

Nonrenormalizability of nonequilibrium quantum field theory in the classical approximation

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*T. Epelbaum, F. Gelis and B. Wu, Phys. Rev. D **90** 065029 (2014).*

*T. Epelbaum, F. Gelis, N. Tanji and B. Wu, Phys. Rev. D **90** 125032 (2014).*



① Introduction

② Nonrenormalizability of the classical statistical approximation (CSA)

- UV divergences in 4-point functions
- UV divergences in self-energies

③ The Boltzmann equation in the classical approximation

- Early-time UV cutoff dependence
- Late-time UV cutoff dependence

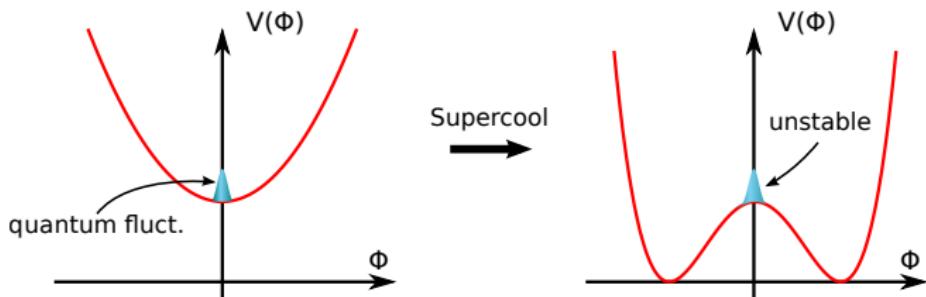
④ Summary

§1 Introduction: applications of the CSA

- Cold Electroweak Baryogenesis in two-Higgs-doublet model

- Supercooling in the early universe:

Instability \Rightarrow large distribution $f \simeq f_0 e^{2\sqrt{\mu^2 - p^2} t} \gg 1$

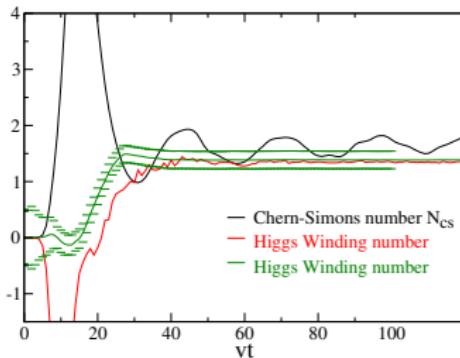


See A. Tranberg and B. Wu, JHEP 1207, 087 (2012); JHEP 1301, 046 (2013).

§1 Introduction: applications of the CSA

- Cold Electroweak Baryogenesis in two-Higgs-doublet model

- Using the classical field simulation (CSA):



- Anomaly \Rightarrow generation of baryon number

$$B(t) - B(0) = L(t) - L(0) = 3 [N_{\text{CS}}(t) - N_{\text{CS}}(0)] .$$

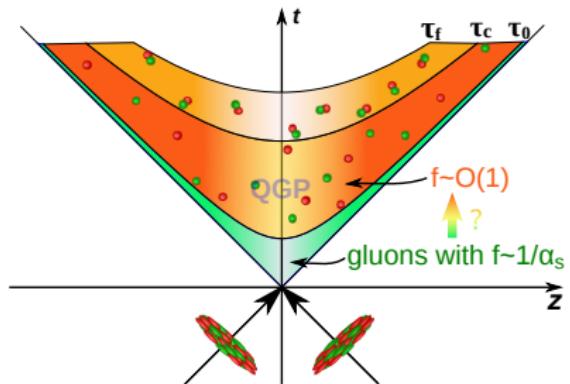
Give a successful explanation of observed cosmological baryon asymmetry.

See A. Tranberg and B. Wu, JHEP 1207, 087 (2012); JHEP 1301, 046 (2013).

