Local RG, Quantum RG, and Holographic RG

Yu Nakayama

Special thanks to Sung-Sik Lee and Elias Kiritsis

Local renormalization group

The main idea dates back to Osborn NPB 363 (1991)

See also my recent review 1302.0884

Some results I discuss today are trivial but have not appeared in literature

Local renormalization group

- Put a QFT on curved space-time with space-time dependent coupling constant

$$\int d^4x \, (L_0[\phi(x)] + gO(x)) \to \int d^4x \sqrt{g} \, (L_0[\phi(x)] + g(x)O(x) + \cdots)$$

- Natural to make (Wilsonian) cut-off space-time dependent (e.g. space-time dependent PV mass)

$$\mu \to \mu(x)$$

- Study the evolution under the change $\delta\mu(x)$
- Vast generalization of renormalization group
- Within power-counting RG scheme as a formal power series, conventionally renormalizable theories are locally renormalizable

Schwinger functional

- Consider the renormalized Schwinger functional for

$$O(x), J_\mu(x), \tilde{J}_\mu(x), O_{\mu\nu}(x) = -O_{\nu\mu}(x)$$

$$e^{-W[g(x),A_\mu(x),\tilde{A}_\mu(x),B_{\mu\nu}(x),g_{\mu\nu}(x)]}$$

$$= \int \mathcal{D}X e^{-S_0[X]+\int d^4x\sqrt{g}\left(g(x)O(x)+A_\mu(x)J^\mu(x)+\tilde{A}_\mu(x)\tilde{J}^\mu(x)+B_{\mu\nu}(x)O^{\mu\nu}(x)+\cdots\right)}$$

- Assume the operator identities

$$D_\mu J^\mu = O(x) \ , \quad D_\mu O^{\mu\nu} = \tilde{J}^\nu \ , \quad D_\mu \tilde{J}^\mu = 0 \ .$$

- $\tilde{J}_\mu$ is a so-called topological current
- Schwinger functional has the "gauge symmetries"

$$\delta g = w \ , \qquad \delta A_\mu = \partial_\mu w$$

$$\delta B_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu \ , \quad \delta\tilde{A}_\mu = b_\mu$$

- Also it is diff invariant

Local RG invariance

- Local RG operator

$$\Delta_\sigma = \int d^4x\sqrt{g}\left[2\sigma(x)g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}} + \beta\sigma(x)\frac{\delta}{\delta g(x)} + \beta_\mu\sigma(x)\frac{\delta}{\delta A_\mu} + \tilde{\beta}_\mu\sigma(x)\frac{\delta}{\delta \tilde{A}_\mu} + \beta_{\mu\nu}\sigma(x)\frac{\delta}{\delta B_{\mu\nu}}\right]$$

- Local renormalizability

$$\Delta_\sigma W[g_{\mu\nu}, g, \ldots] = A_\sigma[g_{\mu\nu}, g, \ldots]$$

- $A_\sigma[g_{\mu\nu}, g, \ldots]$: a local functional (trace anomaly or local renormalization of cc)
- From Schwinger action principle

$$T^\mu_{\ \mu} = \beta O + \beta_\mu J^\mu + \tilde{\beta}_\mu \tilde{J}^\mu + \beta_{\mu\nu}O^{\mu\nu} + \text{anomaly}$$

- Anomaly part (many terms in principle)

$$A_\sigma = \int d^4x\sqrt{g}\sigma\Big(a\text{Euler} - c\text{Weyl}^2 + bR^2 + \tilde{b}\Box R + \\ + \chi^a D^2 g D^2 g + \chi^g D_\mu g D_\nu g G^{\mu\nu} + \cdots \\ + \kappa F_{\mu\nu}F^{\mu\nu} + \tilde{\kappa}(\tilde{F}_{\mu\nu} - B_{\mu\nu})(\tilde{F}^{\mu\nu} - B^{\mu\nu}) + \cdots\Big)$$

Local RG consistency condition

- Local RG operator

$$\Delta_\sigma = \int d^4x \sqrt{g} \left[2\sigma(x) g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} + \beta\sigma(x) \frac{\delta}{\delta g(x)} + \beta_\mu \sigma(x) \frac{\delta}{\delta A_\mu} + \tilde{\beta}_\mu \sigma(x) \frac{\delta}{\delta \tilde{A}_\mu} + \beta_{\mu\nu}\sigma(x) \frac{\delta}{\delta B_{\mu\nu}} \right]$$

