RG as a Hamiltonian flow and the symplectic structure of QFT

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- In the gauge

$$ds^2 = dr^2 + \gamma_{ij}(r,x)dx^i dx^j,$$

the most general asymptotic solutions of Einstein+matter equations of motion with a negative cosmological constant take the Fefferman-Graham form

$$\begin{aligned}\gamma_{ij}(r,x) =& e^{2r}\Big(g_{(0)ij}(x) + e^{-2r}g_{(2)ij}(x) + \cdots \\ & \qquad + e^{-dr}(-2rh_{(d)ij}(x) + g_{(d)ij}(x)) + \cdots\Big), \\ \phi(r,x) =& e^{-(d-\Delta)r}\phi_{(0)}(x) + \cdots + e^{-\Delta r}\phi_{(\Delta)}(x) + \cdots\end{aligned}$$

- The coefficients $g_{(0)ij}(x)$ and $\phi_{(0)}(x)$ are interpreted holographically as sources of local gauge-invariant operators, while $g_{(d)ij}(x)$ and $\phi_{(\Delta)}(x)$ are related to their 1-point functions.
- Given the sources $g_{(0)ij}(x)$, $\phi_{(0)}(x)$ and the 1-point functions $g_{(d)ij}(x)$, $\phi_{(\Delta)}(x)$, the bulk geometry and matter fields are uniquely specified in an open neighborhood of the conformal boundary under radial evolution according to the bulk equation of motion.*

*True strictly for Euclidean signature, but can be extended to Lorentzian signature as discussed in Kostas Skenderis' talk.

- This reconstruction of the bulk amounts to a boundary value problem:

  The bulk solution is determined by evolving boundary data with given bulk equations of motion.

- But can one *derive* the bulk equations of motion?

- There have been various different approaches to address this question in recent years, but the way I would like to proceed is by trying to answer first the question:

What do the bulk equations of motion correspond to on the QFT side?

Outline

1. Hamiltonian boundary value problem and generalized holography

2. Local RG as a Hamiltonian flow

3. Concluding remarks

Outline

Hamiltonian formulation of gravitational theories

- Given a $d+1$-dimensional manifold $\mathcal{M}$ with a d-dimensional boundary $\partial\mathcal{M}$ we can (at least piecewise in the vicinity of the boundary) define a Gaussian normal coordinate r emanating from the boundary $\partial\mathcal{M}$ and write the metric in the ADM form

$$ds^2 = (N^2 + N_i N^i)dr^2 + 2N_i dr dx^i + \gamma_{ij} dx^i dx^j$$

- Inserting this decomposition of the metric (and the corresponding ones for gauge fields) into a generic gravity+matter action in $\mathcal{M}$,

$$S = -\frac{1}{2\kappa^2}\left(\int_{\mathcal{M}} d^{d+1}x\sqrt{g}\left(R[g] - \partial_\mu\varphi\partial^\mu\varphi + V(\varphi)\right) + \int_{\partial\mathcal{M}} d^d x\sqrt{\gamma}2K\right) + h.d.$$

  allows one to write the action in the form

$$S = \int dr L(\dot{\gamma}, \dot{\varphi}; \gamma, \varphi, N, N_i)$$

- Higher derivatives can be included, at least perturbatively.

- The canonical momenta conjugate to the induced fields γ_{ij} and φ can be obtained from this Lagrangian as

$$\pi^{ij} \equiv \frac{\delta L}{\delta \dot{\gamma}_{ij}} = -\frac{1}{2\kappa^2}\sqrt{\gamma}\left(K\gamma^{ij} - K^{ij}\right), \quad K_{ij} = \frac{1}{2N}\left(\dot{\gamma}_{ij} - D_i N_j - D_j N_i\right)$$
$$\pi_\varphi \equiv \frac{\delta L}{\delta \dot{\varphi}} = \frac{1}{\kappa^2}\sqrt{\gamma}\frac{1}{N}\left(\dot{\varphi} - N^i\partial_i\varphi\right)$$

- The Hamiltonian is given by the Legendre transform

$$H = \int d^d x \left(\dot{\gamma}_{ij}\pi^{ij} + \dot{\varphi}\pi_\varphi\right) - L = \int d^d x \left(N\mathcal{H} + N_i\mathcal{H}^i\right)$$

where

$$\mathcal{H} = 2\kappa^2\gamma^{-\frac{1}{2}}\left(\pi^i_j\pi^j_i - \frac{1}{d-1}\pi^2 + \frac{1}{4}\pi^2_\varphi\right) + \frac{1}{2\kappa^2}\sqrt{\gamma}\left(R[\gamma] - \partial_i\varphi\partial^i\varphi + V(\varphi)\right)$$
$$\mathcal{H}^i = -2D_j\pi^{ij} + \pi_\varphi\partial^i\varphi$$