§1 Introduction: applications of the CSA

- To understand thermalization in ultra-relativistic heavy-ion collisions
Important but theoretically challenging!
 - Saturation model (CGC) in QCD:

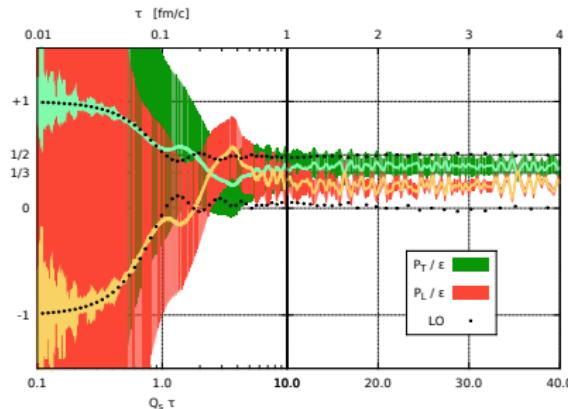
Thermalization of a dense system of gluons $f \sim \frac{1}{\alpha_s}$ at $\tau \sim \frac{1}{Q_s}$



A. H. Mueller, Nucl. Phys. B 572, 227 (2000) [hep-ph/9906322].

§1 Introduction: applications of the CSA

- Isotropization or thermalization using the CSA



Evolution of the ratios $P_{T,L}/\epsilon$ for $g = 0.5$ ($\alpha_s = 2 \cdot 10^{-2}$).

T. Epelbaum and F. Gelis, Phys. Rev. Lett. **111**, 232301 (2013) [[arXiv:1307.2214 \[hep-ph\]](#)].

Motivate us to study UV cutoff dependence of CSA!

See a different result by

J. Berges, K. Boguslavski, S. Schlichting and R. Venugopalan, Phys. Rev. D **89**, 074011 (2014) [[arXiv:1303.5650 \[hep-ph\]](#)].

§1 Introduction: framework for studying non-equilibrium processes

• The Schwinger-Keldysh formalism

- Given $\hat{\rho}(0)$,

$$\begin{aligned}\langle \hat{O}(t) \rangle &= \text{Tr} [\hat{O} \hat{\rho}(t)] = \text{Tr} [\hat{O} \textcolor{red}{U(t)} \hat{\rho}(0) \textcolor{blue}{U(t)}^\dagger] \\ &= \int D\phi_+ D\phi_- \hat{O}[\phi(t)] e^{\frac{i}{\hbar} S[\phi_+]} \langle \phi_{+0} | \hat{\rho}(0) | \phi_{-0} \rangle e^{-\frac{i}{\hbar} S[\phi_-]}.\end{aligned}$$



- In $\frac{g^2}{4!} \phi^4$ theory with $\phi_2 \equiv \frac{1}{2}(\phi_+ + \phi_-)$ and $\phi_1 \equiv \phi_+ - \phi_-$

$$\begin{aligned}\langle \hat{O}(t) \rangle &= \int d\phi_t d\phi_0 \underbrace{\int d\sigma_0 e^{-\frac{i}{\hbar} \sigma_0 \pi_0} \langle \phi_0 + \frac{\sigma_0}{2} | \hat{\rho}(0) | \phi_0 - \frac{\sigma_0}{2} \rangle}_{\rho_W[\phi_0, \pi_0] \text{ with } \pi_0 = \dot{\phi}_0} \hat{O}[\phi_t] \\ &\quad - \frac{i}{\hbar} \int_0^t d^4x \{ \phi_1 [(\square + m^2) \phi_2 + \underbrace{\frac{g^2}{6} \phi_2^3}_{\text{classical vertex}} - j] + \underbrace{\frac{g^2}{24} \phi_1^3 \phi_2}_{\text{quantum vertex}} \} \\ &\times \int D\phi_1 D\phi_2 e^{\phi_1(0) = \sigma_0, \phi_1(t) = 0} \\ &\quad \phi_2(0) = \phi_0, \phi_2(t) = \phi_t\end{aligned}$$

For a review: K. -c. Chou, Z. -b. Su, B. -l. Hao and L. Yu, Phys. Rept. 118, 1 (1985).