- Suppose beta functions take the form (in power-counting RG scheme)

$$\beta = \beta(g) \ , \quad \beta_{\mu\nu} = \gamma(B_{\mu\nu} - \tilde{F}_{\mu\nu}) + \tilde{\gamma} F_{\mu\nu}$$

$$\beta_\mu = \rho D_\mu g = \rho(\partial_\mu g + A_\mu) \ , \quad \tilde{\beta}_\mu = \tilde{\rho} D_\mu g$$

- Local RG transformation is abelian $[\Delta_\sigma, \Delta_{\sigma'}] = 0$

$$\int d^4x \sqrt{g} \left[(\sigma \partial_\mu \sigma' - \sigma' \partial_\mu \sigma) \left(\beta\rho \frac{\delta}{\delta A_\mu} + \beta\tilde{\rho} \frac{\delta}{\delta \tilde{A}_\mu} + (\tilde{\gamma}\rho - \gamma\tilde{\rho}) D_\nu g \frac{\delta}{\delta B_{\mu\nu}} \right) \right] = 0$$

- Requiring $\beta\rho = \beta\tilde{\rho} = \tilde{\gamma}\rho - \gamma\tilde{\rho} = 0$
- Related to consistency of

$$T^\mu_{\ \mu} = \beta O + \beta_\mu J^\mu + \tilde{\beta}_\mu \tilde{J}^\mu + \beta_{\mu\nu} O^{\mu\nu} + \text{anomaly}$$

under local renormalization

On anomaly terms

- Local RG group is abelian $[\Delta_\sigma, \Delta_{\sigma'}] = 0$
- Wess-Zumino consistency condition $\Delta_\sigma A_{\sigma'} = \Delta_{\sigma'} A_\sigma$

$$A_\sigma = \int d^4x\sqrt{g}\sigma \Big(a\text{Euler} - c\text{Weyl}^2 + bR^2 + \tilde{b}\Box R +$$
$$+ \chi^a D^2 g D^2 g + \chi^g D_\mu g D_\nu g G^{\mu\nu} + \cdots$$
$$+ \kappa F_{\mu\nu}F^{\mu\nu} + \tilde{\kappa}(\tilde{F}_{\mu\nu} - B_{\mu\nu})(\tilde{F}^{\mu\nu} - B^{\mu\nu}) + \cdots\Big)$$

- Many constraints on anomalies such as

$$\beta = \chi^g \frac{\partial \bar{a}}{\partial g} \qquad b = \chi^a \beta^2$$

- In relation to (perturbative) "a-theorem"
- No constraint on c , κ , $\tilde{\kappa}$ (as expected)

Relation to anomalous dimensions

- Acting local RG operators on correlation functions

$$\Delta_\sigma = \int d^4x\sqrt{g}\left[2\sigma(x)g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}} + \beta\sigma(x)\frac{\delta}{\delta g(x)} + \beta_\mu\sigma(x)\frac{\delta}{\delta A_\mu} + \tilde{\beta}_\mu\sigma(x)\frac{\delta}{\delta \tilde{A}_\mu} + \beta_{\mu\nu}\sigma(x)\frac{\delta}{\delta B_{\mu\nu}}\right]$$

we have Callan-Symanzik equations

$$\left(\frac{\partial}{\partial\Lambda} + \beta\frac{\partial}{\partial g}\right)\langle O(x)\cdots J_\mu(y)\cdots \tilde{J}_\mu(z)\cdots O_{\mu\nu}(w)\rangle$$
$$= \gamma_O\langle O(x)\cdots J_\mu(y)\cdots \tilde{J}_\mu(z)\cdots O_{\mu\nu}(w)\rangle + \cdots$$

- One can read anomalous dimensions (in flat space limit)

$$\frac{d}{d\log\mu}O = (\partial_g\beta + \rho)O \ , \qquad \frac{d}{d\log\mu}O_{\mu\nu} = \gamma O_{\mu\nu}$$
$$\frac{d}{d\log\mu}J_\mu = \rho J_\mu + \tilde{\rho}\tilde{J}_\mu - \tilde{\gamma}\tilde{J}_\mu \ , \quad \frac{d}{d\log\mu}\tilde{J}_\mu = \gamma\tilde{J}_\mu$$

- Recalling $\beta\rho = \beta\tilde{\rho} = \tilde{\gamma}\rho - \gamma\tilde{\rho} = 0$, we mainly work on $\beta = 0$ case (Stuckerberg mechanism in holography)
- $\beta \neq 0, \rho = \tilde{\rho} = 0$ gives scale but not conformal…

Quantum renormalization group

Developed by Sung-Sik Lee. (See e.g. arXiv:1305.3908 for gravity part)

Operator identities and Higgs/Stuckerberg mechanism have not been discussed in the literature (but are not so non-trivial).

