Quantum Entanglement and Tensor Networks

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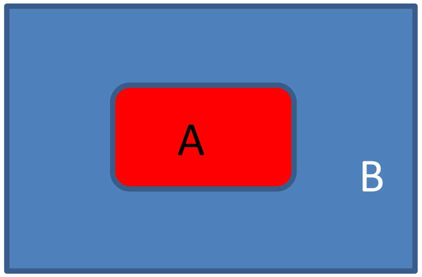
Outline

- Path Integrals, the renormalization group, and Tensor networks
- Topological quantum order, long range entanglement and anyon condensation

Entanglement Entropy

- The quantum correlations of a bipartite system can be quantified using the von Neumann entropy of the reduced density matrix



$$|\psi\rangle = \sum_{ij} X_{ij}|i\rangle|j\rangle \qquad \rho_A = X.X^\dagger$$

$$S(A) = -\mathrm{Tr}\left(\rho.\log\rho\right)$$

  – S has an operational meaning in terms of entanglement of formation / distillation: conversion into EPR pairs

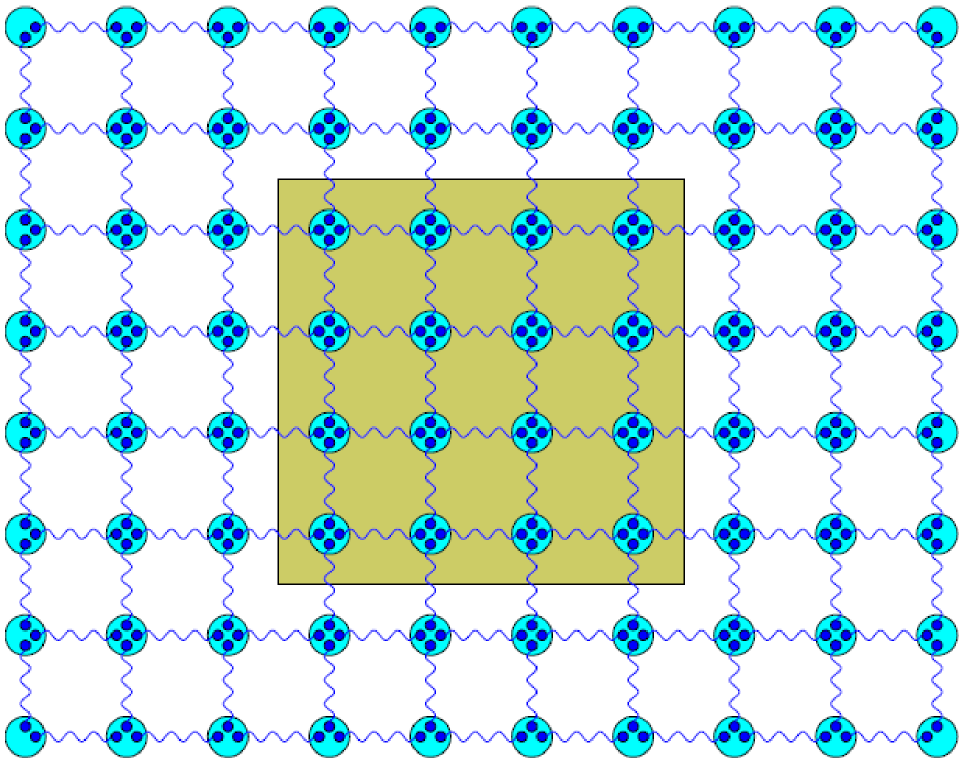
- Page's theorem ['93]: take a random bipartite state according to the Haar measure on a Hilbert space with |A|<<|B| , then with very high probability S(A) is close to its maximal possible value:

$$\int dU\, S(A) = \log|A| - \frac{1}{2}\frac{|A|}{|B|} + \dots$$

  – For a random state of a quantum many body system, this means that the entanglement entropy will scale as the volume of the region where it is calculated

Area laws for the entanglement entropy

- Ground states of quantum many body systems (including quantum field theories) have very few entanglement: area vs. volume entropy

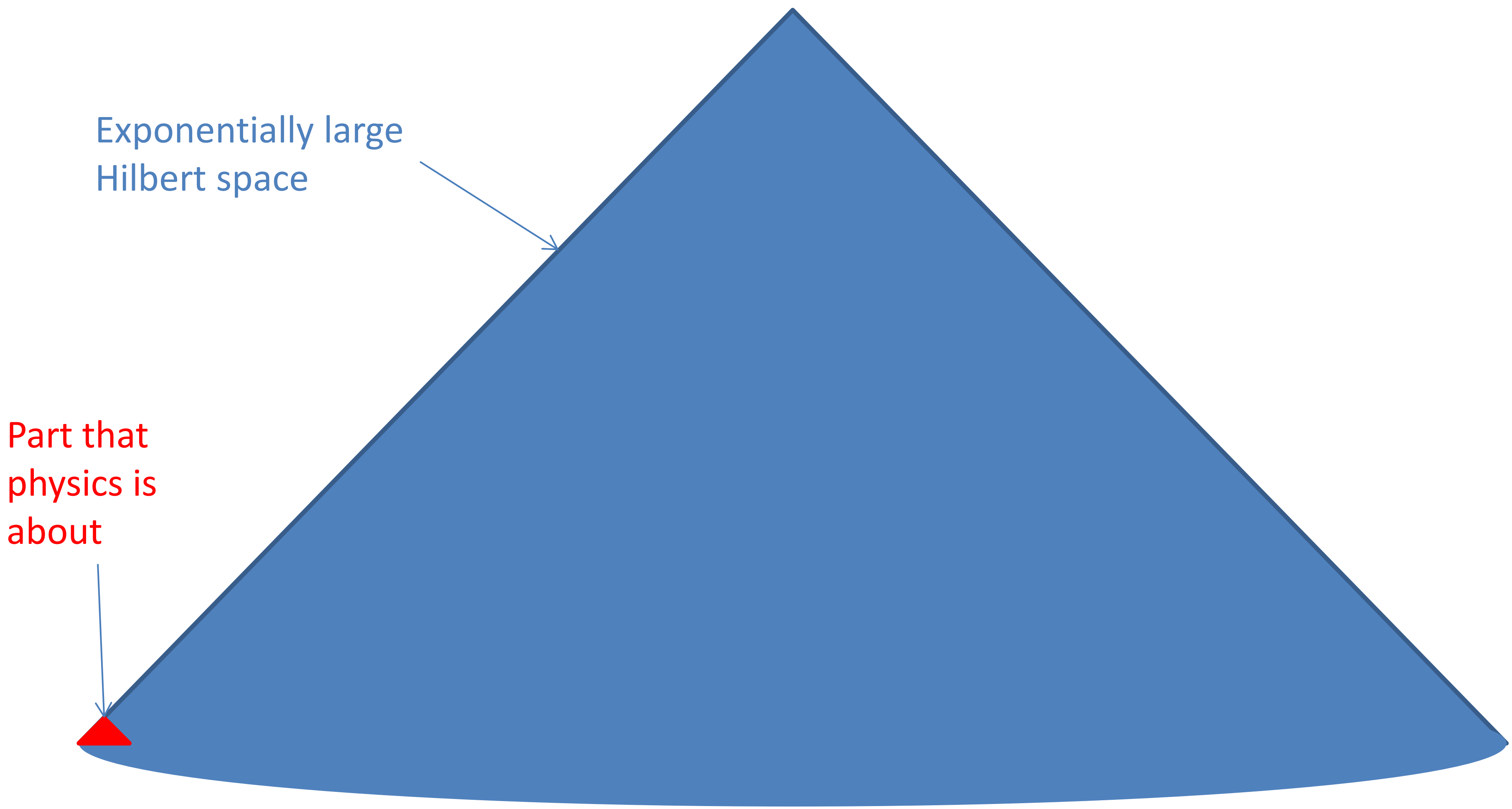


Bombelli et al '86;
Srednicki '93
Vidal, Kitaev '03
Cardy, Calabrese '04
Hastings '06
...

- The leading term of the entanglement corresponds to "local" entanglement, and this suggests that there should be an efficient real-space description of all such quantum many body wavefunctions

Quantum Simulation of Time-Dependent Hamiltonians and the Convenient Illusion of Hilbert Space

David Poulin,[1] Angie Qarry,[2,3] Rolando Somma,[4] and Frank Verstraete[2]

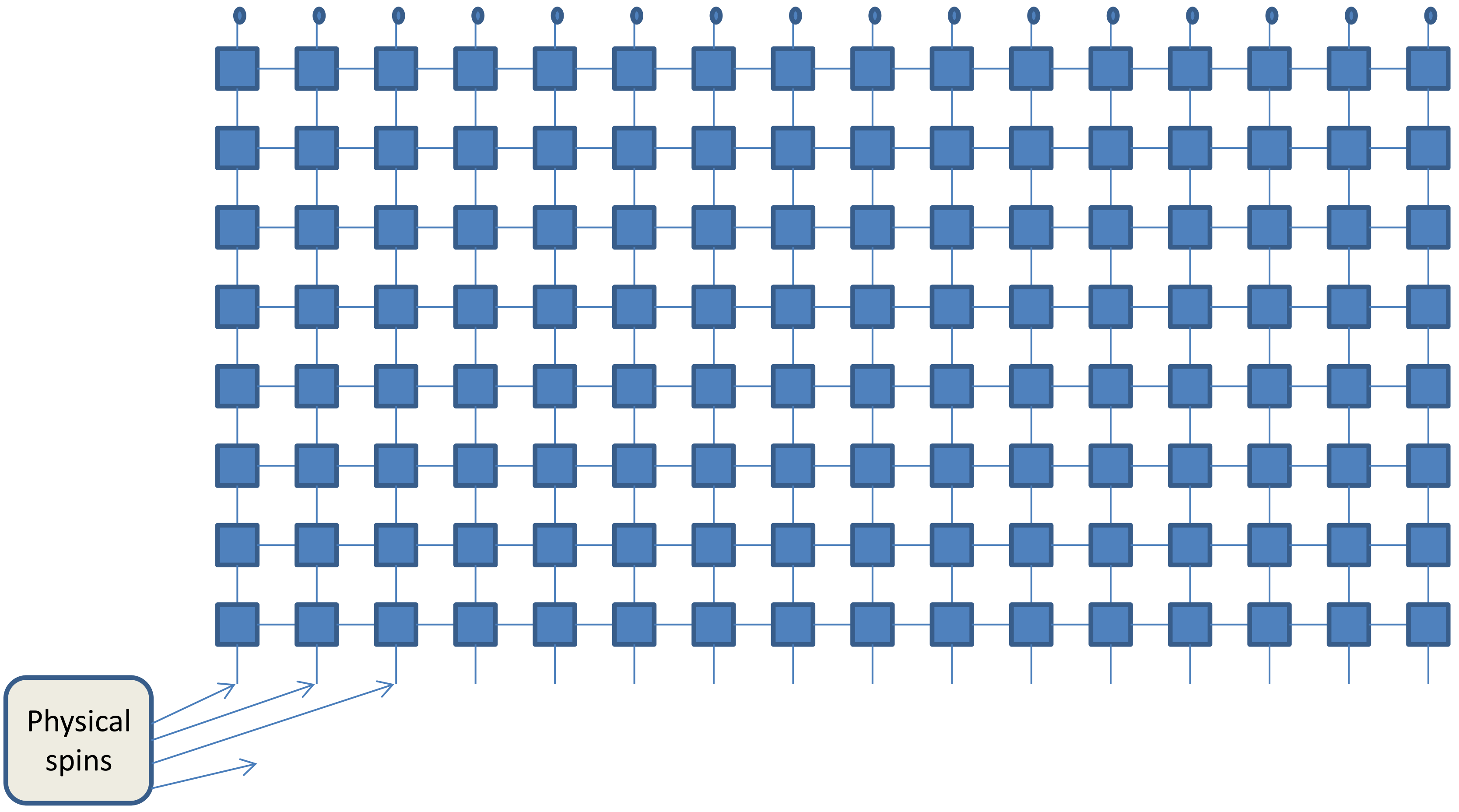


- Can we come up with a systematic way of parameterizing the (low-entanglement) wavefunctions corresponding to ground states of strongly correlated systems?