- Since N and N_i are not dynamical (their momenta vanish identically), the corresponding Hamilton equations lead to the first class constraints

$$\mathcal{H} = \mathcal{H}^i = 0$$

- The symplectic form

$$\Omega = \int_{\Sigma_r} d^d x \left(\delta\pi^{ij} \wedge \delta\gamma_{ij} + \delta\pi_\varphi \wedge \delta\varphi\right)$$

allows us to define the Poisson brackets

$$\left\{\gamma_{ij}(r,x), \pi^{kl}(r,x')\right\} = \delta_i^{(k}\delta_j^{l)}\delta^{(d)}(x-x'), \quad \left\{\varphi(r,x), \pi_\varphi(r,x')\right\} = \delta^{(d)}(x-x')$$

- The phase space function

$$C[\xi] = \int_{\Sigma_r} d^d x \left(\xi\mathcal{H} + \xi^i\mathcal{H}_i\right)$$

generates bulk diffeomorphisms

$$\{C[\xi], \gamma_{ij}\} = \delta_{\tilde\xi}\gamma_{ij}, \quad \{C[\xi], \pi^{ij}\} = \delta_{\tilde\xi}\pi^{ij}, \quad \{C[\xi], \varphi\} = \delta_{\tilde\xi}\varphi, \quad \{C[\xi], \pi_\varphi\} = \delta_{\tilde\xi}\pi_\varphi$$

with parameter

$$\tilde\xi^\mu = \left(\xi/N, \xi^i - \xi N^i/N\right)$$

- Such diffeomorphisms form a closed algebra, albeit a field-dependent one since

$$\left\{C[\xi], C[\xi']\right\} = C[\xi'']$$

with $\xi''^\mu = \left(\xi^i\partial_i\xi' - \xi'^i\partial_i\xi, \quad \xi^i\partial_i\xi^j - \xi'^i\partial_i\xi^j - (\xi D^j\xi' - \xi' D^j\xi)\right)$

- There is an alternative expression for the canonical momenta as gradients of Hamilton's principal function $\mathcal{S}[\gamma, \varphi]$, which can be identified with the on-shell action:

$$\pi^{ij} = \frac{\delta \mathcal{S}}{\delta \gamma_{ij}}, \quad \pi_\varphi = \frac{\delta \mathcal{S}}{\delta \varphi}$$

Aside:

These expressions follow straightforwardly from a variational principle. e.g. for a point particle with action

$$S = \int_{t_1}^{t_2} dt \left(\frac{1}{2}\dot{q} - V(q) \right)$$

we have

$$\delta S = \int_{t_1}^{t_2} dt \left(-\ddot{q} - V'(q) \right) \delta q + p(t_2)\delta q(t_2) - p(t_1)\delta q(t_1)$$

where $p = \dot{q}$. Hence,

$$p(t_2) = \frac{\delta}{\delta q(t_2)} S_{\text{o.s.}}[q(t_1), q(t_2)]$$

- Inserting these in the Hamiltonian constraint $\mathcal{H}=0$ leads to the radial Hamilton-Jacobi (HJ) equation

$$\boxed{\frac{2\kappa^2}{\sqrt{\gamma}}\left(\mathcal{G}_{ikjl}\frac{\delta\mathcal{S}}{\delta\gamma_{ij}}\frac{\delta\mathcal{S}}{\delta\gamma_{kl}}+\frac{1}{4}\left(\frac{\delta\mathcal{S}}{\delta\varphi}\right)^2\right)+\frac{\sqrt{\gamma}}{2\kappa^2}\left(R[\gamma]-\partial_i\varphi\partial^i\varphi+V(\varphi)\right)=0}$$

where

$$\mathcal{G}_{ikjl}\equiv\gamma_{ik}\gamma_{jl}-\frac{1}{d-1}\gamma_{ij}\gamma_{kl}$$

is the de Witt metric on the space of metrics.

- Combining the two expressions for the canonical momenta gives the first order flow equations (cf. BPS/fake supergravity)

$$\boxed{\dot{\gamma}_{ij}=4\kappa^2\mathcal{G}_{ikjl}\frac{1}{\sqrt{\gamma}}\frac{\delta\mathcal{S}}{\delta\gamma_{kl}},\quad\dot{\varphi}=\frac{\kappa^2}{\sqrt{\gamma}}\frac{\delta\mathcal{S}}{\delta\varphi}}$$

- Finding a complete integral of the HJ equation and integrating these first order flow equations is equivalent to finding the most general solution of the second order equations of motion.

This Hamiltonian language allows us to formulate the variational problem for gravitational theories in non-compact manifolds.

- The variational problem at infinity requires that variations are within the space of general asymptotic solutions of the equations of motion.
- This requires that a specific boundary term S_b is added to the action.
- It can be shown in general that $-S_b$ is a (specific asymptotic) solution of the HJ equation [IP, Skenderis '04;IP '10]:

$$S_b = -\mathcal{S}_o$$

  For asymptotically locally AdS spaces this boundary term ensures that the *conformal class* of boundary data is kept fixed and not the conformal represenative.
- The same boundary term ensures that on-shell action remains finite as the radial regulator is removed [de Boer, Verlinde, Verlinde '99].