Local RG is not Holographic RG (yet)

- Both discuss the evolution of Schwinger functional (= GKP-W partition functional)
- Local RG knows something about d+1 dim gauge invariance and diff invariance (as a scheme (in)dependence of RG)

- But, Local RG equations are classical deterministic equations for any QFT (quantum gravity?)

- Related point: (even classically) bulk EOM is not first order… (boundary condition?)

Large N and "second quantization"

- We focus on matrix QFT in large N limit

  Single trace coupling: $g_1(x)\mathrm{Tr}O$

  Multi trace coupling: $g_n(x)(\mathrm{Tr}O)^n$

- Idea: $(\mathrm{Tr}O)^n$ is multi-particle states of $(\mathrm{Tr}O)$ in holography

- Represent multiple-particle states out of single particle states → naturally leads to "second quantization"

- For each single trace operator $\mathrm{Tr}O(x)$, we introduce one bulk quantum field $\Phi(z, x^\mu)$ that represents the local RG flow of single trace coupling (as well as higher trace coupling) as path integral

Quantum RG formula in Hamiltonian form

- The path integral expression of the Schwinger functional in large N matrix theory becomes (c.f. Sung-Sik's talk)

$$e^{-W} = \int \mathcal{D}\Phi(x,z)\mathcal{D}P(x,z)e^{\int dz d^4x P\partial_z \Phi + n(x,z)(\Lambda[\Phi]+\beta_1[\Phi]P+\beta_2[\Phi]P^2)}$$

$$n(z,x) = \log \mu(z,x)$$

$\Lambda[\Phi]$:Local Renormalization of cc (c.f. trace anomaly $A_\sigma[g(x)]$)

$\beta_1[\Phi]$:Beta function for single trace operator

$\beta_2[\Phi]$:Beta function for double trace operator

- Working in local Wilsonian renormalzation → beta functions contain derivatives:

$$\Lambda[g_1] = \Lambda_0(g_1) + (D_\mu g_1)^2 + (D_\mu g_1)^4 + \cdots$$

Quantum RG formula in Lagrangian form

$$e^{-W} = \int \mathcal{D}\Phi(x,z)\mathcal{D}P(x,z)e^{\int dzd^4x P\partial_z\Phi + n(x,z)(\Lambda[\Phi]+\beta_1[\Phi]P+\beta_2[\Phi]P^2)}$$

- One may integrate momentum P to get Lagrangian form
- To get rid of first order derivative in z direction, we need the gradient condition for single trace beta function

$$\beta_1[\Phi] = \beta_2 \frac{\delta}{\delta\Phi} S[\Phi]$$

- Then

$$\int dz \beta_2(\Phi)^{-1}\beta_1(\Phi)\partial_z\Phi$$

$$= \int dz \frac{\delta}{\delta\Phi} S[\Phi]\partial_z\Phi = \int dz \partial_z S[\Phi] = S[\Phi]|_{z=0}$$

$$e^{-W} = \int \mathcal{D}\Phi(x,z)e^{\int dzd^4x L_{\text{bulk}}[\Phi]}$$

$$L_{\text{bulk}}[\Phi] = n^{-1}(\partial_z\Phi)^2 + n\Lambda[\Phi] + n\beta_2^{-1}\beta_1[\Phi]^2 + \cdots$$

- $S[\Phi]$ is a boundary counter-term
- First order derivative may be OK in some cases (c.f. CS action)

Our toy model again

- For now, we neglect energy-momentum tensor (gravity)
- We had single trace operators

$$\mathrm{Tr}O\ ,\quad \mathrm{Tr}J_\mu\ ,\quad \mathrm{Tr}\tilde{J}_\mu\ ,\quad \mathrm{Tr}O_{\mu\nu}$$

- Lowest derivative (power-counting) single trace beta functions

$$\beta = \beta(g)\ ,\quad \beta_{\mu\nu} = \gamma(B_{\mu\nu} - \tilde{F}_{\mu\nu}) + \tilde{\gamma}F_{\mu\nu}$$

$$\beta_\mu = \rho D_\mu g = \rho(\partial_\mu g + A_\mu)\ ,\quad \tilde{\beta}_\mu = \tilde{\rho}D_\mu g$$