§1 Introduction: the quasiparticle approximation

- Green functions $G_{ab}(x, y)$ ($\hat{O} = \phi_a(x)\phi_b(y)$)

- The quasiparticle approximation:

Assume $X^\mu \equiv (x^\mu + y^\mu)/2$ is a slowly varying variable in $G_{ab}(x, y)$

$$G_{22}(X, p) = \left(f_p(X) + \underbrace{\frac{1}{2}}_{\text{vacuum quanta}} \right) 2\pi\delta(p^2 - m^2), \quad G_{11}(X, p) = 0,$$

$$G_{12}(X, p) = \frac{i}{p^2 - m^2 + ip^0\epsilon}, \quad G_{21}(X, p) = \frac{i}{p^2 - m^2 - ip^0\epsilon}.$$

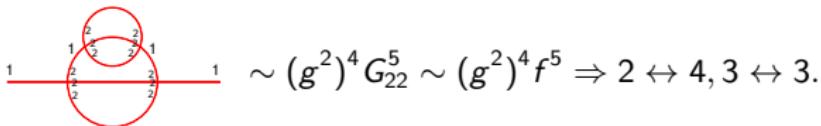
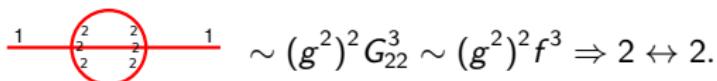
- The Dyson-Schwinger equation \Rightarrow The Boltzmann equation

$$D_t f_p \equiv \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} \right) f_p = -\frac{i}{2\omega_p} \left[(\Sigma_{12} - \Sigma_{21}) \left(f_p + \frac{1}{2} \right) - \Sigma_{11} \right] \equiv \mathcal{C}[f].$$

A. H. Mueller and D. T. Son, Phys. Lett. B 582, 279 (2004) [hep-ph/0212198].

§1 Introduction: the quasiparticle approximation

- The power counting (in Σ_{11}) with the classical vertex $\mathcal{L}_{lc} = -\frac{g^2}{6}\phi_1\phi_2^3$



Graph with n classical vertices $\sim (g^2)^n G_{22}^{n+1} \sim (g^2)^n f^{n+1}$.

Including the quantum vertex reduces the power of f .

- If $f \sim \frac{1}{g^2}$, all the above graphs are of the same order $O(g^{-2})$.

To resum those graphs, neglect the quantum vertex \Rightarrow the CSA!.

§1 Introduction: the classical statistical approximation (CSA)

- The classical statistical approximation(CSA)

- The formalism:

$$\langle \hat{O}(t) \rangle = \int d\phi_0 d\pi_0 \rho_W[\phi_0, \pi_0] \hat{O}[\phi_{cl}(t)].$$

where

$$\square \phi_{cl} + \frac{g^2}{6} \phi_{cl}^3 = j \text{ with } \phi_{cl}(0) = \phi_0, \dot{\phi}_{cl}(0) = \pi_0.$$

Proof: dropping the quantum vertex $-\frac{g^2}{24} \phi_1^3 \phi_2$

$$\begin{aligned} \langle \hat{O}(t) \rangle &= \int d\phi_t d\phi_0 \rho_W[\phi_0, \dot{\phi}_0] \hat{O}[\phi_t] \\ &\times \int D\phi_2 D\phi_1 \underbrace{e^{-\frac{i}{\hbar} \int_{t_0}^t d^4x \left[\phi_1 \left(\square \phi_2 + \frac{g^2}{6} \phi_2^3 - j \right) \right]}}_{\delta \left[\square \phi_2 + \frac{g^2}{6} \phi_2^3 - j \right]} \\ &\phi_1(0) = \sigma_0, \phi_1(t_\infty) = 0 \\ &\phi_2(0) = \phi_0, \phi_2(t) = \phi_t \end{aligned}$$

A. H. Mueller and D. T. Son, Phys. Lett. B 582, 279 (2004) [hep-ph/0212198].

§2 Nonrenormalizability of CSA: coherent states

- The perturbative vacuum: a **coherent state**

$$\rho_0 = |0\rangle\langle 0| \Rightarrow \rho_W[\phi_0, \pi_0] = \mathcal{N} e^{-\int \frac{d^3 p}{(2\pi)^3} \left(p|\phi_0(p)|^2 + \frac{1}{p} |\pi_0(p)|^2 \right)},$$

where interactions switched on adiabatically at $t = -\infty$.