Symplectic space of boundary data

The symplectic space of boundary data from which the bulk can be reconstructed under evolution with the radial Hamiltonian can be obtained by the following steps:

1. Solve the Hamilton-Jacobi equation to obtain $\mathcal{S}_o$ in the form of a covariant asymptotic expansion. This determines the boundary term $S_b = -S_o$ that renders the variational problem well posed and the on-shell action finite.
2. Use this solution in the first order flow equations to obtain the general asymptotic expansions (Fefferman-Graham expansions) for the fields.
3. Evaluate the symplectic form on the space of asymptotic solutions and diagonalize it if necessary in order to identify the symplectic conjugate modes.

Recursive solution of the Hamilton-Jacobi equation

The HJ equation can be solved recursively in the form of a covariant expansion in eigenfunctions of a suitable operator [IP, Skenderis '04; IP '11; Chemissany, IP '14].

- Examples of such operators is the dilatation operator [IP, Skenderis '04] or for the current example the operator

$$\delta_\gamma = \int d^dx 2\gamma_{ij} \frac{\delta}{\delta\gamma_{ij}}$$

  which organizes the HJ solution in a derivative expansion.

- Writing

$$\mathcal{S} = \int_{\Sigma_r} d^dx \mathcal{L}(\gamma, \varphi, \chi)$$

  we formally expand

$$\mathcal{S} = \mathcal{S}_{(0)} + \mathcal{S}_{(2)} + \mathcal{S}_{(4)} + \cdots$$

  where $\delta_\gamma \mathcal{S}_{(2n)} = (d-2n)\mathcal{S}_{(2n)}$

- The leading contribution to the solution of the HJ equation contains no transverse derivatives:

$$\mathcal{S}_{(0)}=\frac{1}{\kappa^2}\int_{\Sigma_r}d^dx\sqrt{\gamma}U(\varphi)$$

where $U(\varphi)$ satisfies

$$(\partial_\varphi U)^2-\frac{d}{d-1}U^2+V(\varphi)=0$$

- The flow equations relate $U(\varphi)$ to the leading asymptotic form of the fields themselves. In fact $U(\varphi)$ can be inferred from the *leading* asymptotic form of the fields.
- Inserting this expansion into the Hamilton-Jacobi equation leads to *linear* equations for $\mathcal{S}_{(2n)}$, $n>0$.
- In particular, applying the identity

$$\pi^{ij}\delta\gamma_{ij}+\pi_\varphi\delta\varphi+\pi_\chi\delta\chi=\delta\mathcal{L}+\partial_i v^i(\delta\gamma,\delta\varphi,\delta\chi)$$

to the variation δ_γ and absorbing the total derivative terms into $\mathcal{L}_{(2n)}$ we obtain

$$2\pi_{(2n)}=(d-2n)\mathcal{L}_{(2n)}$$

- The linear recursion equations for $\mathcal{L}_{(2n)}$, $n>0$ then become

$$\boxed{U'(\varphi)\frac{\delta}{\delta\varphi}\int d^dx\mathcal{L}_{(2n)}-\left(\frac{d-2n}{d-1}\right)U(\varphi)\mathcal{L}_{(2n)}=\mathcal{R}_{(2n)},\quad n>0}$$

where the source terms are given by

$$\begin{aligned}\mathcal{R}_{(2)}=&-\frac{1}{2\kappa^2}\sqrt{\gamma}\left(R[\gamma]-\partial_i\varphi\partial^i\varphi\right),\\ \mathcal{R}_{(2n)}=&-2\kappa^2\gamma^{-\frac{1}{2}}\sum_{m=1}^{n-1}\left(\pi_{(2m)}{}^i_j\pi_{(2(n-m))}{}^j_i-\frac{1}{d-1}\pi_{(2m)}\pi_{(2(n-m))}\right.\\ &\left.+\frac{1}{4}\pi_{\varphi(2m)}\pi_{\varphi(2(n-m))}\right),\quad n>1\end{aligned}$$

- Only need to integrate with respect to φ. In certain cases the recursion relations become algebraic, e.g. for pure AdS gravity.
- Homogeneous solution contributes a finite piece and hence can be discarded – only *inhomogeneous* solution is relevant.

$$
\begin{array}{ccccc}
\mathcal{R}_{(2n)} & \xrightarrow{\int} & \mathcal{L}_{(2n)} & \xrightarrow{\delta} & \{\pi_{(2n)}\} \\
 & & & & \downarrow \\
\{\pi_{(2n+2)}\} & \xleftarrow{\delta} & \mathcal{L}_{(2n+2)} & \xleftarrow{\int} & \mathcal{R}_{(2n+2)} \\
\downarrow & & & & \\
\mathcal{R}_{(2n+4)} & \xrightarrow{\int} & \mathcal{L}_{(2n+4)} & & \cdots
\end{array}
$$

The space of asymptotic solutions

Having obtained the solution of the HJ equation we can integrate the flow equations to obtain the general asymptotic expansions of the fields.