- With local RG constraint $\beta\rho = \beta\tilde{\rho} = \tilde{\gamma}\rho - \gamma\tilde{\rho} = 0$, take $\beta = 0$
- Quantum RG prescription gives

$$\begin{aligned} S = \int d^4x dz \big(& P\partial_z\Phi + n\beta_2 P^2 \\ & + P^\mu(\partial_z A_\mu + n\rho D_\mu\Phi) + n\beta_2^{\mu\nu}P_\mu P_\nu \\ & + \tilde{P}^\mu(\partial_z\tilde{A}^\mu + n\tilde{\rho}D_\mu\Phi) + n\tilde{\beta}_2^{\mu\nu}\tilde{P}_\mu\tilde{P}_\nu \\ & + P^{\mu\nu}(\partial_z B_{\mu\nu} + n\gamma(B_{\mu\nu} - \tilde{F}_{\mu\nu}) + n\tilde{\gamma}F_{\mu\nu}) + n\beta_2^{\mu\nu\rho\sigma}P_{\mu\nu}P_{\rho\sigma} \\ & + n\Lambda[\Phi, F_{\mu\nu}, \tilde{F}_{\mu\nu}, B_{\mu\nu}] + \text{more derivatives}\big)\ . \end{aligned}$$

- We need more date (e.g. double trace beta functions) to fix the complete action

Gauge invariance in 1+4 dimension

- By local RG construction, the action is 4d gauge invariant
- We may do the gauge transformation along with the local RG (scheme choice)
- This 1+4d gauge parameter is arbitrary → extra fields

$$w(z, x^\mu) \ , b_\mu(z, x^\mu) \to A_z \ , B_{z\mu}$$

- Superficially, the single trace beta function depends on the scheme (1+4 d gauge) choice, but physics should not change

$$\partial_z A_\mu \to \partial_z A_\mu - \partial_\mu A_z = F_{z\mu} \ , \quad \partial_z \Phi \to \partial_z \Phi + A_z = D_z \Phi$$

- Everything is d + 1 dim gauge covairantized!

$$\begin{aligned} S = \int d^4 x dz \, \Big(& P D_z \Phi + n\beta_2 P^2 \\ & + P^\mu (F_{z\mu} + n\rho D_\mu \Phi) + n\beta_2^{\mu\nu} P_\mu P_\nu \\ & + \tilde{P}^\mu (\tilde{F}_{z\mu} + n\tilde{\rho} D_\mu \Phi) + n\tilde{\beta}_2^{\mu\nu} \tilde{P}_\mu \tilde{P}_\nu \\ & + P^{\mu\nu} (H_{z\mu\nu} + n\gamma (B_{\mu\nu} - \tilde{F}_{\mu\nu}) + n\tilde{\gamma} F_{\mu\nu}) + n\beta_2^{\mu\nu\rho\sigma} P_{\mu\nu} P_{\rho\sigma} \\ & + n\Lambda[\Phi, F_{\mu\nu}, \tilde{F}_{\mu\nu}, B_{\mu\nu}] + \text{more derivatives} \Big) \ . \end{aligned}$$

- We still need more date (e.g. double trace beta functions)

CC terms and Lagrangian form

- Certain terms in local renormalization of cc $\Lambda[\Phi, F_{\mu\nu}, \cdots]$ is fixed by trace anomaly

$$\Lambda[\Phi, F_{\mu\nu}, \cdots] = \cdots + \chi^a D^2\Phi D^2\Phi + \chi^g D_\mu\Phi D_\nu\Phi G^{\mu\nu} + \cdots$$
$$+ \kappa F_{\mu\nu}^2 + \tilde{\kappa}(\tilde{F}_{\mu\nu} - B_{\mu\nu})(\tilde{F}^{\mu\nu} - B^{\mu\nu}) + \cdots$$

- While power-counting RG doesn't say anything, it is reasonable to assume double trace beta functions have derivative expansions

$$\beta_2 = \chi_0^{-1} + \cdots \ , \quad \tilde{\beta}_2^{\mu\nu\rho\sigma} = K^{-1} g^{\mu\nu} g^{\rho\sigma} + \cdots$$
$$\beta_2^{\mu\nu} = \kappa_0^{-1} g^{\mu\nu} + \cdots \ , \quad \tilde{\beta}_2^{\mu\nu} = \tilde{\kappa}_0^{-1} g^{\mu\nu} + \cdots$$