- This problem is especially relevant for systems with strong interactions (i.e. systems for which perturbations on a fiducial Slater determinant fail)
    - Authoritative examples: Hubbard model, quantum spin systems, strongly coupled (lattice) gauge theories, …
    - There is special interest in getting a deeper understanding of systems exhibiting topological order such as fractional quantum Hall systems and gauge theories with anyonic elementary excitations
        - new phases of matter with applications in quantum computation
        - The low energy wavefunctions in such systems exhibit *long range quantum entanglement*

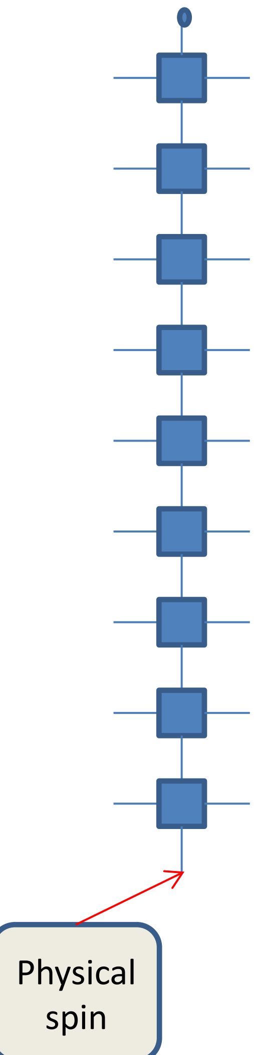
Path Integral representation of ground states

- Let us consider an arbitrarily Hamiltonian of a quantum spin system, and a path integral $\exp\left(-\beta\mathcal{H}\right)|\psi_0\rangle$ representing the ground state for $\beta \to \infty$



Wilson’s RG for quantum impurities

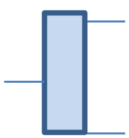
- Let us consider 1 column in this transfer matrix picture:

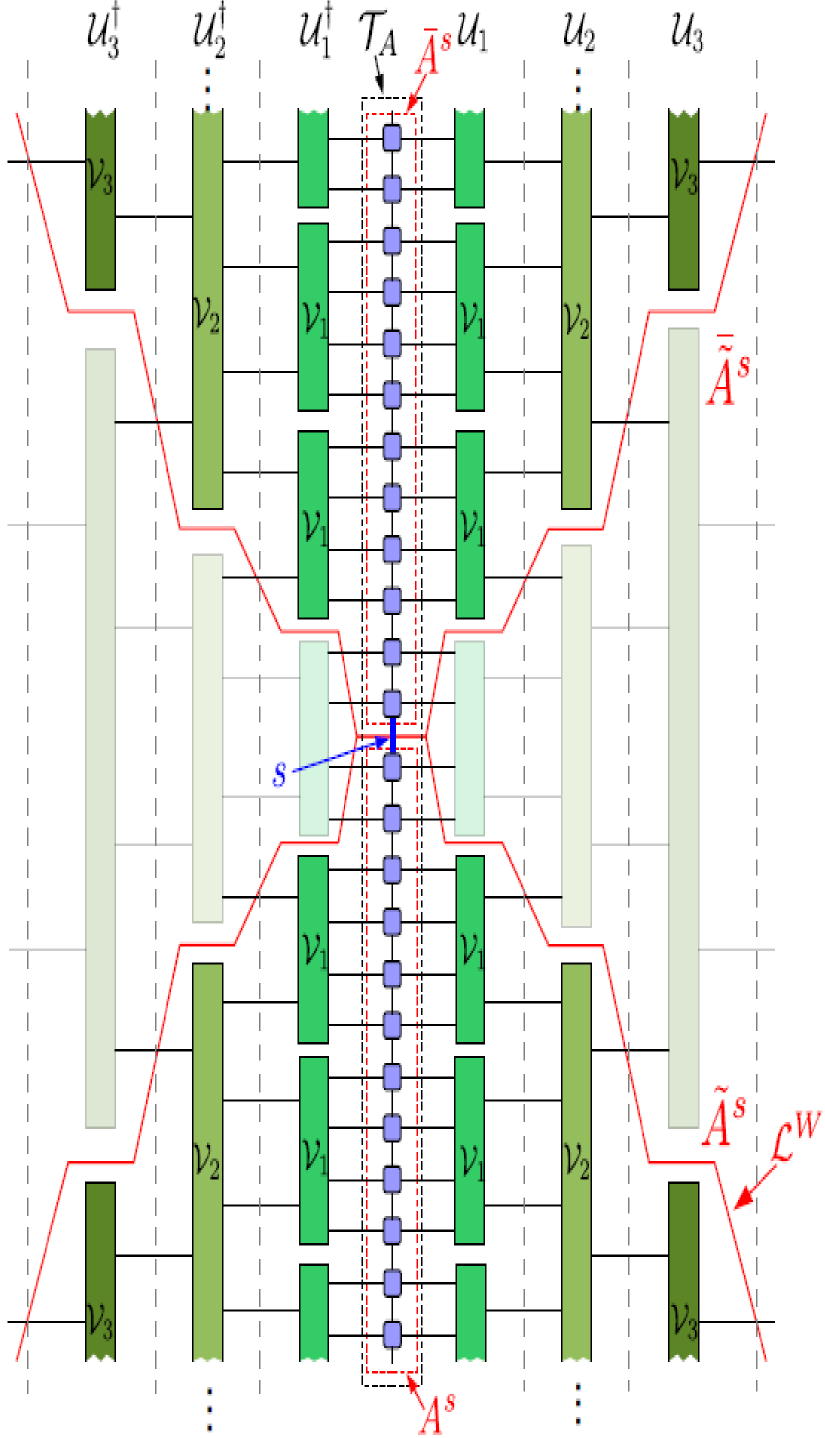


- The physical spin can be understood as a Kondo like impurity attached to a translational invariant system (“conduction electrons”)
- The crucial question that we tackle here: can we compress the information on the “virtual” links such as to obtain a more economical representation of the ground state?
  - Just like Wilson, we can indeed envision devising an RG transformation “compressing” the information of the conduction electrons
  - The dimension of the compressed Hilbert space will be related to the amount of quantum entanglement in the system

Wilson's RG procedure reduces the translational invariant chain to a chain with exponentially fewer degrees of freedom: logarithmic discretization

In terms of quantum gates, this transformation can be understood in terms of isometries





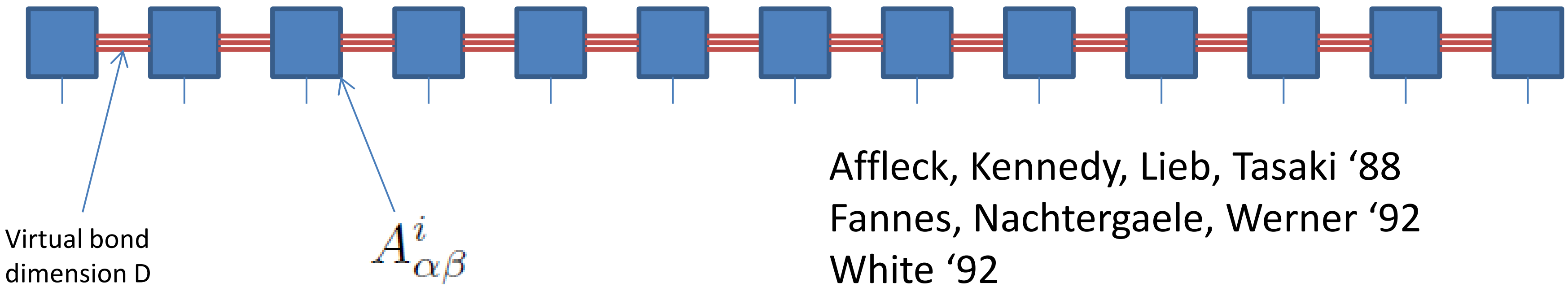
The emanating degrees of freedom contain all the relevant information for describing the physics of the impurity; the different levels correspond to different energy scales

Stopping at some level hence corresponds to introducing a cut-off, but for a gapped system this cut-off is given by the gap and hence does not introduce an extra approximation

V. Zauner et al., arXiv:1408.5140

Finitely Correlated States / Matrix Product States

- The picture that is emerging is that any ground state of a gapped Hamiltonian can be presented by a FCS/MPS:



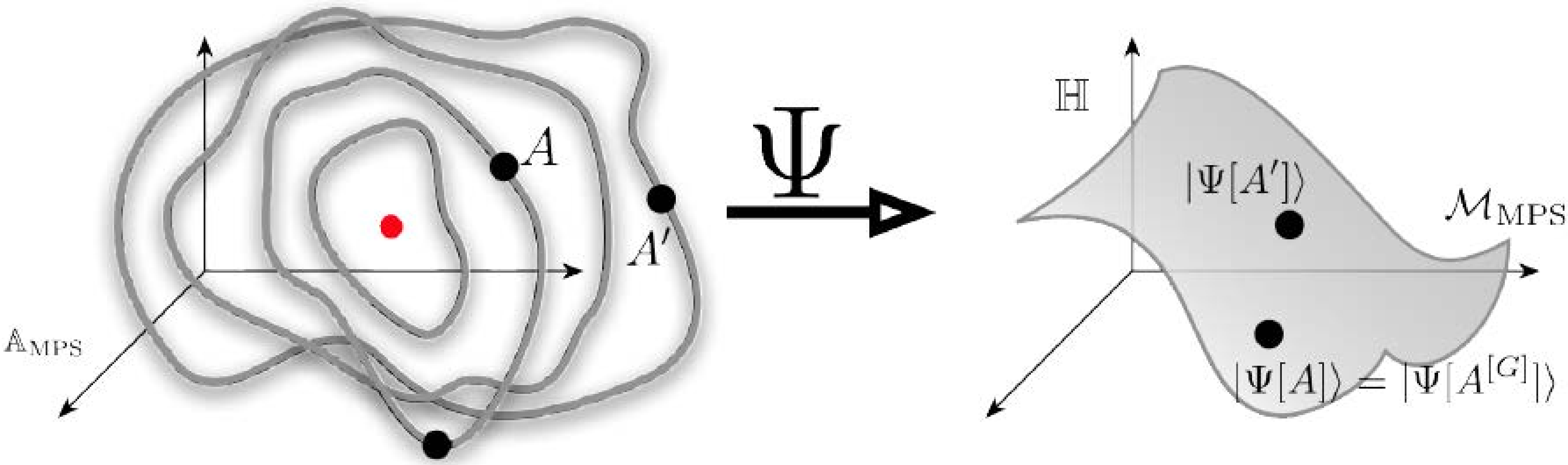
Affleck, Kennedy, Lieb, Tasaki '88
Fannes, Nachtergaele, Werner '92
White '92

$$|\psi\rangle = \sum_{i_1 i_2 \cdots} \mathrm{Tr}\left(A^{i_1}_{\alpha_1\alpha_2} A^{i_2}_{\alpha_2\alpha_3} \cdots\right) |i_1\rangle|i_2\rangle \cdots$$

