_6 = L_1 + i\tfrac{\beta}{2} \pm t\,, \\ \bar{x}_5 &= L_2 - i\frac{\beta}{2} \mp t, \qquad \bar{x}_6 = L_1 - i\tfrac{\beta}{2} \mp t\,. \end{aligned}$$

Result

Using t-channel and assuming the dominant contribution comes from ψ itself, we find (in the large t_ω limit)

$$S_{A\cup B} \simeq 2S_B - \frac{c}{3}\log\left|\frac{x}{1-x}\right| - \frac{1}{n-1}\log\left[|1-z_5|^{4H_\sigma}|1-\tilde{z}_2|^{4H_\sigma}G(z_2,\bar{z}_2)G(z_5,\bar{z}_5)\right]$$

with

$$\left|\frac{x}{1-x}\right| \simeq 4\,e^{-\frac{2\pi t_\omega}{\beta}}\sinh^2\frac{\pi L}{\beta}$$

Scrambling time

It turns out that for large t_ω

$$\Delta I(A:B;t_\omega) = 0 \quad \Leftrightarrow \quad I(A:B;t_\omega) = 0$$

we can derive

$$t_\omega \sim \frac{L}{2} + \frac{\beta}{2\pi}\log\frac{S}{\pi E_\psi} + f(L)$$

where we used

$$\frac{\beta}{\pi\epsilon}\frac{\sin\pi\alpha_\psi}{\alpha_\psi} \sim \frac{\pi E_\psi}{S}$$

with $S = \frac{\pi c}{3\beta}$ and $E_\psi = \frac{h_\psi}{\epsilon}$.

Conclusions & Further directions

- Using conformal maps, we computed the correlators responsible for Renyi entropies (n=2) without using twist operators.
- We studied the time evolution of Renyi entropies (n=2) for localised excitations at finite temperature in 2d CFTs at large c and for the thermofield double
- Using large c techniques should provide more results in the CFT side
- We can exactly match our CFT results for S_A and S_B for the thermofield double set-up in the bulk
- The bulk estimation for the scrambling time reproduces the CFT answer except for the finite function of L.
- Open question : Statistics of OPE coefficients