- The free propagators**

$$G_{22}^0(p) = \pi \delta(p^2 - m^2), \quad G_{11}^0(p) = 0,$$

$$G_{12}^0(p) = \frac{i}{p^2 - m^2 + ip^0\epsilon}, \quad G_{21}^0(p) = \frac{i}{p^2 - m^2 - ip^0\epsilon}.$$

- The Lagrangian mimicking the CGC framework**

$$\mathcal{L} = -\phi_1 [(\square + m^2)\phi_2 + \underbrace{\frac{g^2}{6}\phi_2^3}_{\text{classical}} - \underbrace{j}_{\text{source}}] - \underbrace{\frac{g^2}{24}\phi_1^3\phi_2}_{\text{quantum}}$$

where the last term is dropped in the CSA.

§2 Nonrenormalizability of CSA: 4-point functions at NNLO

- Linear divergence of Γ_{1122}

$$\begin{aligned} -i[\Gamma_{1122}]_{\text{CSA}}^{\text{1 loop}} &= \text{Diagram 1} + \text{Diagram 2} \\ &= -\frac{g^4}{64\pi} \left[\text{sign}(T) + \text{sign}(U) + 2\Lambda_{\text{UV}} \left(\frac{\theta(-T)}{|\mathbf{p}_1 + \mathbf{p}_3|} + \frac{\theta(-U)}{|\mathbf{p}_1 + \mathbf{p}_4|} \right) \right] \end{aligned}$$

Diagram 1: A 4-point function with two external lines labeled p_1 and p_3 meeting at a vertex with indices 1 2. Two internal lines labeled 2 2 meet at a central loop. Diagram 2: Similar to Diagram 1, but the external lines p_1 and p_4 are swapped.

with $T \equiv (p_1 + p_3)^2$, $U \equiv (p_1 + p_4)^2$.

Need non-local counter-term $\sim \phi_1^2 \phi_2^2 \Rightarrow$ nonrenormalizable

§2 Nonrenormalizability of CSA: 4-point functions at NNLO

- Including the quantum vertex

$$-i\Gamma_{1122}^{\text{1 loop}} = \underbrace{\text{S channel}}_{\begin{array}{c} \text{p}_1 \diagdown 11 \diagup \text{p}_3 \\ \diagup 12 \quad \diagdown 22 \\ \diagup 12 \quad \diagdown 22 \\ \text{p}_2 \quad \text{p}_4 \end{array}} + \underbrace{\text{T channel}}_{\begin{array}{c} \text{p}_1 \diagdown 12 \diagup \text{p}_3 \\ \diagup 22 \quad \diagdown 11 \\ \diagup 22 \quad \diagdown 11 \\ \text{p}_2 \quad \text{p}_4 \\ \text{p}_1 \diagdown 12 \diagup \text{p}_3 \\ \diagup 22 \quad \diagdown 11 \\ \diagup 22 \quad \diagdown 11 \\ \text{p}_2 \quad \text{p}_4 \end{array}} + \underbrace{\text{U channel}}_{\begin{array}{c} \text{p}_1 \diagdown 12 \diagup \text{p}_4 \\ \diagup 22 \quad \diagdown 11 \\ \diagup 22 \quad \diagdown 11 \\ \text{p}_2 \quad \text{p}_3 \\ \text{p}_1 \diagdown 12 \diagup \text{p}_4 \\ \diagup 11 \quad \diagdown 22 \\ \diagup 11 \quad \diagdown 22 \\ \text{p}_2 \quad \text{p}_3 \end{array}} + \frac{g^4}{32\pi} [\theta(T) + \theta(U)].$$

Finite as the full theory is renormalizable!

§2 Nonrenormalizability of CSA: 4-point functions at NNLO

- Could it be fixed?

① Minimal requirement for avoiding the strong cutoff dependence

$$[G_{22}]_{\text{CSA}}^{\text{LO}} = \begin{array}{c} \text{Diagram: two horizontal lines with red dots at vertices, indices 2 below lines} \end{array} \propto \frac{Q^2}{g^2}, [G_{22}]_{\text{CSA}}^{\text{NLO}} = \begin{array}{c} \text{Diagram: two horizontal lines with red dots, indices 2 below lines, a diagonal line connecting them with index 1} \end{array} \propto Q^2,$$
$$[G_{22}]_{\text{CSA}}^{\text{NNLO}} = \begin{array}{c} \text{Diagram: two horizontal lines with red dots, indices 2 below lines, a loop between them with indices 1} \end{array} \propto g^2 \Lambda_{UV} Q \ll [G_{22}]_{\text{CSA}}^{\text{NLO}} \Leftrightarrow \Lambda_{UV} \ll \frac{Q}{g^2}$$