- For pure AdS gravity this gives the well-known Fefferman-Graham expansion

$$\gamma_{ij}(r,x) = e^{2r}\Big(g_{(0)ij}(x) + e^{-2r}g_{(2)ij}(x) + \cdots \\ + e^{-dr}\left(-2rh_{(d)ij}(x) + g_{(d)ij}(x)\right) + \cdots\Big)$$

  where $g_{(0)ij}$ is arbitrary, $g_{(2)ij}, \ldots, h_{(d)ij}$ are determined in terms of the Riemann tensor of $g_{(0)ij}$ and the tensor

$$\mathcal{T}_{ij} = \frac{d}{2\kappa^2}\Big(g_{(d)ij} - g_{(0)}{}^{kl}g_{(d)kl}g_{(0)ij}\Big) + X_{ij}[g_{(0)}]$$

  satisfies the constraints

$$D_{(0)}{}^{i}\mathcal{T}_{ij}(x) = 0, \quad \mathcal{T}^i_i(x) = \mathcal{A}(x)$$

- The symplectic form is r-independent and can be evaluated in the limit $r\to\infty$ with the result

$$\Omega = \int d^dx\,\delta\pi_{(d)}{}^{ij}\wedge\delta g_{(0)ij}, \quad \pi_{(d)}{}^{ij} \equiv -\frac{1}{2}\sqrt{g_{(0)}}\mathcal{T}^{ij}$$

S_b as the generator of a canonical transformation

I mentioned above that the boundary term S_b ensures that the variational problem at infinity is well posed and renders the on-shell action finite. We can now reveal a third property of this boundary term:

- Adding S_b to the action shifts the canonical momentum according to

$$\Pi^{ij} = \pi^{ij} + \frac{\delta S_b}{\delta \gamma_{ij}}$$

  while leaving the induced fields unchanged.
- Such a transformation preserves the symplectic form

$$\Omega = \int d^d x \delta \Pi^{ij} \wedge \delta \gamma_{ij} = \int d^d x \delta \pi^{ij} \wedge \delta \gamma_{ij}$$

  and is therefore a canonical transformation.
- The third crucial property of S_b is that it induces a canonical transformation such that the canonical momentum becomes asymptotically proportional to the tensor $\mathcal{T}^{ij}$, namely

$$\begin{pmatrix} \delta \Pi^{ij} \\ \delta \gamma_{ij} \end{pmatrix} = \begin{pmatrix} e^{-2r} \delta {\pi_{(d)}}^{ij} + \cdots \\ e^{2r} \delta g_{(0)ij} + \cdots \end{pmatrix}.$$

Generalized PBH diffeomorphisms

The symplectic space of asymptotic solutions is therefore parameterized in terms of the conjugate variables $g_{(0)ij}$ and $\mathcal{T}^{ij}$. However, this phase space description is still gauge redundant since there are bulk diffeomorphisms that preserve the form of the asymptotic solutions, while $\mathcal{T}^{ij}$ satisfies the two constraints found above.

- Considering a generic infinitesimal diffeomorphism $\delta_\xi g_{\mu\nu} = -\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu$ and requiring that it preserves the gauge $N = 1$, $N^i = 0$ leads to the two equations

$$\delta_\xi g_{rr} = -\mathcal{L}_\xi g_{rr} = -2\dot{\xi}^r = 0$$
$$\delta_\xi g_{ri} = -\mathcal{L}_\xi g_{ri} = -\gamma_{ij}(\dot{\xi}^j + \partial^j \xi^r) = 0$$

- Solving these conditions we obtain

$$\xi^r = -\sigma(x), \quad \xi^i = \xi^i_o(x) - \frac{1}{2}e^{-2r}\left(g_{(0)}{}^{ij} - \frac{1}{2}e^{-2r}g_{(2)}{}^{ij} + \mathcal{O}(e^{-4r})\right)\partial_j\sigma(x)$$

where $\sigma(x)$ and $\xi^i_o(x)$ are arbitrary.

- Under such diffeomorphisms the induced metric then transforms as

$$\delta_\xi \gamma_{ij} = -\mathcal{L}_\xi g_{ij} = -\left(L_\xi \gamma_{ij} + 2K_{ij}\xi^r\right) = -\left(D_i\xi_j + D_j\xi_i - 2K_{ij}\sigma\right)$$

which implies

$$\begin{aligned}\delta_\xi g_{(0)ij} &= -\left(D_{(0)i}\xi_{oj} + D_{(0)j}\xi_{oi}\right) + 2\sigma g_{(0)ij}\\ \delta_\xi \pi_{(d)}{}^{ij} &= -\left(D_{(0)k}\left(\pi_{(d)}{}^{ij}\xi_o^k\right) - \pi_{(d)}{}^{ik}D_{(0)k}\xi_o^j - \pi_{(d)}{}^{jk}D_{(0)k}\xi_o^i\right)\\ &\quad - 2\sigma(x)\pi_{(d)}{}^{ij} - \frac{\delta}{\delta g_{(0)ij}}\int d^dx\sqrt{g_{(0)}}\mathcal{A}\sigma.\end{aligned}$$