- Integrating out P, we get Lagrangian form (with gradientness)

$$S = \int d^4x dz n \Big(\kappa_0 (D_z\Phi)^2 + \kappa (D_\mu\Phi)^2$$
$$+ \kappa_0 F_{z\mu}^2 + \rho^2 (D_\mu\Phi)^2 + \kappa F_{\mu\nu}^2$$
$$+ \tilde{\kappa_0} \tilde{F}_{z\mu}^2 + \tilde{\rho}^2 (D_\mu\Phi)^2 + \tilde{\kappa} \tilde{F}_{\mu\nu}^2$$
$$+ K_0 H_{z\mu\nu}^2 + H_{\mu\nu\rho}^2 + (\gamma(B_{\mu\nu} - \tilde{F}_{\mu\nu}) + \tilde{\gamma} F_{\mu\nu})^2 + \text{more derivatives}\Big) \ .$$

- Higher derivative terms are not suppressed

“Strongly coupled assumption”

- From quantum RG viewpoint alone, there is no reason why higher derivative terms are suppressed
- Gradient condition e.g. must give $\beta_g = \beta(g) + \bar{\chi}^a D^2 g + \cdots$
- But we do expect that higher derivative terms are gone in “strongly coupled CFTs”
- Some terms are further constraint from 1+4 d diffeormorphism, but we need more non-trivial cancellation mechanism (discuss later)

$$\Lambda[\Phi] = \cdots + \chi^a D^2\Phi D^2\Phi + \chi^g D_\mu \Phi D_\nu \Phi G^{\mu\nu} + \chi^c (D_\mu \Phi)^4 + \cdots$$

must cancel against other terms (quite non-trivial)

- Assuming they all contrive to happen, we should get

$$S = \int d^4x dz \sqrt{g} \Big(\chi (D_M \Phi)^2 + \kappa F_{MN}^2 + \tilde{\kappa} \tilde{F}_{MN}^2 + H_{MNL}^2 + (\gamma (B_{MN} - \tilde{F}_{MN}) + \tilde{\gamma} F_{MN})^2 \Big)$$

- Stuckerberg action with massive vector/two-forms $D_M \Phi = \partial_M \Phi + A_M$
- Anomalous dimension and current central charges are naturally obtained

Holographic renormalization group

Mainly from my paper **arXiv:1401.5257**

CFT with only EM tensor

Apply the recipe to CFT with only single trace Energy-momentum tensor (and its multi- trace)

- Schwinger functional is 4 dim diff invariant

$$e^{-W[g_{\mu\nu}]} = \int \mathcal{D}X e^{-S_0 + h_{\mu\nu}T^{\mu\nu}}$$

$$\delta g_{\nu\nu} = D_\mu \xi_\nu + D_\nu \xi_\mu$$

- The local renormalization group functions such as double trace beta function are model dependent, but the form is more or less fixed:

$$\delta S = \Lambda + \bar{\beta}^{\mu\nu} T_{\mu\nu} + \bar{\beta}^{\mu\nu;\rho\sigma} T_{\mu\nu} T_{\rho\sigma} + \cdots$$

$$\Lambda = \Lambda_0 + R + \cdots$$

$$\bar{\beta}^{\mu\nu} = c g^{\mu\nu} + \cdots \qquad \bar{\beta}^{\mu\nu;\rho\sigma} = a g^{\mu\nu} g^{\rho\sigma} + b g^{\mu\rho} g^{\nu\sigma} + \cdots$$

CFT with only EM tensor 2

- One thing to note is that we can simultaneously change the coordinate along the local RG

$$g_{\mu\nu} \to g_{\mu\nu} + dz(D^\mu N^\nu + D^\nu N^\mu)$$

- The effect is add $\bar{P}_{\mu\nu}(D_\mu N_\nu)$.
- We will regard N^μ as shift vector
- A freedom in local renormalization factor $n(x, z)$ is regarded as Lapse function

$$ds^2 = n^2 dz^2 + g_{\mu\nu}(dx^\mu + N^\mu dz)(dx^\nu + N^\nu dz)$$

- The resulting theory is nothing but Hamiltonian form of 1+4 dimensional (Einstein) gravity (if everything works fine…)

$$S = \int dz d^4x \left(\bar{P}^{\mu\nu}\partial_z g_{\mu\nu} - N^\mu D^\nu \bar{P}_{\mu\nu} - nH\right)$$

$$H = \sqrt{g}(\Lambda_0 + R + \cdots) - \bar{\beta}^{\mu\nu}\bar{P}_{\mu\nu} - \bar{\beta}^{\mu\nu;\rho\sigma}\bar{P}_{\mu\nu}\bar{P}_{\rho\sigma}$$

- Some claims are based on “optimism” (e.g. gradientness, consistency of constraint, cancellations of higher derivatives).
- Better to check if possible.