② Introducing counter-term

$$\Delta S \equiv -\frac{i}{2} \int d^4x d^4y [\phi_1(x)\phi_2(x)] v(x,y) [\phi_1(y)\phi_2(y)]$$

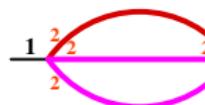
with

$$v(x,y) \equiv \frac{g^4}{64\pi^3} \frac{\Lambda_{UV}}{|\mathbf{x}-\mathbf{y}|} \delta((x^0-y^0)^2 - (\mathbf{x}-\mathbf{y})^2).$$

§2 Nonrenormalizability of CSA: self-energies

- Self-energies at two loop (NNLO)

$$-i[\Sigma_{11}(P)]_{\text{CSA}}^{\text{2 loop}} = -\frac{1}{1024\pi^3} \left(\Lambda_{\text{UV}}^2 - \frac{2}{3}p^2 \right),$$


$$\text{Im} [\Sigma_{12}(P)]_{\text{CSA}}^{\text{2 loop}} = -\frac{g^4}{1024\pi^3} \left(\Lambda_{\text{UV}}^2 - \frac{2}{3}p^2 \right)$$


Contributing to the collision term of the Boltzmann equation!

- Required counter-terms

$\mathcal{O} = i\phi_1^2$ or $i\phi_1\phi_2$ for Σ_{11} or Σ_{12} , not exist in the CSA Lagrangian!

Nonrenormalizable: UV cutoff dependence (starting) at NNLO!

§3 The Boltzmann equation in the classical approximation

- Equivalence between the Boltzmann equation and the CSA

$$\frac{1}{g^2} \gtrsim f \gg 1.$$

Use the Boltzmann equation to study Λ_{UV} dependence in the CSA!

- In the classical approximation \mathcal{C}^1 :

By neglecting the quantum vertex,

$$\begin{aligned} D_t f_p &\equiv \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} \right) f_p = -\frac{i}{2\omega_p} \left[(\Sigma_{12} - \Sigma_{21}) \left(f_p + \frac{1}{2} \right) - \Sigma_{11} \right] \\ &= \frac{g^4}{4\omega_p} \int_{\mathbf{p}' \mathbf{k} \mathbf{k}'} (2\pi)^4 \delta(P + K - P' - K') \left[-\frac{1}{4} (f_p + f_k - f_{p'} - f_{k'}) \right. \\ &\quad \left. + f_{p'} f_{k'} (1 + f_p)(1 + f_k) - f_p f_k (1 + f_{p'})(1 + f_{k'}) \right] \equiv C^{\mathcal{C}^1}. \end{aligned}$$

- Thermal equilibrium distribution: $f_p = \underbrace{\frac{T}{\omega_p - \mu}}_{O(\hbar^0)} - \underbrace{\frac{1}{2}}_{O(\hbar)}$

§3 The Boltzmann equation in the classical approximation

- Early-time UV cutoff dependence ($\Lambda_{UV} \gtrsim \frac{Q}{g^2}$)

- Analytically

In a spatially homogeneous system

$$\frac{\partial}{\partial t} f_p \simeq -\frac{g^4 \Lambda_{UV}^2}{1024\pi^3} \frac{1}{\omega_p} f(p)$$

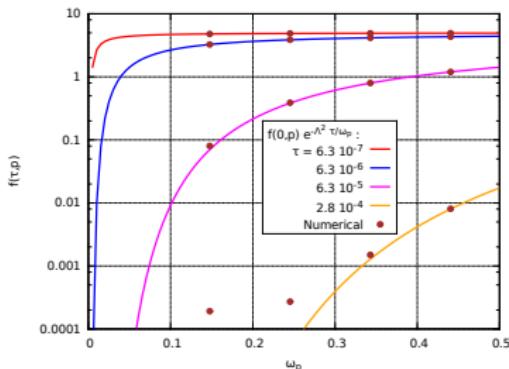
which gives

$$f_p \simeq f_p(0) e^{-\frac{g^4 \Lambda_{UV}^2}{1024\pi^3} \frac{t}{\omega_p}}.$$