- The entanglement entropy is related to the virtual dimension D, and is related to the gap through the cut-off (the smaller the gap, the higher the number of levels to keep)
  - Vice versa: any state satisfying an area law for the entanglement entropy has an efficient representation in terms of MPS
  - Also, any injective MPS is unique ground state of a gapped parent H
  - FCS/MPS allow to break exponential wall in 1D

Manifold of MPS

Haegeman et al. '12

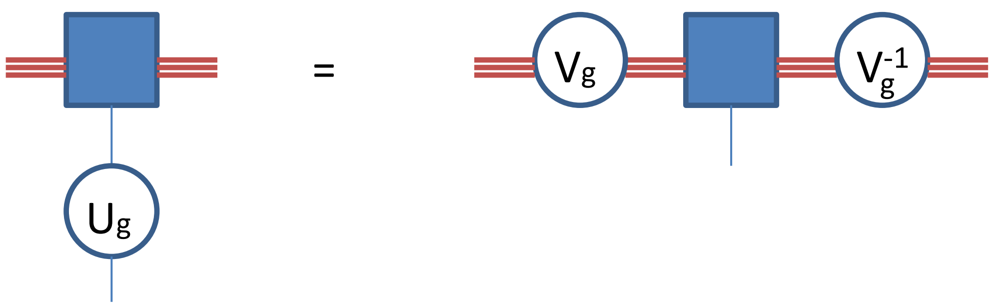


- Set of MPS form a low-dimensional manifold in the humongous physical Hilbert space: parameterization of <u>*all*</u> ground states
- Instead of solving (linear) Schrodinger equations on the large Hilbert space, we project those equations onto the manifold of MPS and we get nonlinear differential equations for the parameters $A^i_{\alpha\beta}$: time dependent variational principle
  - Ground states can be found by evolving in imaginary time: MPS form perfect variational states (and basis for DMRG of White)
  - Effective Hamiltonians are obtained by tangent spaces of the respective ground states (hence allowing to determine gap, dispersion relations, …)

1D SPT phases of matter

- Classification of phases of matter of 1-D spin chains under adiabatic paths preserving a symmetry: symmetry protected topological order

- Pollmann, Turner, Berg, Oshikawa '10; Chen, Gu, Wen '11; Fidkowksi, Kitaev '12; Schuch, Perez-Garcia, Cirac '12: all those different phases can be classified according to the projective representations of the global symmetry as manifested on the virtual level of the MPS

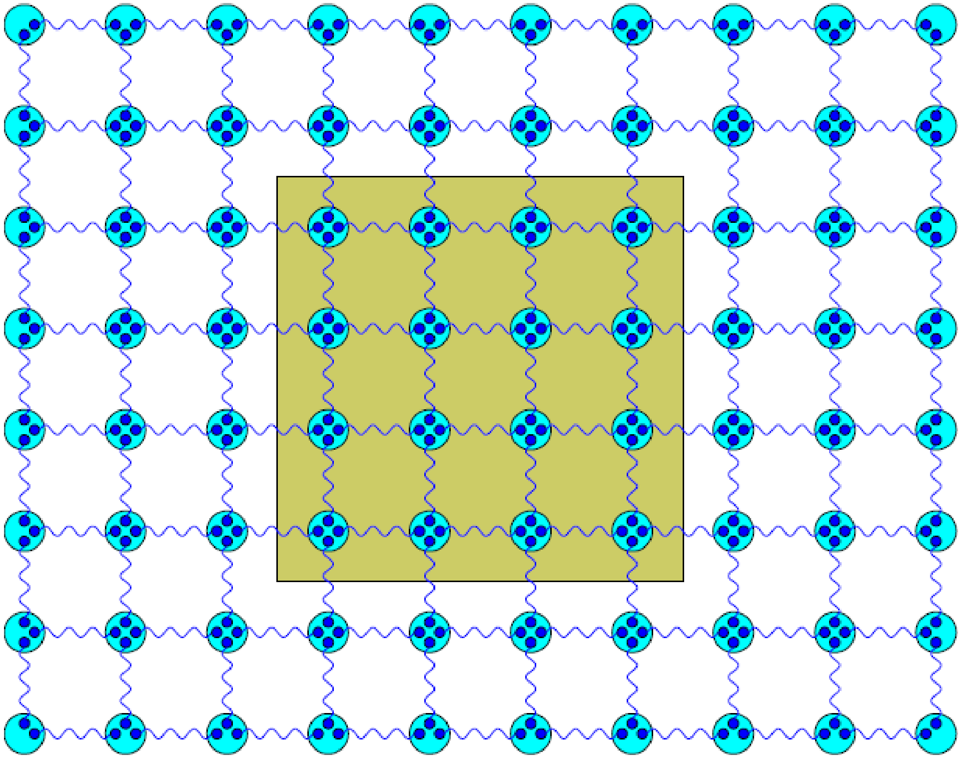


Pérez-García et al. '08

  - V_g forms a projective representation of the group representing the global symmetry of the state, and the corresponding 2-cocycle cannot change on an adiabatic path (gap has to close when interpolating 2 states with different cocycle)

Projected Entangled Pair States

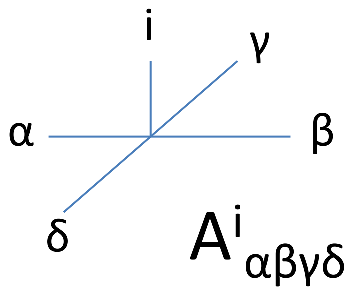
- 2-dimensional version of MPS: PEPS

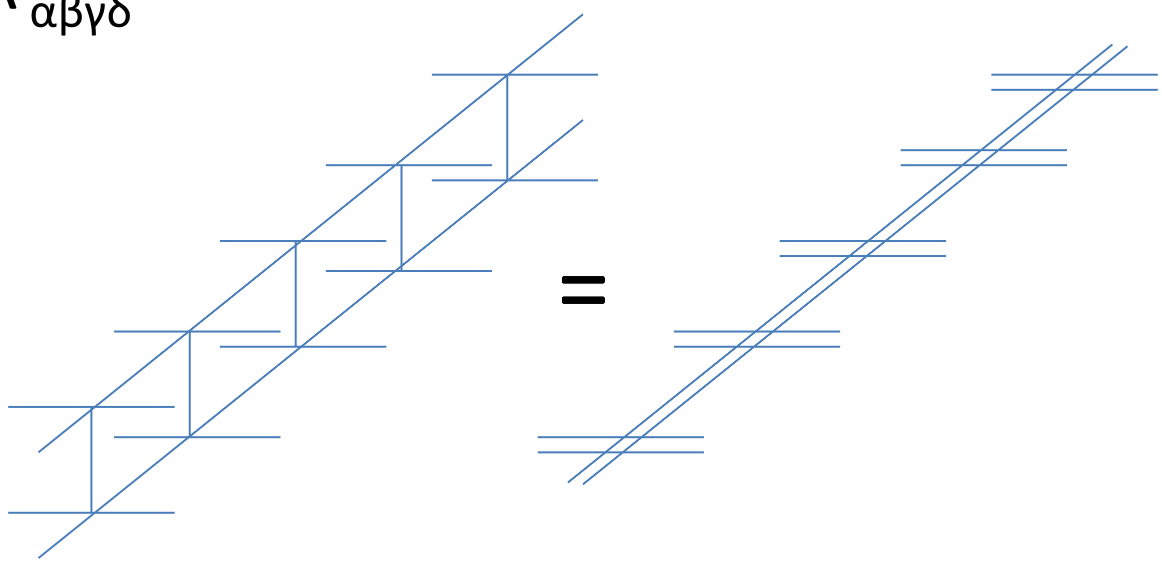


- Satisfies area law by construction; is ground state of local Hamiltonian; …

FV, Cirac '04

- The properties of PEPS are encoded in the eigenvalues and eigenvectors of the corresponding transfer matrices (similar to classical statistical physics, although here we have a “double layer” structure and here we deal with a state as opposed to a Hamiltonian)

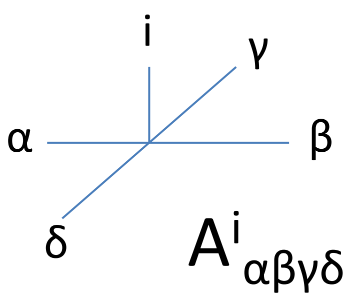




Example: Resonating Valence Bond State

Anderson, Baskaran '87

- Equal superposition of all possible singlets on a square lattice: simple PEPS with bond dimension 3