- Does it reproduce what we know in AdS/CFT?

a = c test of quantum local RG → AdS/CFT

- Consider d = 4 CFT with $a = c$
- Dimensionless local renormalization functions are universally determined by trace anomaly

$$\Lambda[g_{\mu\nu}] = \Lambda_0 + R + a\text{Euler} - c\text{Weyl}^2 + bR^2$$
$$= \Lambda_0 + R + \left(\frac{c}{3} - a + b\right) R^2 + (-2c + a)R_{\mu\nu}^2 + (c-a)R_{\mu\nu\rho\sigma}^2 \ .$$
$$\bar{\beta}_{\mu\nu} = 2g_{\mu\nu} + \bar{\beta}_1 R g_{\mu\nu} + \bar{\beta}_2 R_{\mu\nu}$$
$$\bar{\beta}_{\mu\nu;\rho\sigma} = g_{\mu\rho}g_{\nu\sigma} - \lambda g_{\mu\nu}g_{\rho\sigma} \qquad \mathcal{G}^{\mu\nu;\rho\sigma} = \bar{\beta}^{-1}_{\mu\nu;\rho\sigma}$$

- Consistency of Hamiltonian requires $b = 0 \qquad \lambda = \frac{1}{3}$
- Suppose we'd like to construct the gravity dual with Einstein-Hilbert action (with no higher derivative). Then, we need to cancel $R_{\mu\nu}^2 - \frac{1}{3}R^2$ terms in $\Lambda[g_{\mu\nu}]$ with $\bar{\beta}_{\mu\nu}\mathcal{G}^{\mu\nu;\rho\sigma}\bar{\beta}_{\rho\sigma}$.

$$\bar{\beta}_{\mu\nu} = 2g_{\mu\nu} + \sqrt{a}\left(R_{\mu\nu} - \frac{R}{6}g_{\mu\nu}\right)$$

This appearance of Schouten tensor was computed by Verlinde et al, Kiritsis-Li-Nitti in AdS/CFT

- On the other hand, the metric beta function must be gradient

$$\bar{\beta}_{\mu\nu} = \bar{\beta}_{\mu\nu;\rho\sigma}\frac{\delta S_{\text{EH}}}{\delta g_{\rho\sigma}}$$
$$S = \int dz d^d x \left(\bar{P}^{\mu\nu}\partial_z g_{\mu\nu} - N^\mu D^\nu \bar{P}_{\mu\nu} - NH\right)$$
$$H = \sqrt{g}(\Lambda_0 + R + \cdots) - \bar{\beta}^{\mu\nu}\bar{P}_{\mu\nu} - \bar{\beta}^{\mu\nu;\rho\sigma}\bar{P}_{\mu\nu}\bar{P}_{\rho\sigma}$$

a - c test of quantum local RG → AdS/CFT

- Consider d = 4 CFT with $a \neq c$
- Dimensionless local renormalization functions are universally determined by trace anomaly

$$\Lambda[g_{\mu\nu}] = \Lambda_0 + R + a\text{Euler} - c\text{Weyl}^2 + bR^2$$
$$= \Lambda_0 + R + \left(\frac{c}{3} - a + b\right) R^2 + (-2c + a)R_{\mu\nu}^2 + (c - a)R_{\mu\nu\rho\sigma}^2 \ .$$

$$\bar{\beta}_{\mu\nu} = 2g_{\mu\nu} + \bar{\beta}_1 R g_{\mu\nu} + \bar{\beta}_2 R_{\mu\nu} \qquad S = \int dz d^d x \left(\bar{P}^{\mu\nu}\partial_z g_{\mu\nu} - N^\mu D^\nu \bar{P}_{\mu\nu} - NH\right)$$
$$\bar{\beta}_{\mu\nu;\rho\sigma} = g_{\mu\rho}g_{\nu\sigma} - \lambda g_{\mu\nu}g_{\rho\sigma} \qquad H = \sqrt{g}(\Lambda_0 + R + \cdots) - \bar{\beta}^{\mu\nu}\bar{P}_{\mu\nu} - \bar{\beta}^{\mu\nu;\rho\sigma}\bar{P}_{\mu\nu}\bar{P}_{\rho\sigma}$$