- It can be now checked that these transformations are generated by the constraints

$$C[\xi_o,\sigma] = \int d^dx\sqrt{g_{(0)}}\left(\xi_o^i(x)D_{(0)}{}^j\mathcal{T}_{ij} + \sigma(x)\left(\mathcal{T}_i^i - \mathcal{A}\right)\right)$$

namely,

$$\left\{C[\xi_o,\sigma], g_{(0)ij}(x)\right\} = \delta_\xi g_{(0)ij}(x), \quad \left\{C[\xi_o,\sigma], \pi_{(d)}{}^{ij}(x)\right\} = \delta_\xi \pi_{(d)}{}^{ij}(x).$$

- The algebra generated by the constraints on the symplectic space of asymptotic solutions closes

$$\{C[\xi_o,\sigma],C[\xi'_o,\sigma']\}=C[\xi''_o,\sigma'']$$

  with

$$\xi''^i_o=\xi^j_o\partial_j\xi'^i_o-\xi'^j_o\partial_j\xi^i_o,\quad \sigma''=\xi^j_o\partial_j\sigma'-\xi'^j_o\partial_j\sigma$$

  which is now independent of the phase space variables.
- In the case of AdS_3 this algebra reproduces the Brown-Henneaux result.

- This 3-step procedure provides a general algorithm for constructing the symplectic space of (generally constrained) asymptotic data from which the bulk can be reconstructed by evolution under the radial Hamiltonian.
- It can be applied equally well to asymptotically non AdS backgrounds, e.g. non-conformal branes [IP '11], Schrödinger [Guica '13], Lifshitz and hyperscaling violating Lifshitz backgrounds [Chemissany, IP '14].
- It is also directly applicable to higher derivative theories of gravity, as well as to the dynamics of extended objects such as probe branes and extremal surfaces [IP '10].

Outline

In the previous section I formulated the bulk dynamics as a Hamiltonian flow of some "initial" data that are evolved by a prescribed radial Hamiltonian. Can we formulate local QFTs in the same language without any reference to holography?

- I will assume the existence of a relativistic UV fixed point that is in general strongly coupled, and so I will not assume there is a weakly coupled Lagrangian description of the QFT.
- I will also keep the discussion totally classical, assuming some large-N saddle point on the QFT side – in any case the bulk dynamics we want to match is classical, although it could be generalized to the full quantum theory (see Sung-Sik Lee's talk).
- I will sketch an answer to this question based on a Hamiltonian formulation of the renormalization group [B. Dolan '94].
- This description can a priori be applied to different versions of the renormalization group, namely the usual global RG, the local RG [Osborn '91; de Boer, Verlinde, Verlinde '99; Erdmenger '01], and the quantum RG [S.S. Lee '12, S.S. Lee '13].

Work with Paolo Benincasa to appear.

QFT and symplectic geometry

- In the local RG formulation of QFT with a Lagrangian description the dynamics is encoded in the partition function

$$Z[J(x), \gamma_{ij}(x)]$$

where all the couplings, including the background metric γ_{ij} and any gauge fields coupling to conserved currents, are made local.

- In the quantum RG we only need to include couplings for single-trace operators.
- Since the couplings are local they act like sources and so we can define the dual operators as

$$\mathcal{O}(x) \sim \frac{\delta W[J,\gamma]}{\delta J(x)}, \quad T^{ij}(x) \sim \frac{\delta W[J,\gamma]}{\delta \gamma_{ij}(x)}$$

where $W[J,\gamma] \equiv \log Z[J,\gamma]$ is the generating function of connected correlators.

- In a QFT without a Lagrangian formulation $W[J]$ (I will suppress γ_{ij} from now on) can be defined through the connected correlators as

$$W[J] = \sum_{k=0}^{\infty} \frac{1}{k!} \prod_{\ell=1}^{k} \int d^d x_\ell J(x_\ell) \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\ldots\mathcal{O}(x_k)\rangle_c$$

- Treating the sources J and the 1-point functions in the presence of sources (i.e. the dual operators)

$$\langle\mathcal{O}(x)\rangle_J = \frac{\delta W[J]}{\delta J(x)}$$

as independent variables, they can be thought of as symplectic coordinates on an abstract symplectic space associated to any local QFT.