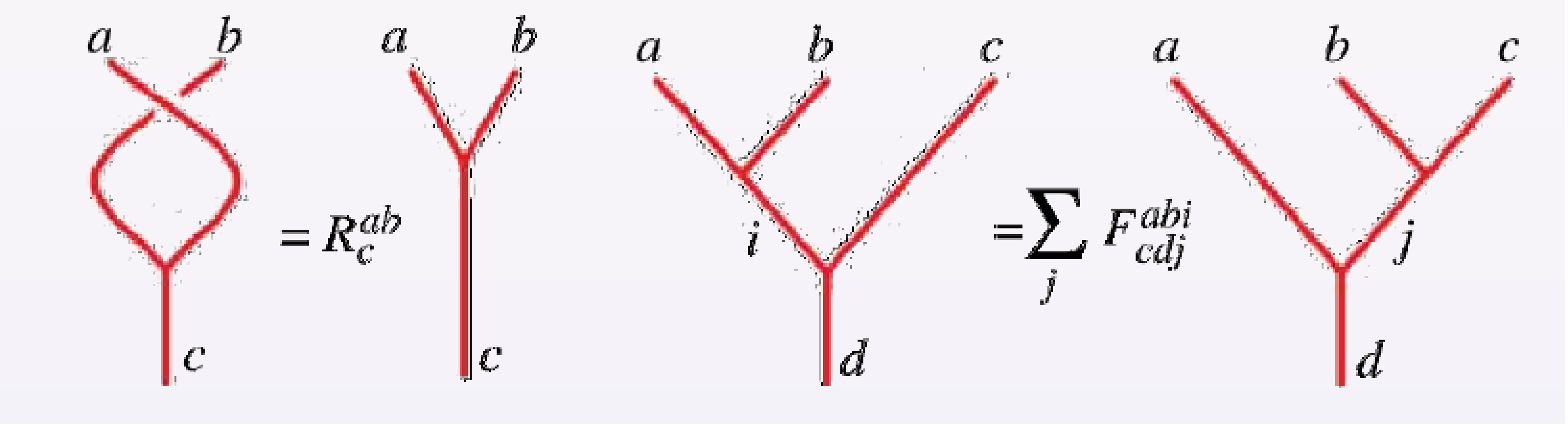


$$A^i \equiv |0\rangle\left(\langle 0222| + \langle 2022| + \langle 2202| + \langle 2220|\right)$$
$$+|1\rangle\left(\langle 1222| + \langle 2122| + \langle 2212| + \langle 2221|\right)$$

$$|I\rangle = |0\rangle|1\rangle - |1\rangle|0\rangle + |2\rangle|2\rangle$$

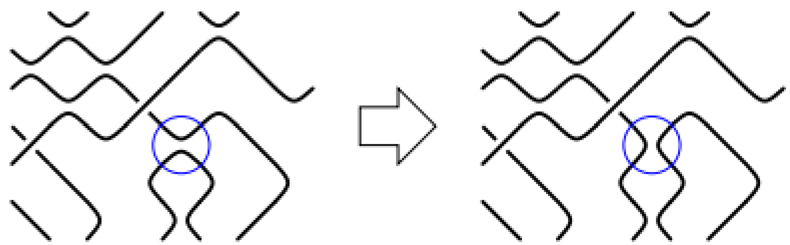
- This state is actually much more interesting on a frustrated lattice (e.g. Kagome) because it is then in a topological nontrivial phase ("G-injective")

- Such topological phases are stable under perturbations (Hastings) and exhibit “long-range” entanglement (Wen) instead of having a local order parameter
- elementary excitations (anyons) have nontrivial statistics and typically come in pairs: anyons
  - Consistency of a set of fusion rules of anyons is dictated by so-called R and F tensors (Pentagon equation)



Levin-Wen models

- Levin and Wen constructed quantum spin systems realizing any such topological theory (elementary excitations are anyons whose statistics is governed by F)

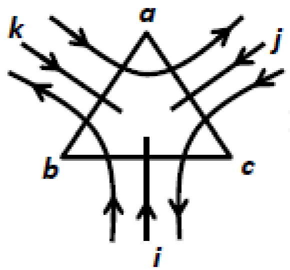


  - Those are all real-space RG fixed points with zero correlation length, but nevertheless long-range entanglement

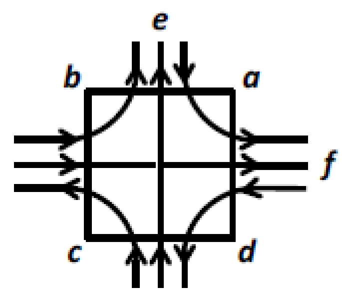
  - In topological QFT : Dijkgraaf-Witten realizes this for the group case with unique fusion rules (quantum doubles and twisted quantum doubles)

PEPS for Levin-Wen string nets

- Define the tensors

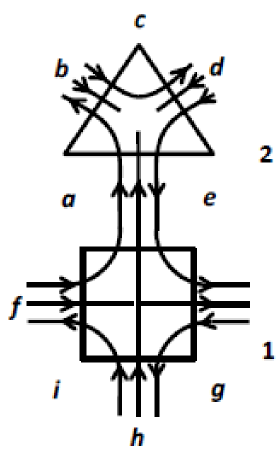


$$:= \sqrt{v_i v_j v_k}\, G^{ijk}_{abc}$$



$$:= G^{ab^{*}e}_{cdf}$$

- The Pentagon equation in tensor network language:



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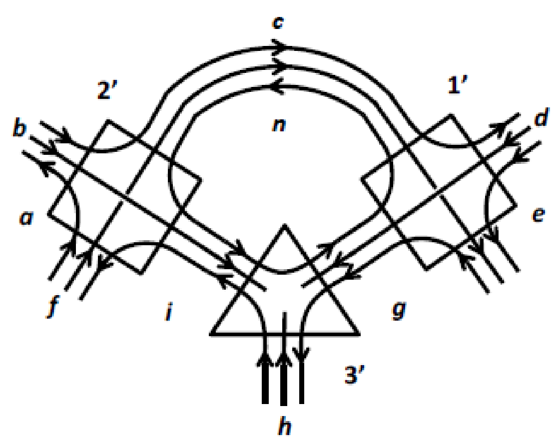
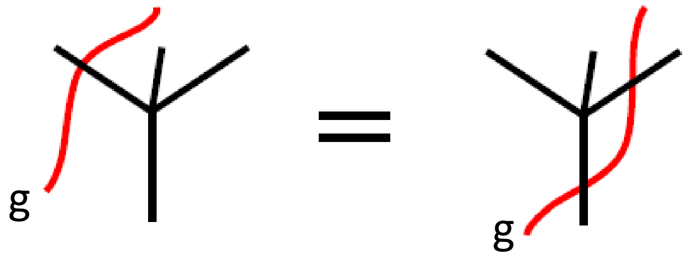
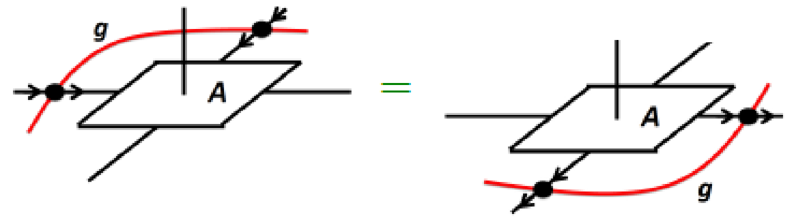


Sahinoglu et al.
arXiv:1409.2150

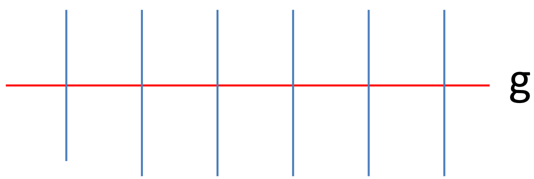
PEPS for topological ordered systems

- Topological order is manifested locally in the PEPS as a symmetry of the tensors:



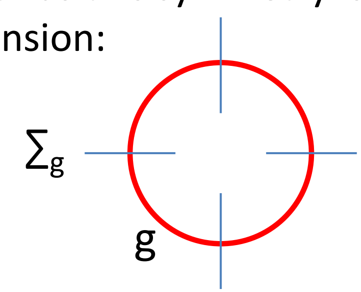


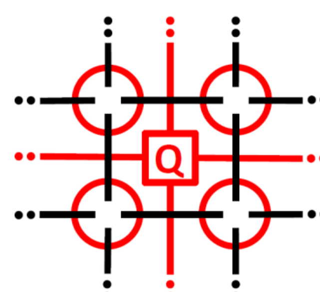
  - Global order parameter for topological ordered systems is hence manifested LOCALLY as a symmetry on the virtual level: the Hilbert space on the virtual level is constrained (subspace), and this subspace is determined by Matrix Product Operators (MPO)



    - The label g labels the anyon type, and those MPO's form a representation

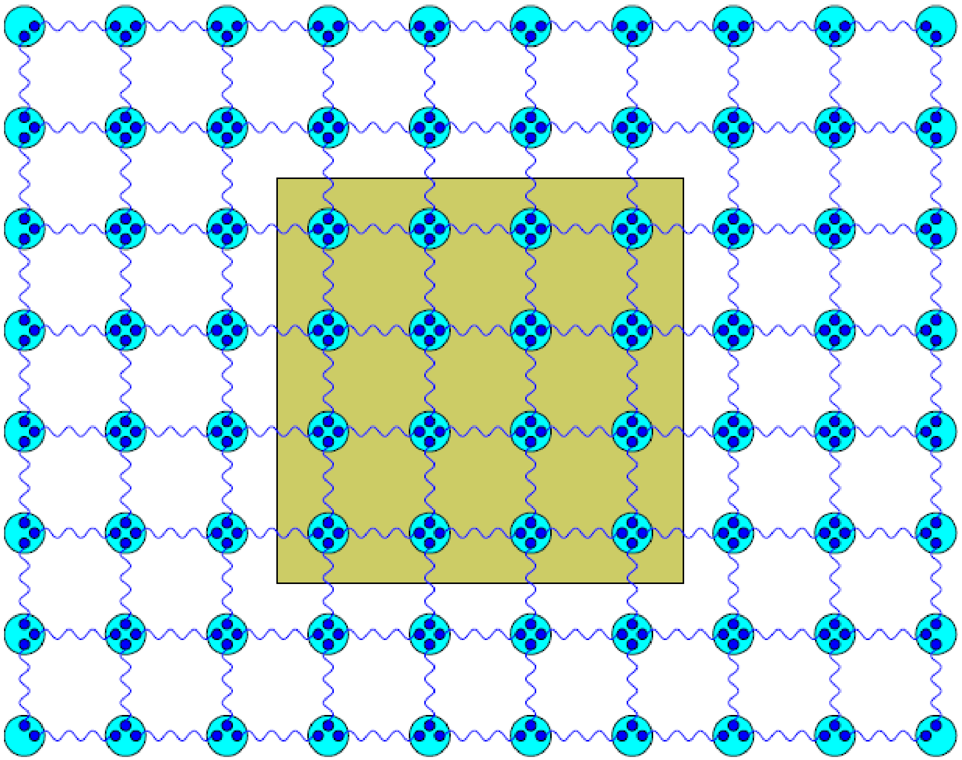
- The simplest PEPS that has this symmetry is of clearly the sum of all those MPO's (=fixed point tensor)+string tension:





Topological Quantum Entropy

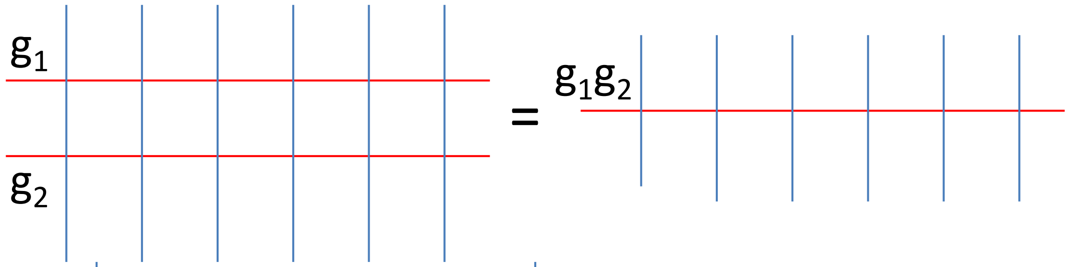
- Topological quantum entropy (Kitaev-Preskill / Levin-Wen ‘06): the entanglement entropy has a correction related to the quantum dimension of the anyons; indeed, the symmetry dictates that the virtual system is constrained on a subspace



$$S(A) = c.L - \gamma$$

$$\gamma = \sqrt{\sum_i d_i^2}$$

- In the group case (quantum doubles and twisted quantum doubles) different topological theories are characterized by different 3-cocycles by which the “group” is realized