- Within $O(R^2)$, $R^2_{\mu\nu\rho\sigma}$ cannot be cancelled from $\bar{\beta}_{\mu\nu}\mathcal{G}^{\mu\nu;\rho\sigma}\bar{\beta}_{\rho\sigma}$. This is the only origin of $\int d^4x dz \sqrt{g_5} R^2_{IJKL}$ term in d=1+4 from quantum local RG construction
- On the other hand, holographic Weyl anomaly computation says that the source of a-c only comes from $\int d^4x dz \sqrt{g_5} R^2_{IJKL}$ term
- "Derivation" is consistent at $O(R^2)$
- If we wish to retain bR^2 anomaly at the lowest derivative, we may change λ, but we had work harder to get consistency

… and Holographic SFT?

I wish the content would be published somewhere soon…

- Going back to local RG consistency condition we discussed, $\beta\rho = 0$

  allows the other possibility

$$\rho = 0 \ , \beta \neq 0$$

$$\mathrm{Tr} D^\mu J_\mu = \mathrm{Tr} O$$

- It means $T^\mu_{\ \mu} = \beta O = \beta D^\mu J_\mu$

- When trace of EM tensor is a divergence of virial current, it is scale invariant but not conformal

$$\delta S = \int d^4x \sqrt{g}\sigma(x) T^\mu_{\ \mu} = \int d^4x \sqrt{g}\sigma(x)\beta D^\mu J_\mu$$

Quantum RG construction

- We focus on $\mathrm{Tr}O$ and $\mathrm{Tr}J_\mu$ sector (and multi-trace)
- Single trace beta function

$$\delta S_{\text{single}} = \int d^4x \beta n(x)((1-s)\mathrm{Tr}O + s\mathrm{Tr}D^\mu J_\mu)$$

- This is ambiguous because of the operator identity

$$\mathrm{Tr}D^\mu J_\mu = \mathrm{Tr}O$$

(with original 4 dimensional gauge invariance of the Schwinger functional)

$$S = \int d^4x \sqrt{g}\,(L_0 + g(x)\mathrm{Tr}O(x) + A^\mu(x)\mathrm{Tr}J_\mu)$$

$$g(x) \to g(x) + w(x)$$
$$A_\mu(x) \to A_\mu + \partial_\mu w(x)$$

- This gives 1+4 dimensional gauge symmetry of the bulk action

Bulk effective action

$$\delta S_{\text{single}} = \int d^4x \beta n(x)((1-s)\text{Tr}O + s\text{Tr}D^\mu J_\mu)$$

- Following the recipe, we get the following bulk matter action

$$\beta n s \to A_z$$

$$S = \int d^4x dz \left(P(\partial_z \Phi + \beta n - A_z) + n\beta_2 P^2 \right.$$
$$+ P^\mu(\partial_z A_\mu - \partial_\mu A_z) + n\beta_2^{\mu\nu} P_\mu P_\nu$$
$$\left. + n(\Lambda_0 + (\partial_\mu \Phi + A_\mu)^2 + F_{\mu\nu}^2 + \cdots\right)$$

- We have introduced the RG scale dependent gauge transformation → (1+4) dim Stuckelberg symmetry
- Freedom of beta function is given by gauge symmetry on A_z

Various gauge fixing

$$S = \int d^4x dz \left(P(\partial_z \Phi + \beta n - A_z) + n\beta_2 P^2 + P^\mu(\partial_z A_\mu - \partial_\mu A_z) + n\beta_2^{\mu\nu} P_\mu P_\nu + n(\Lambda_0 + (\partial_\mu \Phi + A_\mu)^2 + F_{\mu\nu}^2 + \cdots \right)$$

- In unitary gauge $\Phi = 0$, a configuration $A_z dz = \beta n dz \sim \beta \frac{dz}{z}$ would be a natural solution (AdS breaking vector VEV but scale inv)

- In axial gauge $A_z = 0$, we instead obtain $\Phi \sim \beta \log z$ ("cyclic RG" that cannot be cancelled)

  c.f. $T^\mu_\mu = \beta O = \beta D^\mu J_\mu$

- In "aether gauge" $A_z = \beta n$ we get "healthy extension of Horava gravity" (Foliation preserving diff gravity with $(n^{-1}\partial_\mu n)^2$ terms)
- We did not ask for unitarity. Consistency of Hamiltonian constraint, energy-condition etc. To be checked.