- In particular, the manifold parameterized by the coordinates

$$\{\langle\mathcal{O}_\alpha(x)\rangle, J^\alpha(x)\}$$

can be endowed with a symplectic structure via the closed 2-form

$$\Omega := \int d^dx\delta\langle\mathcal{O}_\alpha(x)\rangle \wedge \delta J^\alpha(x)$$

- This 2-form vanishes "on-shell"

$$\Omega = \int d^dx \int d^dx' \,\frac{\delta^2 W[J]}{\delta J^\beta(x')\delta J^\alpha(x)}\delta J^\beta(x') \wedge \delta J^\alpha(x) = 0$$

Point particle analogy

- A useful analogy is provided by a point particle described by generalized coordinates q^α and momenta p_α, whose symplectic form is

$$\Omega = dp_\alpha \wedge dq^\alpha$$

- Off-shell q^α and p_α are independent coordinates, but on-shell we always have

$$p_\alpha = \frac{\partial \mathcal{S}(q)}{\partial q^\alpha}$$

  where $\mathcal{S}(q)$ is Hamilton's principal function – or the on-shell action – which we have seen before.
- As in the QFT setup, the symplectic form vanishes on-shell

$$\Omega = \delta p_\alpha \wedge dq^\alpha = \frac{\partial^2 \mathcal{S}(q)}{\partial q^\beta \partial q^\alpha} dq^\beta \wedge dq^\alpha = 0$$

- This example implies that there is a mathematical *analogy* between the following quantities in any local QFT and a classical point particle:

$$
\begin{aligned}
J^{\alpha}(x) &\longleftrightarrow q^{\alpha} \\
\langle \mathcal{O}_{\alpha}(x) \rangle &\longleftrightarrow p_{\alpha} \\
W[J] &\longleftrightarrow \mathcal{S}(q) = S_{\mathrm{on-shell}}(q)
\end{aligned}
$$

- Can this analogy be pushed further? In particular, what is the analogue of the point particle *Hamiltonian* in the QFT side and what is the corresponding "time".

Renormalization group flow as a Hamiltonian flow

- In fact, we could take "time" to be any continuous parameter the QFT depends on, but there is a universal parameter that plays a fundamental role in any QFT: *the energy scale* $\tau = \log \mu$.

- The corresponding Hamiltonian, $\mathbb{H}$, then governs the evolution of the symplectic variables under the RG flow via the Hamilton equations [B. Dolan '94]

$$j^\alpha = \frac{\delta \mathbb{H}}{\delta \langle \mathcal{O}_\alpha \rangle} \quad \langle \dot{\mathcal{O}}_\alpha \rangle = -\frac{\delta \mathbb{H}}{\delta J^\alpha}$$

where all quantities here are "bare". In particular,

$$j^\alpha \equiv \beta^\alpha$$

are the *beta functions*.

Hamilton-Jacobi equation as the RG equation

- Using the point particle example for guidance, we can now derive an equation for the RG flow of the generating function $W[J]$, i.e. the RG equation.
- Recalling that $\mathcal{S}(q) = S_{\text{on-shell}}$, we have

$$\dot{\mathcal{S}} = L = \frac{\partial \mathcal{S}}{\partial t} + \frac{\partial \mathcal{S}}{\partial q^\alpha}\dot{q}^\alpha$$

  or

$$p_\alpha \dot{q}^\alpha - L + \frac{\partial \mathcal{S}}{\partial t} = H\left(\frac{\partial \mathcal{S}}{\partial q^\beta}, q^\gamma\right) + \frac{\partial \mathcal{S}}{\partial t} = 0.$$

- The corresponding equation on the QFT side is

$$\boxed{\mathbb{H}\left(\frac{\delta W[J]}{\delta J^\beta}, J^\gamma\right) + \frac{\partial W}{\partial \tau} = 0}$$

- This Hamilton-Jacobi equation is the exact renormalization group equation for the generating function $W[J]$.

Non-perturbative renormalization

- In a renormalizable QFT, renormalization amounts to adding local counterterms to the generating functional and renormalizing the couplings such that

$$W[J] \to W_R[J_R] \equiv W[J] + W_{ct}[J]$$

is free from UV divergences. i.e. $W_R[J_R]$ has a well defined limit as $\tau \to \infty$.

- Normally, $W_{ct}[J]$ is computed perturbatively by adding local counterterms at each order in perturbation theory and absorbing the UV divergences into a redefinition of the couplings. In a strongly coupled QFT, possibly without a Lagrangian microscopic description, this option is not available.

Renormalization from a variational principle

- $W_{ct}[J]$ can be understood in the context of the variational principle for the action

$$\mathbb{S} = \int^{\infty} d\tau \mathbb{L}$$

where

$$\mathbb{L} := \int d^d x J^\alpha(x) \langle \mathcal{O}_\alpha(x) \rangle - \mathbb{H}$$

- The counterterms $W_{ct}[J]$ is the "boundary term" that needs to be added to the action $\mathbb{S}$ at some UV cut-off τ_o in order for the variational problem at $\tau_o = \infty$ to be well posed, in terms of the renormalized couplings $J_R^\alpha(x)$ [I.P. '10].

- The addition of $\mathbb{S}$ implements a canonical transformation from the bare couplings $J^\alpha(x)$ to the renormalized couplings $J_R^\alpha(x)$.