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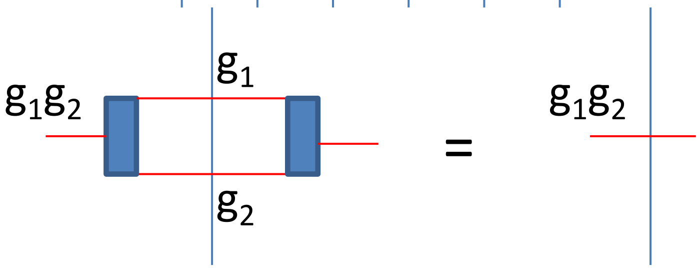
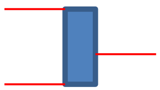
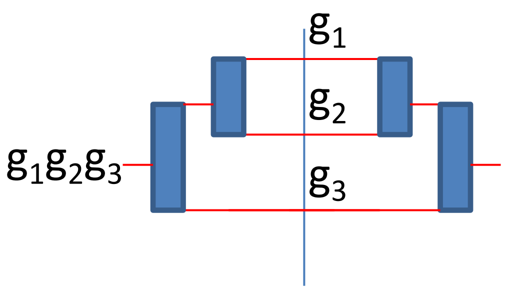


with

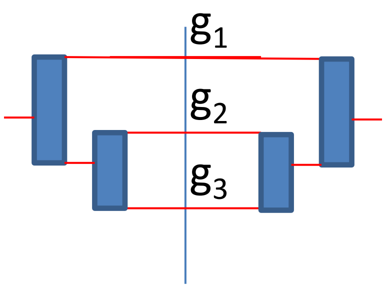


unique up to $e^{i\varphi(g_1,g_2)}$

3-cocycle:



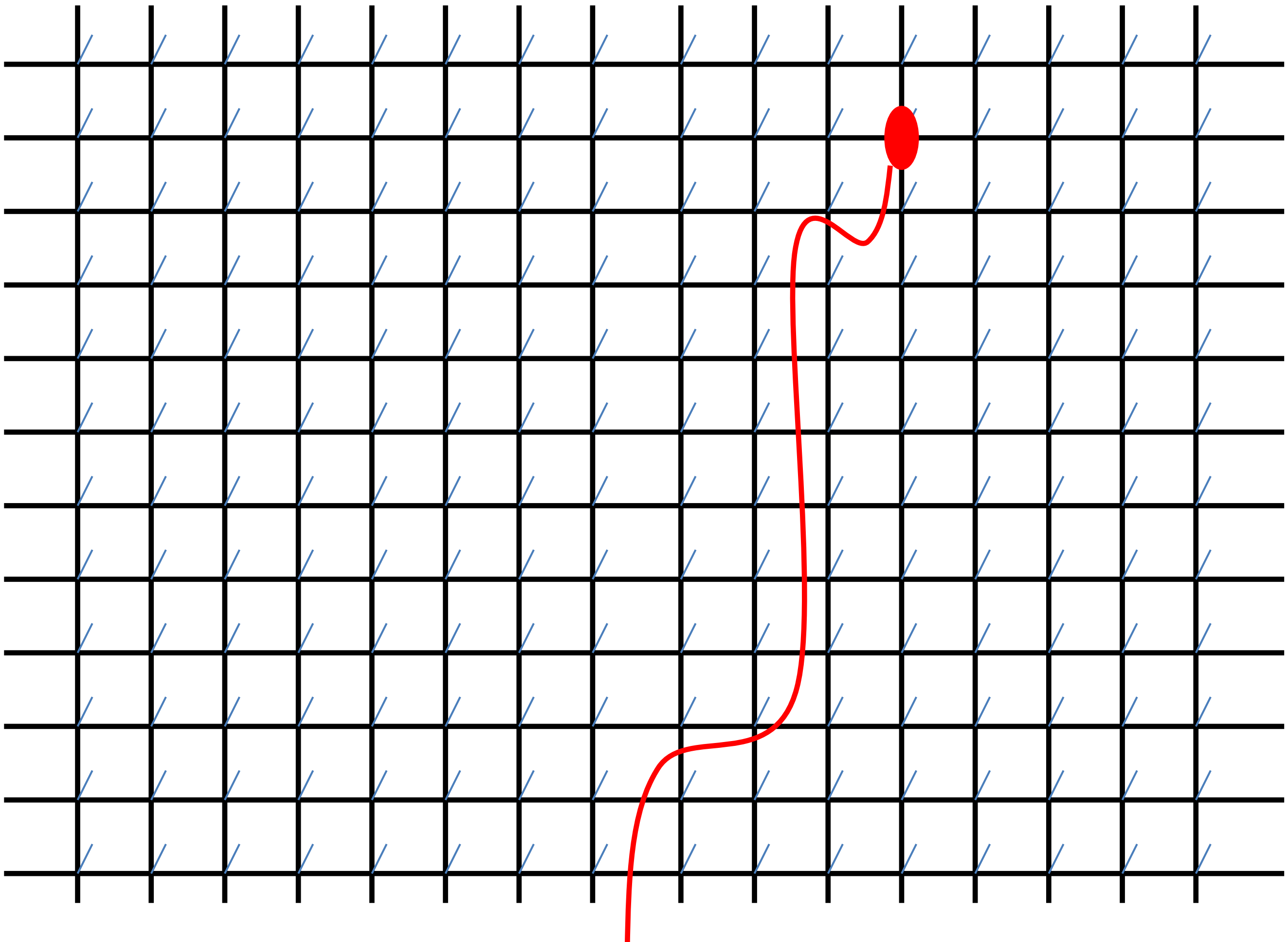
$= e^{i\varphi(g_1,g_2,g_3)}$



Chen et al. ‘12 (for SPT phases)
Buershaper ‘14; Sahinoglu et al. ’14 (for topological phases)

Anyons in PEPS

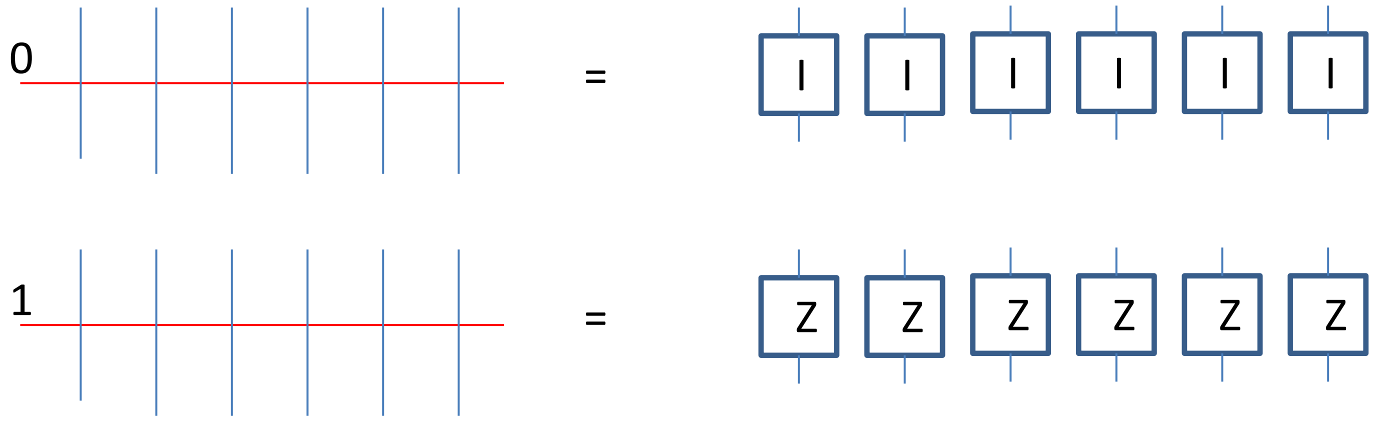
- Anyons "have a string attached to them", and the "pulling through" equations guarantee that it is immaterial where these strings are (but topological properties cannot be changed!)
  - As string has to end, they have to come in pairs (or choice of BC)



There are 2 types of excitations: vertex and plaquette ones; only the latter comes with a string attached on the *virtual* level

Example 1: toric code (trivial)

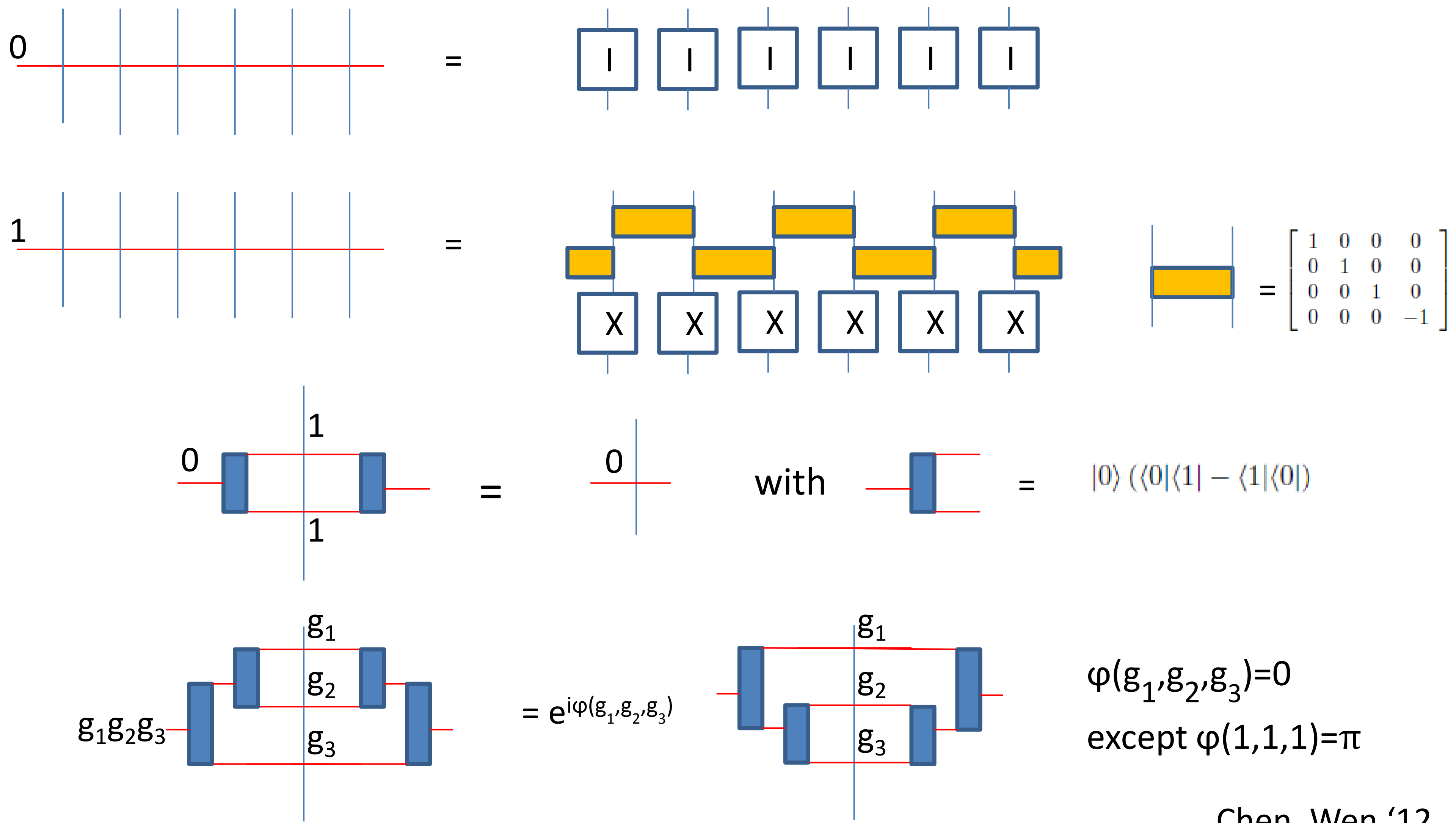
- Toric code: strings are product states