Conclusion

- Local RG → Quantum RG → Holographic RG seem consistent and constraints are beautifully realized
- Need better to understand "strongly coupled assumption" in CFT that assures the bulk locality
- Need better to understand "Wilsonian" local RG constraint beyond power-counting
- Need better understand the bulk role of "unitarity" and "causality" of the boundary QFT (e.g. energy condition)
- Local/Qunatum/Holographic renormalization of non-local operators (Wilson loop, defects) and entanglement entropy

Derivation of quantum RG

“Derivation” of AdS/CFT 1

1st step: Remove multi trace operator

- Start with multi-trace Schwinger functional

$$e^{-W[g_0(x),g_n(x)]} = \int \mathcal{D}X \exp\left(-S_0 + \int g_0(x)\mathrm{Tr}O + \int g_n(x)(\mathrm{Tr}O)^n\right)$$

- Introduce two fields P_0 Φ_0 to get rid of multi-trace terms

$$e^{-W} = \int \mathcal{D}X\mathcal{D}\Phi_0\mathcal{D}P_0 e^{\int P_0(g_0-\Phi_0)+\int g_n P_0^n} e^{-S_0+\int \Phi_0 \mathrm{Tr}O}$$

- Single trace action. The price to pay is that coupling constant is dynamical and quantum mechanical.

“Derivation” of AdS/CFT 2

2nd step: Local renormalization

- Change (local) renormalization scale with $dzn(x,z)$

- Action is renormalized. Generation of double (or higher) trace operators.

$$\delta S_0 = dz \int d^d x n(x,z) \left(\Lambda[g_0] + \beta_0[g_0]\mathrm{Tr}O + \beta_2[g_0](\mathrm{Tr}O)^2 + \cdots \right)$$

- Local renormalization means beta functions can contain derivative terms:

$$\Lambda[g_0] = \Lambda_0(g_0) + (D_\mu g_0)^2 + \cdots$$

“Derivation” of AdS/CFT 3

3rd step: Get rid of RG generated double trace term

- With the same trick, we get rid of RG generated double trace term with two new fields Φ_1 P_1

$$\delta S_0 = dz \int d^d x n(x,z) \left(\Lambda[g_0] + \beta_0[g_0] \mathrm{Tr} O + \beta_2[g_0] (\mathrm{Tr} O)^2 + \cdots \right)$$

$$e^{-W_{\mathrm{eff}}} =$$

$$\int \mathcal{D}\Phi_1 \mathcal{D}P_1 e^{\int P_1(\Phi_0 - \Phi_1) + dz \int n(x,z)\Lambda[\Phi_0] + \beta_0[\Phi_0]P_1 + \beta_2[\Phi_0]P_1^2} e^{-S_0 + \int \Phi_1 \mathrm{Tr} O}$$

- At each step in RG, we can get rid of the generated multi-trace operator with two new dynamical fields
- This is what is called quantum RG (Sung-Sik Lee).

“Derivation” of AdS/CFT 4

4th step: Repeat RG and removal of double trace ops

- Now, along the finite local RG transformation, coupling constant becomes d+1 dimensional dynamical field

$$e^{-W} = \int \mathcal{D}\Phi(x,z)\mathcal{D}P(x,z)e^{\int dz d^dx P\partial_z\Phi + n(x,z)(\Lambda[\Phi]+\beta_0[\Phi]P+\beta_2[\Phi]P^2)}$$

- This is “Hamiltonian form” of path integral. One may integrate out P to get “Lagrangian” form

$$e^{-W} = \int \mathcal{D}\Phi(x,z)e^{\int dz d^dx L_{\mathrm{bulk}}[\Phi]}$$

$$L_{\mathrm{bulk}}[\Phi] = n^{-1}(\partial_z\Phi)^2 + n\Lambda[\Phi] + n\beta_2^{-1}\beta_0[\Phi]^2 + \cdots$$

- When single trace beta function is a gradient flow, it looks like d+1 dimensional 2nd order action for field $\Phi(z,x)$

Recall $\Lambda[g_0] = \Lambda_0(g_0) + (D_\mu g)^2 + \cdots$