- If the Hamiltonian $\mathbb{H}$ is known, one can systematically determine $W_{ct}[J]$ by solving the Hamilton-Jacobi equation asymptotically as $\tau \to \infty$.

Non-Lagrangian derivation of the Ward identities

- In order to derive the Ward identities we will work with renormalized variables at the far UV, i.e. in terms of the variables of the UV fixed point.

- Among all operators in any QFT there is always the stress tensor, T_{ij}. We will consider in addition an internal $U(1)$ symmetry giving rise to a current, J^i, in the spectrum of operators, as well as scalar operator, $\mathcal{O}$, transforming trivially both under the Poincaré group and the $U(1)$ symmetry, but with a definite scaling dimension Δ.

- The generating functional of connected correlation functions a function of the sources, ${g_{(0)}}^{ij}, A_{(0)i}, \varphi_{(0)}$, respectively for the stress tensor, the current, and for the scalar operator, as well as of all other operators in the theory which we will suppress:
$$W[{g_{(0)}}^{ij}, A_{(0)i}, \varphi_{(0)}, \ldots]$$

- We now gauge the global symmetries by promoting the Poincaré transformations to diffeomorphisms and the internal global symmetry to a local gauge symmetry.

- Correspondingly, the sources ${g_{(0)}}^{ij}$ and $A_{(0)i}$ are promoted gauge fields.

Under infinitesimal diffeomorphisms, parameterized by the vector $\xi^i(x)$, the sources then transform as

$$\delta_\xi g^{ij}_{(0)} = \mathcal{L}_\xi g^{ij}_{(0)} = -(D^i\xi^j + D^j\xi^i)$$
$$\delta_\xi A_{(0)i} = \mathcal{L}_\xi A_{(0)i} = A_{(0)j} D_i \xi^j + \xi^j D_j A_{(0)i}$$
$$\delta_\xi \varphi_{(0)} = \mathcal{L}_\xi \varphi_{(0)} = \xi^j D_j \varphi_{(0)}$$

while under infinitesimal $U(1)$ gauge transformations, parameterized by the gauge function $\alpha(x)$, they transform as

$$\delta_\alpha g_{(0)ij} = 0$$
$$\delta_\alpha A_{(0)i} = D_i \alpha(x)$$
$$\delta_\alpha \varphi_{(0)} = 0$$

where D_i denotes the covariant derivative with respect to the metric $g_{(0)ij}$.

- In the absence of quantum anomalies, the Ward identities for Poincaré and $U(1)$ invariance are

$$\delta_\xi W = 0, \quad \delta_\alpha W = 0, \quad \forall \quad \xi^i, \alpha$$

- Using the above gauge transformations of the sources, we can rewrite these in terms of the 1-point functions:

$$\langle T_{ij}(x)\rangle_s = -\frac{2}{\sqrt{g_{(0)}}}\frac{\delta W}{\delta g^{ij}_{(0)}(x)}$$

$$\langle J^i(x)\rangle_s = -\frac{1}{\sqrt{g_{(0)}}}\frac{\delta W}{\delta A_{(0)i}(x)}$$

$$\langle \mathcal{O}(x)\rangle_s = -\frac{1}{\sqrt{g_{(0)}}}\frac{\delta W}{\delta \varphi_{(0)}(x)}$$

where the subscript s indicates that the sources are kept arbitrary.

$U(1)$ invariance

- From the $U(1)$ invariance we get

$$\delta_\alpha W = 0 \Leftrightarrow \int d^dx \left(\delta_\alpha {g_{(0)}}^{ij} \frac{\delta W}{\delta {g_{(0)}}^{ij}} + \delta_\alpha A_{(0)\,i} \frac{\delta W}{\delta A_{(0)\,i}} + \delta_\alpha \varphi_{(0)} \frac{\delta W}{\delta \varphi_{(0)}} \right) = 0$$

$$\Leftrightarrow \int d^dx D_i \alpha(x) \frac{\delta W}{\delta A_{(0)\,i}} = 0 \Leftrightarrow \int d^dx \alpha(x) D_i \left(\frac{\delta W}{\delta A_{(0)\,i}} \right) = 0$$

- Since $\alpha(x)$ is arbitrary, it follows that

$$D_i \left(\frac{\delta W}{\delta A_{(0)\,i}} \right) = 0$$

or

$$\boxed{D_i \langle J^i(x) \rangle_s = 0}$$

Diffeomorphism invariance

- From diffeomorphism invariance we get similarly

$$\delta_\xi W = 0 \Leftrightarrow D^i \left(2 \frac{\delta W}{\delta g_{(0)}{}^{ij}} \right) - F_{(0)\,ij} \frac{\delta W}{\delta A_{(0)\,i}} + \frac{\delta W}{\delta \varphi_{(0)}} D_j \varphi_{(0)}(x) = 0,$$

where $F_{(0)\,ij} = \partial_i A_{(0)\,j} - \partial_j A_{(0)\,i}$ is the field strength of the gauge field $A_{(0)\,i}$.