- The virtual system of any block only has support on the even parity subspace (hence the edge modes realize a structure that cannot be obtained by any local Hamiltonian)

Example 2: double semion (nontrivial cocycle $H^3(Z_2,U(1))$)

- Double semion:

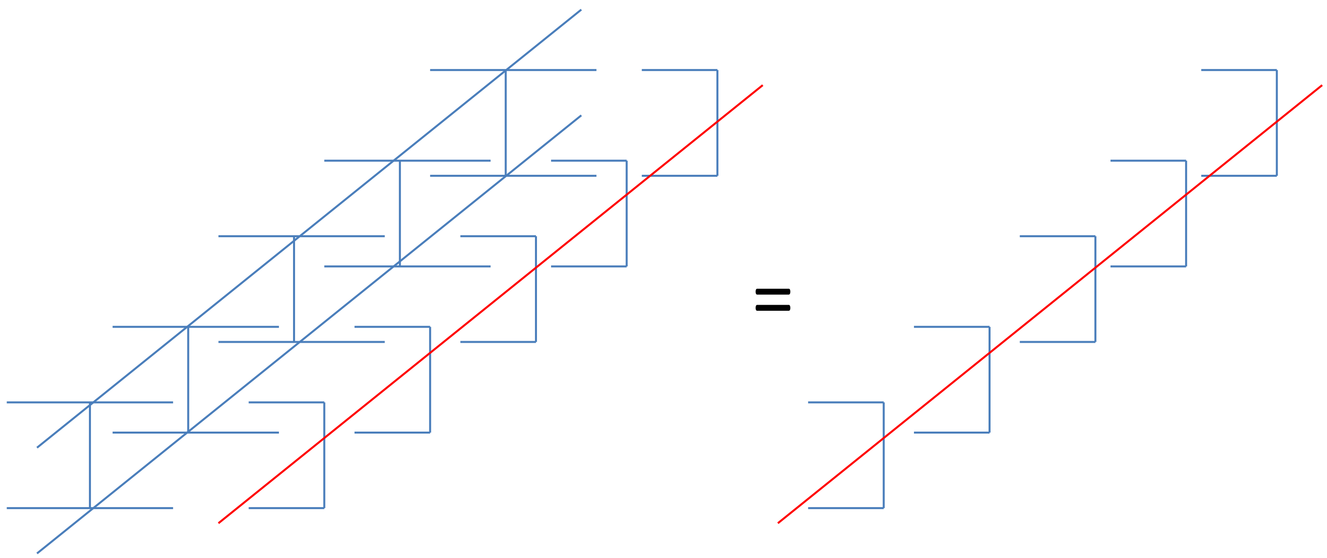


$\varphi(g_1,g_2,g_3)=0$

except $\varphi(1,1,1)=\pi$

Chen, Wen ‘12

Eigenstructure of the doubled transfer matrix



* At the fixed point and in the toplological phase, the degeneracy of the largest eigenvalue 1 is exactly equal to the number of anyon types N + 1 (and not $(N+1)^2$ as could naively be expected from the symmetries)

* The excitations are domain wall excitations tunneling between the different symmetry sectors reflecting the anyons in the true 2D theory

Dispersion relations of the transfer matrix of the RVB on the Kagome lattice

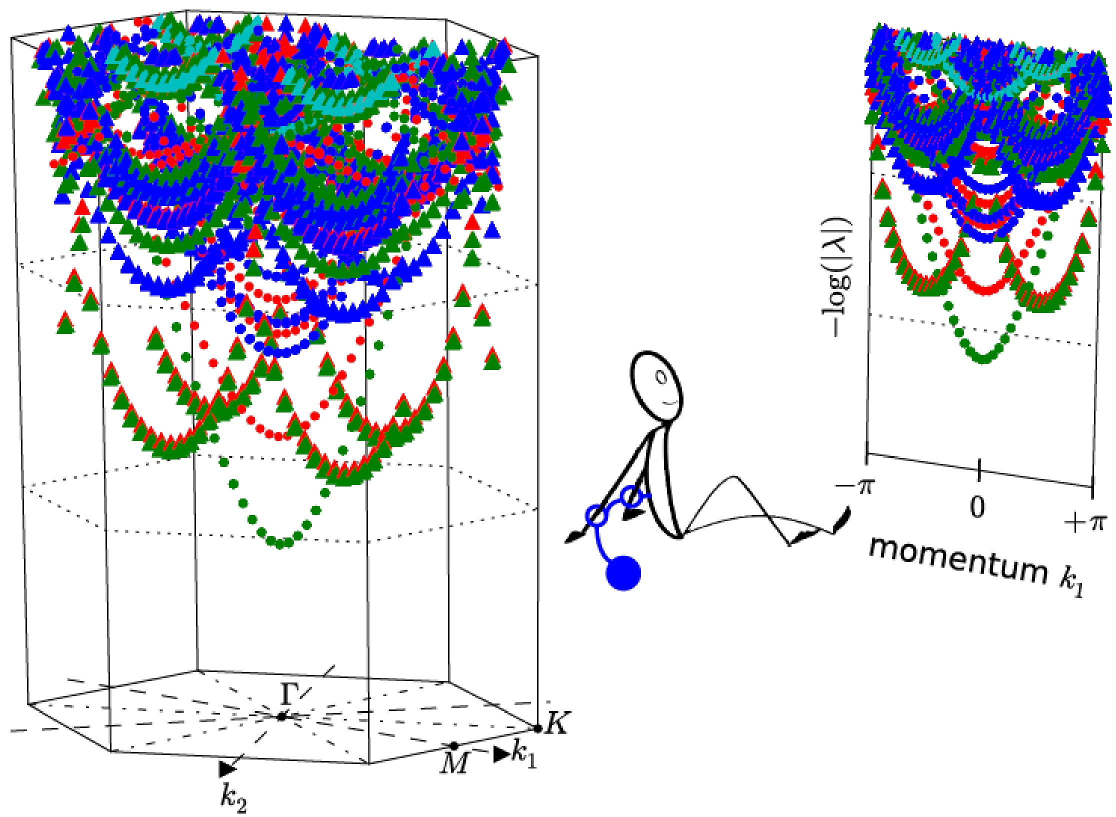
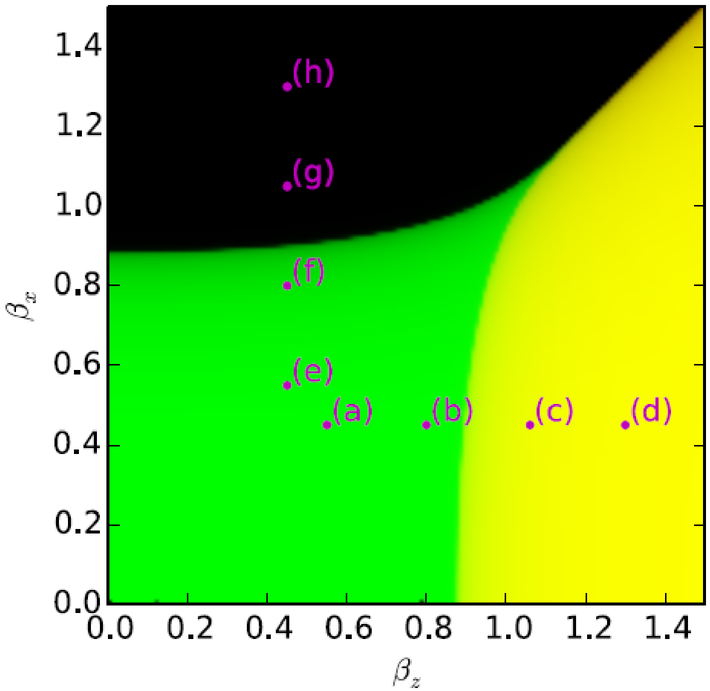


FIG. 8: A condensed matter physicist in Plato's cave, chained by tensor networks, is able to extract the anyons in the real world from their shadow in the spectrum of the transfer matrix. Colors indicate total spin S (red: 0, green: 1/2, blue: 1, cyan: 3/2) whereas marker indicates eigenvalues with trivial (dot) or non-trivial (triangle).