- Requiring

$$\boxed{\text{Linear terms} \ll \text{Cubic terms} \Leftrightarrow \Lambda_{UV} \ll \frac{Q}{g^2}!}$$

- Numerically



T. Epelbaum, F. Gelis, N. Tanji and B. Wu, Phys. Rev. D 90 125032 (2014) [arXiv:1409.0701 [hep-ph]].

§3 The Boltzmann equation in the classical approximation

- Late-time UV cutoff dependence

- The classical approximation \mathcal{C}^0

If cubic terms dominate,

$$\begin{aligned} C_{\mathbf{p}}^{\mathcal{C}^0}[f] &= \frac{g^4}{4\omega_{\mathbf{p}}} \int_{\mathbf{k}} \int_{\mathbf{p}'} \int_{\mathbf{k}'} (2\pi)^4 \delta(P + K - P' - K') \\ &\times [f(\mathbf{p}')f(\mathbf{k}')(f(\mathbf{p}) + f(\mathbf{k})) - f(\mathbf{p})f(\mathbf{k})(f(\mathbf{p}') + f(\mathbf{k}'))] . \end{aligned}$$



Max Planck

- Thermal equilibrium distribution:

$$f_p = \frac{T}{\omega_p - \mu}, \text{ purely classical}$$

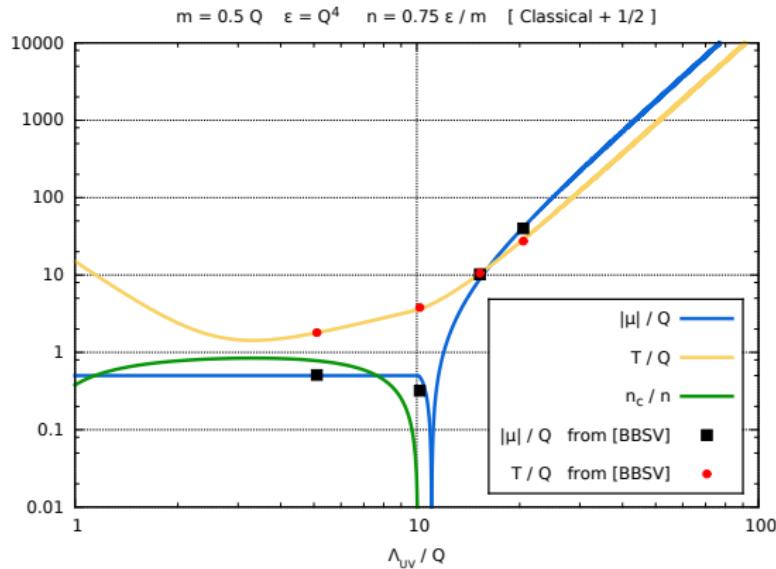
- Scaling solution: no thermalization

- UV cutoff dependent thermalization:

Thermalized only if Λ_{UV} is finite!

§3 The Boltzmann equation in the classical approximation

- Late-time UV cutoff dependence
 - Equivalence to the CSA



Here, [BBSV] stands for the results in the CSA in Ref. J. Berges, K. Boguslavski, S. Schlichting and R. Venugopalan, JHEP 1405, 054 (2014) [[arXiv:1312.5216 \[hep-ph\]](https://arxiv.org/abs/1312.5216)].

Λ_{UV} dependence of the CSA at late times \Rightarrow Using Boltzmann equation!

Summary and perspective

- **About the CSA:**

- CSA: **unique method** for real-time evolution of systems with $f \sim \frac{1}{g^2}$.

$$\langle \hat{O}(t) \rangle = \int d\phi_0 d\pi_0 \rho_W[\phi_0, \pi_0] \hat{O}[\phi_{cl}(t)]$$

where

$$\square \phi_{cl} + \frac{g^2}{6} \phi_{cl}^3 = j \text{ with } \phi_{cl}(0) = \phi_0, \dot{\phi}_{cl}(0) = \pi_0.$$

- **Broad range of applications of CSA**

- ① Cosmology: cold electroweak baryogenesis, reheating ...
- ② Heavy-ion physics: evolution at the early stage ...

- **Limitations of CSA**

- Nonrenormalizability: $\Lambda_{UV} \ll \frac{Q}{g^2}$
- The thermal fixed point depends on Λ_{UV} : **not for near thermal equilibrium**