- Hence,

$$\boxed{D^i \langle T_{ij}(x) \rangle_s - \langle J^i(x) \rangle_s F_{(0)\,ij} + \langle \mathcal{O}(x) \rangle_s D_j \varphi_{(0)}(x) = 0}$$

Weyl invariance & the trace Ward identity

- The last Ward identity, which is particularly important for the renormalization group, is the trace Ward identity, reflecting invariance under scale transformations.

- The local version of a scale trnasformation is a Weyl transformation which acts on the sources as

$$\delta_\sigma {g_{(0)}}^{ij} = -2\delta\sigma(x){g_{(0)}}^{ij} \quad \delta_\sigma A_{(0)i} = 0 \quad \delta_\sigma \varphi_{(0)} = -(d-\Delta)\delta\sigma(x)\varphi_{(0)}$$

  where Δ is the conformal dimension of the operator $\mathcal{O}(x)$.

- In the absence of anomalies the generating functional is invariant under Weyl transformations. However, in theories with a conformal anomaly we have instead

$$\delta_\sigma W = \int d^dx \sqrt{g_{(0)}}\delta\sigma(x)\mathcal{A}$$

  where the anomaly density, $\mathcal{A}$ is a local function of the sources.
- Using the above transformation of the sources, this then leads to the trace Ward identity

$$\boxed{\langle T^i_i(x)\rangle_s = -(d-\Delta)\varphi_{(0)}\langle\mathcal{O}(x)\rangle_s + \mathcal{A}}$$

Trace Ward identity & RG flow Hamiltonian

- An interesting observation is that the trace Ward identity can be written as

$$\langle T_i^i(x)\rangle_s = \sum_{\mathcal{O}} \beta_{\mathcal{O}}|_\infty \langle \mathcal{O}(x)\rangle_s + \mathcal{A}$$

  where

$$\beta_{\mathcal{O}}|_\infty = \lim_{\tau\to\infty} \dot{J}_{\mathcal{O}}$$

  is the beta function of the coupling of the operator $\mathcal{O}$ and the sum is over all relevant operators in the spectrum of the theory.

- Finally, under a constant Weyl rescaling, $\sigma = \tau = \log\mu$, scale invariance implies

$$\int d^d x\mathcal{A} = \dot{W} = \frac{\partial W}{\partial\tau} - \sum_{\mathcal{O}} \int d^d x\ \beta_{\mathcal{O}}|_\infty \langle \mathcal{O}(x)\rangle_s = \frac{\partial W}{\partial\tau} - \int d^d x \langle T_i^i(x)\rangle_s + \int d^d x\mathcal{A}$$

- Comparing this with the RG Hamilton-Jacobi equation we obtain [B. Dolan '94]

$$\boxed{\mathbb{H}_R|_\infty = -\int d^d x \langle T_i^i(x)\rangle_s}$$

Holographic dictionary

Radial coordinate	r	$\leftrightarrow$	$\tau = \log \mu$	RG "time"
Induced fields	ϕ	$\leftrightarrow$	J	Bare couplings (sources)
On-shell action	$S[\phi]$	$\leftrightarrow$	$W[J]$	Generating function
Radial Hamiltonian	H	$\leftrightarrow$	$\mathbb{H}$	RG Hamiltonian
Radial momenta	π_ϕ	$\leftrightarrow$	$\langle \mathcal{O} \rangle$	Bare one-point functions
Non-normalizable modes	$\phi_{(0)}$	$\leftrightarrow$	$J_R\|_\infty$	Renormalized couplings at ∞
Renormalized momenta	$\widehat{\pi}_{(\Delta)}$	$\leftrightarrow$	$\langle \mathcal{O}_R \rangle\|_\infty$	Renormalized 1-point fns at ∞
Integrated & renormalized induced metric momentum	$-\int d^dx \widehat{\pi}_{(d)}{}^i_i$	$\leftrightarrow$	$\mathbb{H}_R\|_\infty$	Renormalized RG Hamiltonian at ∞

How do we compute $\mathbb{H}$?

- Quantum RG coarse graining [S. S. Lee '13, S. S. Lee '14]?
- Other ways? Beta functions and Casimir energy are contained in $W_{ct}[J]$...

Outline

Concluding remarks

- Any local QFT can be described in terms of the symplectic space of sources and 1-point functions.
- The scale evolution is governed by a Hamiltonian on this symplectic space, according to the corresponding Hamilton-Jacobi equation, which plays the role of an exact RG equation.
- In QFTs that admit a holographic dual, this RG Hamiltonian is identified with the bulk radial Hamiltonian.
- Renormalizing the theory amounts to determining the generating function of a canonical transformation from the bare symplectic coordinates to the renormalized ones, which is achieved by solving the corresponding Hamilton-Jacobi equation.
- Would be interesting to compute explicitly the RG Hamiltonian in concrete examples beyond matrix models. 2d CFTs?