Anyon condensation

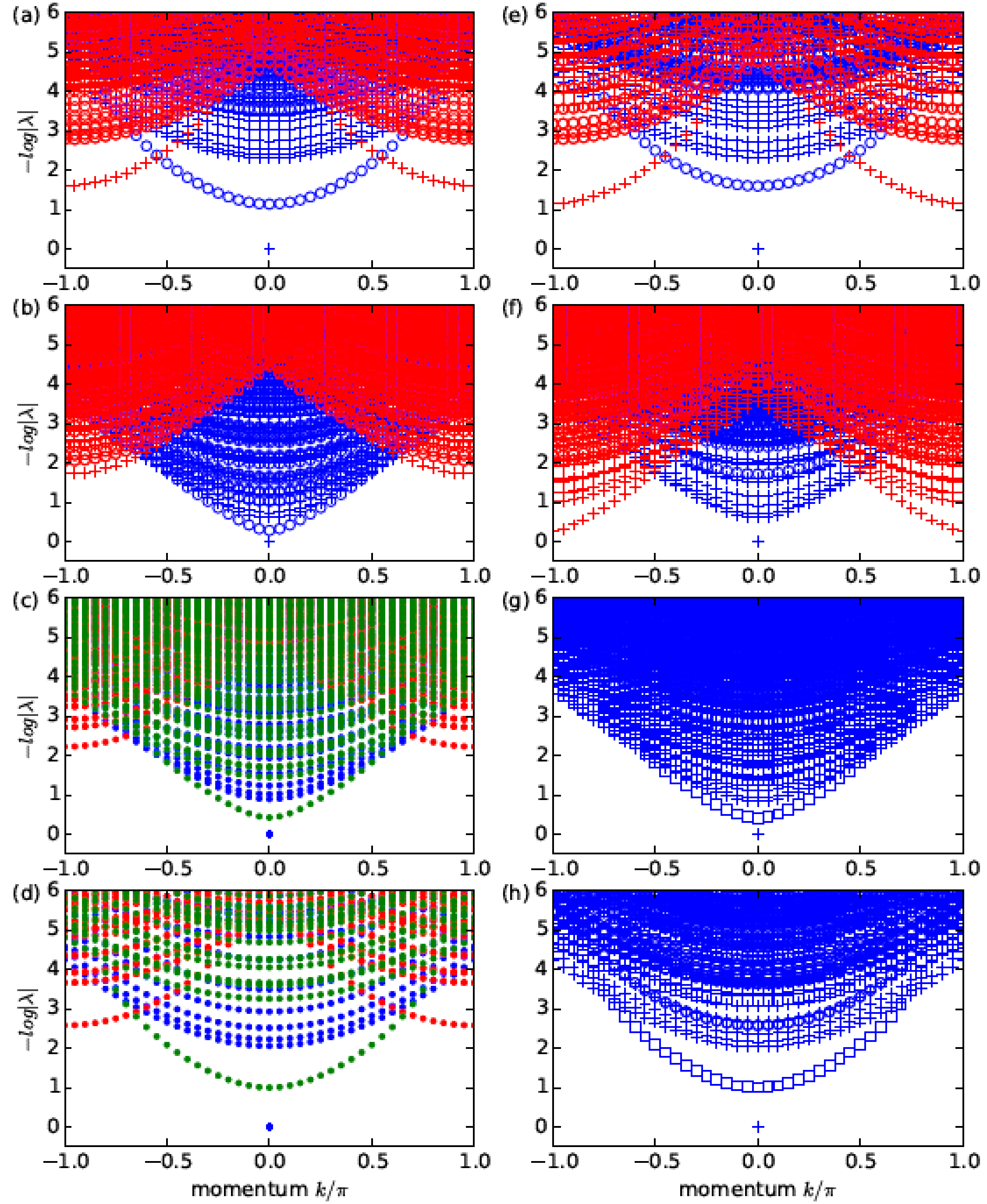
- What happens when anyons condense?
  - Symmetry breaking phase transition at the virtual level: in Plato's cave, the prisoners witness a standard Landau phase transition with a local order parameter
  - Depends strongly whether the vertex or plaquette anyons condense:
    - In one case, we get a unique fixed point
    - In the other case, we get $(N+1)^2$ fixed points
    - This means that the topological phase must break a subgroup of the symmetry

Example: toric code with string tension

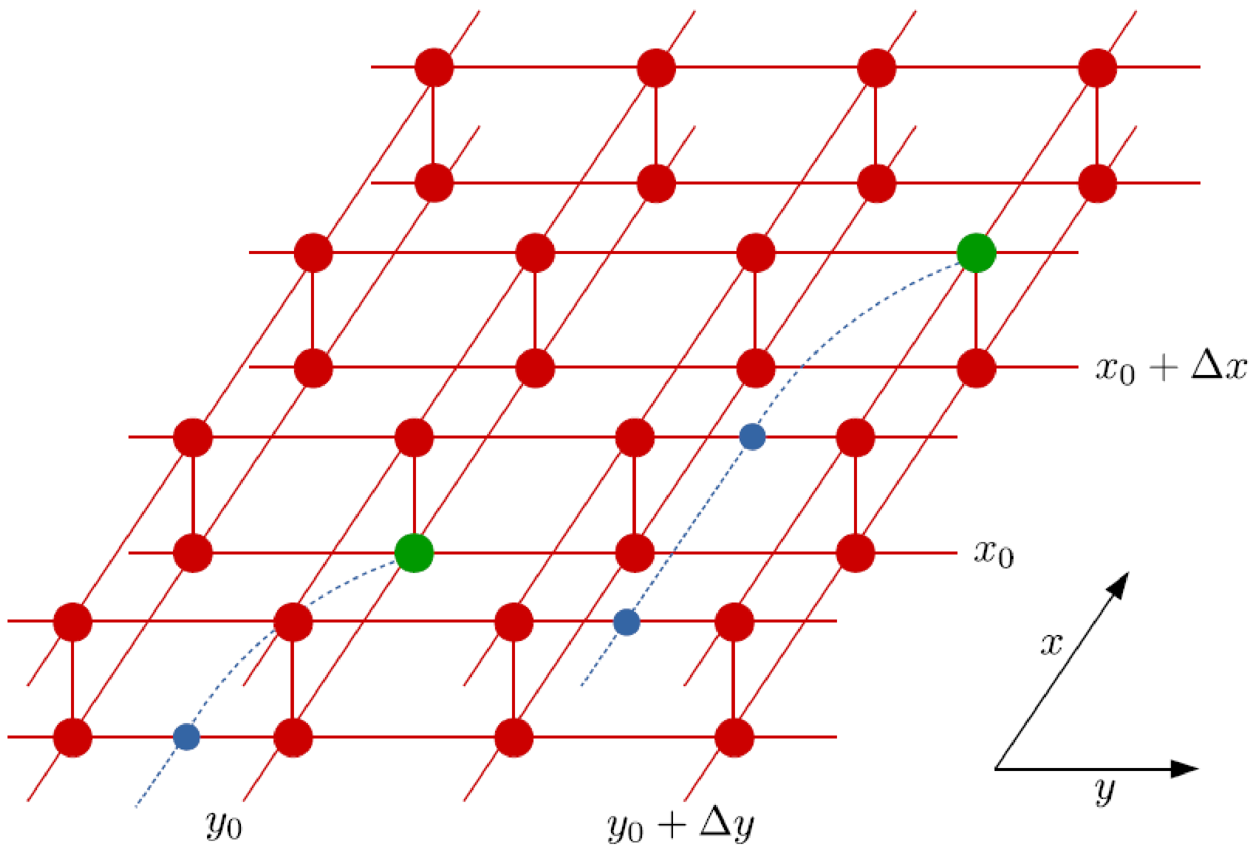


Blue: top. Trivial (no string)
Red: top. Nontrivial (string)
+ : equal charge in bra and ket
o : unequal charge in bra and ket

Green: string in bra + ket: confinement
□: charge def. in bra/ket: confinement

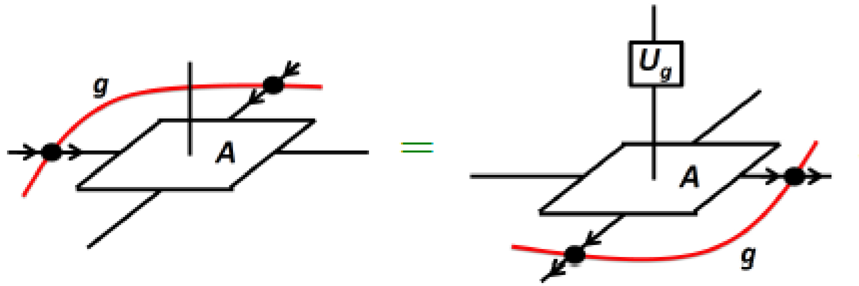


- Anyons are only possible if the eigenspace breaks a sub-group (e.g. Z2)
  - Anyon condensation when Z2 -> Z2xZ2 or Z2 -> 1



Interacting SPT phases in PEPS

- Similar story, but with 1 big difference:



  - Hence e.g. no corrections to the entanglement entropy, G.S. degeneracy

- Nontrivial 3-cocycles indicate the presence of nonlocal edge modes that cannot be realized by a local Hamiltonian in 1D: hence SPT phase
  - Injective MPS cannot exhibit a symmetry related to nontrivial cocycle: degeneracy of edge modes Chen, Xie, Wen '12

- Topological theories are obtained by "gauging" the global symmetry (which hence becomes a constraints as opposed to a symmetry)

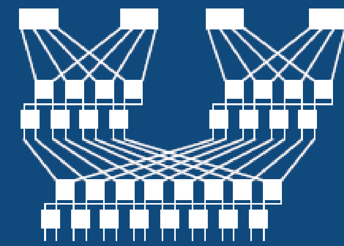
- What about Symmetry Enhanced Topological (SET) Phases?
  - Gauge a subgroup

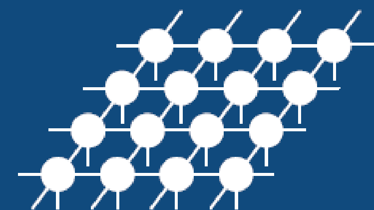
Williamson et al., '14

Conclusion

- Tensor networks provide a natural description of the ground state wavefunction for gapped phases of matter
  - except for chiral phases: status not settled
- Long-range entanglement in topological quantum phases is manifested locally into the symmetries of the tensors
  - Eigenstructure of the 1D transfer matrix reflects the structure op the topological phase in 2D
  - Topological phase transitions correspond to symmetry breaking phase transitions
- Starting point for studying anyon condensation for all string nets

TENSOR NETWORK SUMMER SCHOOL





Theoretical and computational aspects of matrix product states (MPS), projected entangled pair states (PEPS) and the multiscale entanglement renormalization ansatz (MERA)

JUNE 1-5, 2015
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Conclusion

- Tensor Networks are providing a new language for describing and simulating strongly correlated systems in a systematic way

- Entanglement properties are the crucial ingredient characterizing exotic phases of matter, and the MPS/PEPS/tensor network language allows to explore concepts like anyon condensation

cMPS for quantum field theories

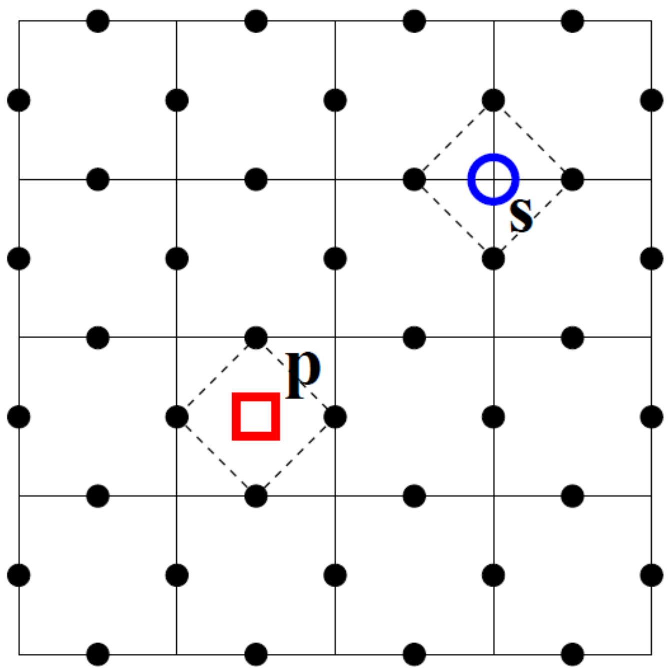
- It is possible to take the limit of the lattice spacing going to zero, and obtain a continuous version of MPS which generalized the concept of coherent states:

$$|\psi\rangle = \mathcal{T} \exp\left(\int dx \;\; Q(x) \otimes I + R(x) \otimes \psi_x^\dagger\right) |\Omega\rangle$$

  - Note that Q and R are DxD matrices

- TDVP equations give a non-commutative version of Gross-Pitaevskii equations in Q,R
- TDVP allows to use those cMPS as variational states, and they have a natural cut-off build in such that they even work for relativistic theories (allowing to surpass no-go arguments of Feynman)

Example: Kitaev’s toric code / Z2 gauge theory

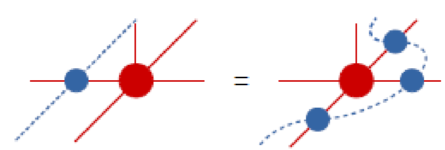


$$A_s = \prod_{i\in \mathrm{star}(s)} \hat{\sigma}_i^{x}$$

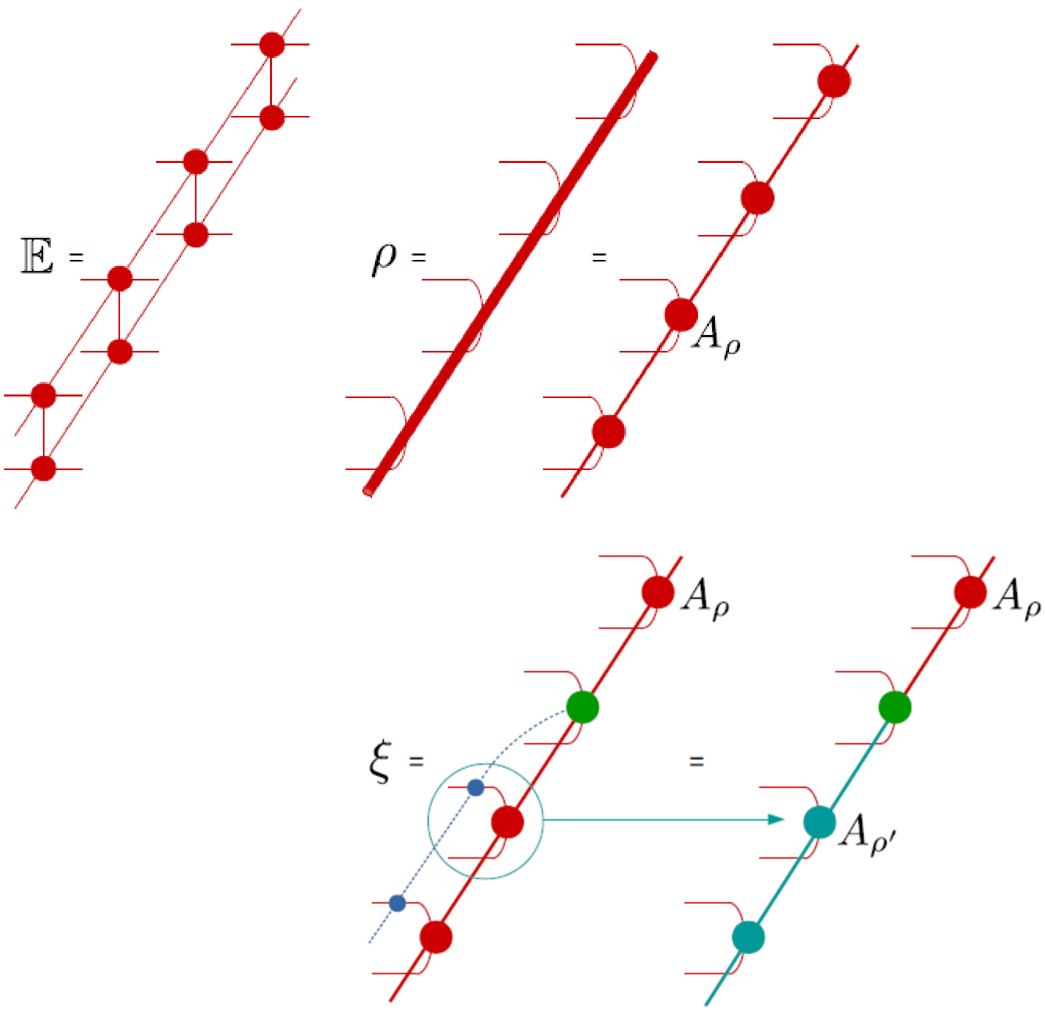
$$B_p = \prod_{i\in \mathrm{plaquette}(p)} \hat{\sigma}_i^{z}$$

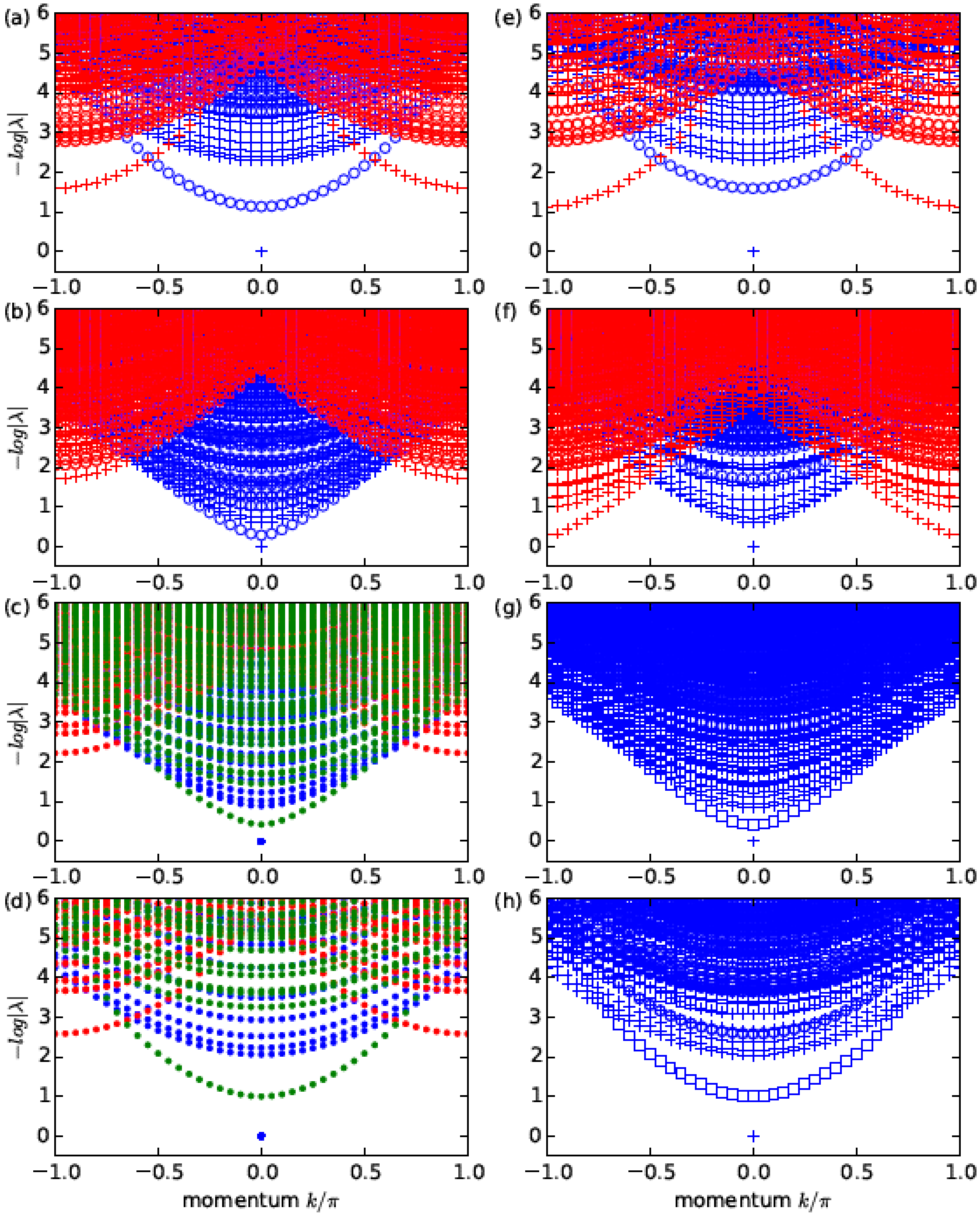
$$\begin{aligned} H &= -\lambda_0 \sum_p B_p - \lambda_1 \sum_s A_s + \lambda_1 \sum_s e^{-\beta \sum_{i\in s} \hat{\sigma}_i^z} \\ &= H_{\text{Kitaev}} + \lambda_1 \sum_s e^{-\beta \sum_{i\in s} \hat{\sigma}_i^z}, \end{aligned}$$

- This model represents the simplest topological phase; β adds string tension, and if it is large enough, the topological phase breaks down to a confined phase
- The ground state as a function of β is an exact PEPS with D=2
- The symmetry MPO is just a product of Z’s:



- There are 2 types of excitations: vertex and plaquette ones
- The physics of PEPS is encoded into the eigenstructure of the transfer matrix; the MPO symmetry dictates a degenerate fixed point (“edge modes”)





Anyon excitations correspond to domain wall excitations in the transfer matrix; a topological quantum phase transition is hence reflected as a symmetry breaking phase transition on the virtual level (anyon condensation / confinement)

"Shadows of Anyons", Haegeman et al. '14

FIG. 7: Spectra of (minus logarithm of) the eigenvalues λ of the transfer matrix as function of momentum k for the different points (β_z, β_x) indicated by the markers (a) - (f) in Fig. 6(a). In the topological phase (a,b,e,f), colors indicate topologically trivial (blue) or non-trivial (red) excitations (flux difference between ket and bra), while the symbol refers to equal (plus sign) or unequal (circle) charge between ket and bra level. In the charge condensed phase (c,d), charges can no longer be measured (dot symbols) and the $Z^{\otimes N} \otimes Z^{\otimes N}$ is broken. This results in a new topologically non-trivial excitation (green) with a string in both ket and bra and indicates that flux excitations become confined, because there is a nonzero string tension. In the flux condensed phase, the full symmetry is restored and no domain wall excitations of the transfer matrix exist, since they are equivalent to local excitations. In addition, charge can be measured in both ket and bra separately. The states with charge difference (circle) can have individual ket and bra charges $+,-$ and $-,+$ but remain degenerate. States with no charge difference can have ket and bra charges $+,+$ (plus sign) or $-,-$ (square) and the higher energy of the latter indicates string tension between charges and thus charge confinement.